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Quaternary quadratic forms and an associated lattice constant

Terence Jackson

Abstract. For an indefinite quadratic form \( f(x_1, \ldots, x_n) \) of discriminant \( d \) let \( P(f) \) denote the greatest lower bound of the positive values assumed by \( f \) for integers \( x_1, \ldots, x_n \). This paper uses recent isolation results about ternary forms of signature \(-1\) to reduce the previously known upper bound of \( P^4/d \) for non-zero quaternary forms of signature \(-2\). This gives a new bound for the lattice constant of the body \( 0 \leq x^2 + y^2 + z^2 + t^2 < 1 \).

1. INTRODUCTION

This paper is concerned with the lattice constants of bodies in \( \mathbb{R}^n \) associated with quadratic forms. So we begin with some relevant geometrical definitions.

A lattice \( \Lambda \) in \( \mathbb{R}^n \) is the set of all integral linear combinations of linearly independent vectors \( u_1, \ldots, u_n \) and its determinant \( d(\Lambda) \) is \( |\det\{u_1, \ldots, u_n\}| \). If \( K \) is a body in \( \mathbb{R}^n \) that is centred at the origin then the lattice \( \Lambda \) is admissible for \( K \) if the origin is the only point of \( \Lambda \) in \( K \) (see eg [1]). The lattice constant of \( K \) is defined as

\[
\Delta(K) = \inf\{d(\Lambda) : \Lambda \text{ is admissible for } K\}
\]

So every lattice with determinant less than \( \Delta(K) \) contains a point of \( K \) other than the origin. If \( d(\Lambda) = \Delta(K) \) then \( \Lambda \) is a critical lattice for \( K \).

Beginning with the work of Hurwitz and Markoff in the 19th century a great deal of effort has been put into finding the lattice constants of bodies associated with quadratic forms. For a given \( n \geq 2 \) and \( r < n \) write

\[
F(X_1, \ldots, X_n) = X_1^2 + \cdots + X_r^2 - \cdots - X_n^2.
\]

The signature \( s \) of \( F \) is \( 2r - n \) and satisfies \( |s| < n \) and \( s \equiv n \pmod{2} \). The lattice constants of the bodies \( |F| < 1 \) are now known for every \( n \) and \( s \) (both for

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$0 \leq |F| < 1$ and for $0 < |F| < 1$. The last results here are due to Margulis (see [2]). In the 1940s Segre, Mahler and Davenport introduced the asymmetric problems of finding the lattice constants of the bodies $0 < F < 1$ and $0 < F < 1$. The lattice constants of the bodies $0 < F < 1$ are also now known for every $n$ and $s$ (and for most $n$ are the same as for the bodies $0 < |F| < 1$. However for $n = 4$ the bodies given by $0 < F < 1$ have proved difficult and $n = 4, s = -2$ is the one case still outstanding. So we concentrate on this body which is

$$0 < X_1^2 - X_2^2 - X_3^2 - X_4^2 < 1.$$  

An admissible lattice is known for this body because in 1931 Oppenheim [8] showed in effect that the lattice with basis

$$(1.4) \quad (1,0,0,0), \left(\frac{1}{2}, \frac{\sqrt{5}}{2}, 0, 0\right), \left(\frac{1}{2}, \frac{\sqrt{5}}{2}, \frac{\sqrt{6}}{2}, 0\right), \left(\frac{1}{2}, \frac{\sqrt{5}}{2}, 0, \frac{\sqrt{6}}{2}\right)$$

is admissible for the body in (1.3). This lattice has determinant $\frac{\sqrt{7}}{2} = 1.32\ldots$ but it is not known whether it is a critical lattice. The most that is known up to now is the 1971 result of Worley (see [12]) that the lattice constant must be at least $\frac{9}{8\sqrt{2}} = 0.795\ldots$. In this paper we prove the following result.

**Theorem 1.** The lattice constant of the body $0 < X_1^2 - X_2^2 - X_3^2 - X_4^2 < 1$ is at least $\frac{\sqrt{3}}{3} = 0.918\ldots$.

Statements about the form $F$ in (1.2) and different lattices are equivalent to statements about different quadratic forms of dimension $n$ and signature $s$ and the one lattice of integer vectors. This is because when $f(x_1, \ldots, x_n)$ is a quadratic form of signature $s$ there are linear forms $X_i = \sum b_j x_j$ such that $f(x_1, \ldots, x_n) = X_1^2 + \cdots + X_s^2 - \cdots - X_n^2$. So the values of $f$ at integral points $(x_1, \ldots, x_n)$ are the values of $F$ at points $(X_1, \ldots, X_n)$ of the lattice $\Lambda$ with basis $u_j = (b_{1j}, b_{2j}, \ldots, b_{nj})$. Here the discriminants of $f$ and $F$ are related by $d(f) = d(\Lambda^2 d(F))$. We use the non-Gaussian discriminant defined as in [11] and in particular the discriminant of $X_1^2 - X_2^2 - X_3^2 - X_4^2$ is $-16$. So the lattice $\Lambda$ of determinant $\delta$ will have a non-zero point in the body given by (1.3) if and only if for the corresponding form $f$ there is an integer vector $x \neq 0$ giving a non-negative value of $f(x)$ with

$$0 \leq f^*(x) < \frac{1}{16\delta^2} d(f).$$

This enables us to use the machinery of quadratic forms. If for a fixed $\delta$ we have (1.5) for all forms with $(n, s) = (4, -2)$ that will show that any admissible lattice for the body (1.3) has to have determinant greater than $\delta$. In trying to establish (1.5) we need only consider those forms for which there is no non-zero $x$ with $f(x) = 0$. For these non-zero forms we define

$$(1.6) \quad \nu(f) = \inf\{\text{positive values of } f\}, \quad \mu(f) = \sup\{\text{negative values of } f\}.$$  

Theorem 1 will then follow immediately from the following result about quadratic forms.
Theorem 2. If $f$ is a real non-zero quaternary form of signature $-2$ and discriminant $d$ then

\[
\varphi(f) = P^4/|d| < \frac{2}{27}.
\]

2. PRELIMINARY RESULTS

Lemma 1. If $g$ is a nonzero ternary form of signature $-1$ with $\varphi(g) = P^3(g)/|d(g)|$ \(\geq 1/2\) then it must be equivalent to a multiple of one of the following forms:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$g_i$</th>
<th>$P(g_i)$</th>
<th>$\varphi(g_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-(x + \frac{1}{3}z)^2 + 15(y + \frac{1}{3}z)^2 - \frac{25}{3}z^2$</td>
<td>6</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>2</td>
<td>$-(x + \frac{1}{3}y)^2 + \frac{11}{4}(y + \frac{1}{3}z)^2 - \frac{120}{13}z^2$</td>
<td>9</td>
<td>$\frac{27}{40}$</td>
</tr>
<tr>
<td>3</td>
<td>$-x^2 + 8(y + \frac{1}{3}z)^2 - 3z^2$</td>
<td>4</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>4</td>
<td>$-x^2 + 24(y + \frac{1}{3}z)^2 - \frac{26}{3}$</td>
<td>8</td>
<td>$\frac{8}{13}$</td>
</tr>
<tr>
<td>5</td>
<td>$-(x + \frac{1}{3}y)^2 + \frac{9}{5}(y + \frac{1}{3}z)^2 - \frac{34}{5}z^2$</td>
<td>5</td>
<td>$\frac{126}{27}$</td>
</tr>
<tr>
<td>6</td>
<td>$-(x + \frac{1}{3}z)^2 + 15(y + \frac{1}{3}z)^2 - \frac{25}{3}z^2$</td>
<td>6</td>
<td>$\frac{15}{13}$</td>
</tr>
<tr>
<td>7</td>
<td>$-x^2 + 15(y + \frac{1}{3}z)^2 - \frac{25}{3}z^2$</td>
<td>6</td>
<td>$\frac{9}{13}$</td>
</tr>
<tr>
<td>8</td>
<td>$-x^2 + \frac{34}{5}(y + \frac{1}{3}z)^2 - \frac{25}{5}z^2$</td>
<td>6</td>
<td>$\frac{19}{5}$</td>
</tr>
<tr>
<td>9</td>
<td>$-(x + \frac{1}{3}y)^2 + \frac{9}{5}(y + \frac{1}{3}z)^2 - \frac{34}{5}z^2$</td>
<td>5</td>
<td>$\frac{1}{3}$</td>
</tr>
</tbody>
</table>

Proof. This is Theorem 2 of [6] (with the values of $P(g_i)$ coming also from [4] and [5]).

Lemma 2. Suppose $F = F(x, y)$ is an indefinite binary form that does not represent 0 non-trivially and, as in (1.6), let $P(F)$ be the infimum of the positive values of $F$ with $N(F) = P(-F)$. Then

\[
P^2(F)N(F) \leq \frac{1}{\sqrt{105}} d^{1/2}(F)
\]

Proof. This is the case $n = 2$ of the first part of Theorem 2 in [3].

Lemma 3. Let $f = f(x, y, z, t)$ be a nonzero quaternary form of signature $-2$ with $\varphi(f) = P^4/|d| \geq \frac{1}{2}$. Then we may assume that

\[
f(x, y, z, t) = -(x + ay + bz + ct)^2 + g(y, z, t)
\]

where

\[
0 \leq a \leq \frac{1}{2}, \quad -\frac{1}{2} < b \leq \frac{1}{2}, \quad -\frac{1}{2} < c \leq \frac{1}{2}
\]

and

\[
N(f) = 1
\]

We may also assume that the non-zero ternary form $g$ is equivalent to a positive multiple of one of the forms $g_i$ in Lemma 1.
Proof. We may assume $P(f) > 0$ since otherwise (1.7) certainly holds; and then Theorem 1 of [9] gives $N(f) \neq 0$. We can then scale $f$ to have $N(f) = 1$. Since $N(f)$ is non-zero and is rational, the form $f$ must have rational coefficients by [2].

So $f$ takes the value $-N(f)$ and a unimodular transformation puts it in the shape (2.2). Then simple parallel transformations give (2.3) and $g$ is a non-zero form as otherwise we would have $f(x) \in [-\frac{1}{2}, 0]$ for some $x \neq 0$. We may also suppose that $P(g) = g(l, m, n) = v$ say. Now consider the binary section $F$ of $f$ given by $F(x, y) = f(x, yl, ym, yn)$. Since $f$ represents all the values of $F$ we have $F$ a non-zero form, $P(f) \leq P(F)$ and $N(f) = 1 = N(F)$. So, using the asymmetric inequalities about binary forms in Lemma 2,

$$P^4(f) \leq P^4(F)N^2(F) \leq \frac{2}{6^3}d^2(f) = \frac{16}{27}v^3.$$ 

When $g$ is not equivalent to a multiple of one of the forms $g_1, \ldots, g_9$ we have $v^3 < \frac{1}{3}|d(g)|$ and therefore

$$P^4(f) < \frac{8}{27}|d(g)| = \frac{2}{27}|d(f)|.$$ 

We now use the results of Lemma 3 to begin the proof of Theorem 2. In particular we will always take $f$ to be of the shape (2.2) and can assume that for some $i = 1, \ldots, 9$ we have $g = \frac{k}{P(g)}g_i$ for a positive parameter $k$. So $\varphi(g) = \varphi(g_i)$ and

$$P(g) = k \text{ and } N(g) = -g(1, 0, 0) = \frac{k}{P(g_i)}$$

because each $g_i$ has $N(g_i) = 1$. Choosing $z$ such that $|x + a| \leq \frac{1}{2}$ then gives

$$\frac{1}{4} - \frac{k}{P(g_i)} \leq f(x, 1, 0, 0) = -(x + a)^2 - \frac{k}{P(g_i)} < 0$$

so $N(f) = 1$ forces

$$k > \frac{3}{4}P(g_i).$$

We can also suppose that $\frac{1}{4}a^2 - 1 < k \leq \frac{1}{4}(a + 1)^2 - 1$ for some $a \geq 0$, so that choosing $z$ with $\frac{1}{2} \leq |x + ay + \beta z + \gamma t| \leq \frac{3}{4}$ gives

$$P(f) \leq k - \frac{(a - 1)^2}{4}.$$ 

Since $|d(f)| = 4|d(g)| = 4k^3/\varphi(g_i)$ we therefore have

$$\frac{P^4(f)}{|d(f)|} \leq \frac{\varphi(g_i)}{4} \left( k - \frac{(a - 1)^2}{4} \right)^4$$

and so

$$\varphi(f) \leq \frac{16\varphi(g_i)(a - 1)}{(a + 3)^3}.$$ 

This makes $\varphi(f) < \frac{2}{3}$ for sufficiently large $a$. Indeed when $i = 9$ the inequality (2.6) gives $k \geq \frac{1}{4}$, with consequently $a \geq 4$, and then (2.9) gives $\varphi(f) < \frac{2}{3}$. For
i < 9 we try to improve (2.7) and thus (2.8). We do this by looking for small positive values of $g$ other than the value $k$.

### 3. COMPLETION OF THE PROOF

When $g = \frac{3}{5}g_1$ the bounds (2.6) and (2.9) give (1.7) except for $\frac{27}{4} \leq k \leq \frac{45}{4}$ with $a = 5$ or 6. In this interval (2.8) gives (1.7) for $k < \frac{31}{4}$ when $a = 5$ and for $8 < k < 11.25$ when $a = 6$. But $g_1(6, 1, 1) = 11$ so that $g$ represents $\frac{11k}{6} > |\frac{15(k+1)}{4}|$ and then, with $x$ so that $\frac{3}{5} \leq |x + ay + \beta z + \gamma t| \leq \frac{9}{5}$, we have $P(f) \leq \frac{11k}{6} - \frac{4}{5}$.

This inequality gives (1.7) for the remaining possibilities for $k$.

For $4 < z < 8$ we have $\frac{1}{5}g_1(6, 1, 1) = \frac{31}{15}, \frac{1}{5}g_1(2, 1, 1) = \frac{11k}{6}, \frac{1}{5}g_1(4, 1, 1) = \frac{29k}{24}, \frac{1}{5}g_1(4, 1, 1) = \frac{31}{15}$. When reduced by an appropriate $|x + ay + \beta z + \gamma t|$ each time, we get new upper bounds for $P(f)$.

In a similar manner to the argument for $i = 2$ these new bounds can then be used in conjunction with (2.7) to eliminate the possibility $g = \frac{3}{5}g_1$ for each $i > 4$.

When $g = \frac{3}{5}g_1$, inequalities (2.6) and (2.9) show that we only need to consider the range $3 \leq k \leq 8$ with $a = 3$ when $k = 3$, $a = 4$ or 5 otherwise. Here (2.8) immediately gives (1.7) for $3 < k < 4.95$. For $4.95 < k \leq 5.15$ we have $8.6 < g(1,1,0) = \frac{31}{15} < 9.1$ whence, with $x$ such that $2.5 \leq |x + ay + \beta z + \gamma t| \leq 3$, we get $P(f) \leq \frac{17k}{4} - 6.25$ and then $\varphi(f) \leq \frac{(\frac{17k}{4} - 6.25)^4}{6k^2} < \frac{2}{27}$.

For $5.25 < k < 7.8$ the inequality (2.8) again gives (1.7); and for $7.8 < k \leq 8$ we have $11.7 < g(3,1,1) = \frac{31}{15} < 12$, leading to $P(f) < \frac{31}{3} - 9$ and $\varphi(f) \leq \frac{(\frac{31k}{3} - 10)^4}{6k^2} < \frac{2}{3}$. This leaves the possibilities $k = 3$ or $5.15 < k \leq 5.25$.

When $5.15 < k \leq 5.25$ we may suppose that $P(f) > 2.75$ or else we have at once $P(f)/6k^2 < \frac{2}{3}$. Then using $g(1,1,0) = \frac{31}{15}, g(2,1,0) = k$ and $g(3,1,1) = \frac{31}{15}$ we see that we must have $||a+\beta|| > 0.46, ||2a+\beta|| > 0.47$ and $||3a+\beta+\gamma|| < 0.05$ or else we would have one of $-1 < f(x,1,1,0) < 2.75, -1 < f(x,2,1,0) < 2.75$ or $-1 < f(x,3,1,1) < 2.75$ for some $x$. Hence $\alpha = (2a + \beta) - (a + \beta) < 0.07$ and $||a+\beta+\gamma|| = ||3a+\beta+\gamma-2a|| < 0.19$ which gives $0.4 < f(x,1,1,1) = -(x+a+\beta+\gamma)^2 + \frac{7k}{2} \leq 2.4$ for suitable $x$.

When $k = 3$ then for suitable choice of $x$ we would have either $-1 < -(x+2a+\beta)^2 + k \leq 0.75, -1 < -(x+4a+\beta+2\gamma)^2 + k \leq 0.75$ or $-1 < -(x+a+\beta)^2 + \frac{4}{3}k \leq 1.25$, giving (1.7) immediately unless $2a + \beta \equiv 0 \pmod{1}, a + \beta \equiv 0 \pmod{1}$ and $4a + \beta + 2\gamma \equiv 0 \pmod{1}$. These make $\alpha = \frac{1}{3}, \beta = 0$ and $2\gamma \equiv 0 \pmod{1}$. For $\gamma = 0$ we would then have $-(2 + 2a + \beta + \gamma)^2 + \frac{11k}{6} = -\frac{4}{3} \leq -\frac{4}{3}$ contradicting $N(f) = 1$; and for $\gamma = \frac{1}{3}$ we would have $-3a + \beta + \gamma \equiv \frac{3}{2}$ which gives (1.7).

When $g = \frac{3}{5}g_1$, the inequalities (2.6) and (2.9) contradict that we only need to consider the range $4.5 \leq k \leq 15$ with $4 \leq a \leq 7$. We split this range into 10

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1The notation $||t||$ denotes the distance from $t$ to the nearest integer.
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subintervals in each of which we use different estimates for $P(f)$. Firstly (2.8), with $a = 4, 5, 6, 7$ in turn, gives (1.7) for $4.5 < k < 7.5$, $5.25 < k < 10.9$ and $11.25 < k < 14.9$. The gaps $7.55 < k < 8$ and $10.9 < k < 11.25$ correspond to $a = 5$ and $a = 6$ respectively and in each of these cases we have $g(4,1,1) = \frac{3}{2} > \frac{(a+3)^2}{4} - 1$. Choosing $x$ so that $\frac{a+1}{2} \leq |x + 4a + \beta + \gamma| \leq \frac{a+2}{2}$ then gives $P(f) \leq \frac{a}{2} - \frac{(a+1)^2}{4}$. This is enough to make $\varphi(f) < \frac{1}{\sqrt{3}}$ each time.

For $4.85 < k < 4.95$ we similarly get $P(f) \leq \frac{a}{2} - 9$ which again gives $\varphi(f) < \frac{1}{\sqrt{3}}$. This leaves the intervals $4.75 < k < 4.85$, $4.95 < k < 5.25$ and $14.9 < k < 15$.

For $k \in (4.75, 4.85)$ we may assume that $P(f) > 2.5$ and we have $g(1,1,0) = \frac{3}{4} \in (11.11.32)$ and $g(2,1,0) = \frac{5}{4} \in (8.7,8.9)$. These imply $||\alpha + \beta|| > 0.46$, $||2\alpha + f|| > 0.47$ and so $||\beta|| > 0.39$. Since $g(0,1,0) = \frac{5}{4} \in (11.8,12.2)$ we then have $-0.45 < f(x,0,1,0) < 0.75$ for some $x$.

Finally for $k \in (14.9,15)$ we may assume that $P(f) > 5.9$ and we have $g(1,1,0) = (34.76,35)$, $g(2,1,0) = (27.3,27.5)$, $g(3,1,1) = \frac{11}{4} \in (42.2,42.5)$, $g(4,1,1) = \frac{3}{2} \in (22.35,22.5)$. These imply $||\alpha + \beta|| < 0.025$, $||2\alpha + \beta|| > 0.35$, $||3\alpha + \beta + \gamma|| < 0.05$, and $||4\alpha + \beta + \gamma|| < 0.075$. The first two make $\alpha > 0.32$ and the second two give the contradiction $\alpha < 0.2$.

References


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