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FROM AN ALTERNATIVE MODEL OF ROUGH SETS TO FUZZY SETS

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Instead of the classical definition of rough sets based on an indiscernibility relation, an alternative approach is suggested, based on a relation which quantifies the effort and other demands necessary to decide the membership predicate for a given element and a given set. The points for which this effort exceeds a threshold value are taken as undecidable. A simply defined inverse mapping of this quantifying relation may be taken as a membership degree function which defines fuzzy sets in the universe of discourse.

1. INTRODUCTION – THE CLASSICAL MODEL OF ROUGH SETS

The classical set theory, either in its naive conception or in one of its axiomatizations, is based on the classical propositional and predicate logic and transforms, through the axiom of extensionality, the two truthvalues of this logic into the sharp and crisp nature of classical classes and sets (cf. [1] or [3] for more detailed philosophical and methodological discussions). Classes and sets are defined as collections of objects possessing a property expressed by a classical two-valued predicate, hence, each object either is in the defined class or set or it is not-tertium non datur. This crispness and unambiguity is taken in a purely Platonistic sense, not supposing one's ability to decide, in general, whether a particular object belongs to a given class or set. In what follows, we shall limit ourselves just to sets, as the difference between sets and classes, important for some axiomatizations of set theory, is irrelevant in our context.

In spite of its theoretical soundness and formal consistence the classical notion of set must be considered as an abstraction and idealization inadequate from a less Platonistic viewpoint. Every reader can easily introduce a number of cases when the theoretical or practical decidability of the membership predicate is far from being simple or even possible, consider, e. g., neither recursive nor recursively enumerable sets of nonnegative integers supposing that our decision tools are limited to the algorithmizable ones. An intuitively arising idea is to approximate the set in question by two sets: one containing the elements which are surely in the original set (say, those for which we are able to verify it within our scopes of abilities or possibilities), the other set contains the elements the membership of which in the original set cannot be excluded within the same scopes. The approximating pair of sets is called the *rough set* generated by the original set or by the predicate which defines this set, and by the supposed scope of decision abilities.

In the classical model the limited decision abilities are formalized by *indiscernibility relations*. Considering a nonempty basic space X, an indiscernibility relation is simply an equivalence relation $\approx$ on X interpreted in such a way that if $x \approx y$, then x and y cannot be discerned from each other. Consequently, if P is a unary predicate such that $P(x)$ holds but $P(y)$ does not hold, we are not able to decide about the validity of P neither for x nor for y. Translating this idea into the set-theoretic language we may define, given a subset $V \subset X$, two sets

$$V_* = \bigcup\{[x] : [x] \subset V\} = \{x \in V : (\forall y \in X)(y \approx x \Longrightarrow y \in V)\}, \tag{1}$$

$$V^* = \bigcup\{[x] : [x] \cap V \neq \emptyset\} = \{x \in V : (\exists y \in V)(x \approx y)\}, \tag{2}$$

where $[x]$ denotes the equivalence class in $X \mid \approx$ containing x. Evidently, if $x \in V_*$, then certainly $x \in V$, if $x \in X - V^*$, we can be sure that $x \in X - V$, for $x \in V^* - V_*$ we cannot decide whether $x \in V$ or not. The pair $\langle V_*, V^* \rangle$ is called the (classical) rough set generated or defined by V and $\approx$ in X, cf. [4], [5], or [6] for more details. The equivalence nature of indiscernibility relations simplifies the construction and further considerations, but may be subjected to serious and far going objections.

2. TWO NON-CLASSICAL MODELS OF ROUGH SETS

As before, let X be a nonempty basic set (space, universe of discourse), let B be another nonempty set, let $\preceq$ be a *partial-ordering relation* defined on B. I.e., for each $x, y, z \in B$, $x \preceq x$, $x \preceq y$ and $y \preceq x$ imply $x = y$ in the sense of identity relation on B and, finally, $x \preceq y$ and $y \preceq z$ imply $x \preceq z$. Let $\mathcal{L}$ be a formalized first-order language with the set *Pred* of all unary predicates (well-formed formulas with a single free indeterminate). Finally, let

$$\rho : \mathcal{P}(X) \times \mathit{Pred} \times X \longrightarrow B \tag{3}$$

be a *partial* mapping, called *reference system* ascribing to (some, in general) subsets of X, predicates P, and elements $x \in X$ the value $\rho(A, P, x) \in B$. The intuition and interpretation behind is straighforward: $\rho(A, P, x)$ expressed the degree of effort and expenses necessary to decide, within a given scope of abilities corresponding to the reference system ρ, whether $P(x)$ holds or not supposing we know that $x \in A$. If $\rho(A, P, x)$ is not defined, it is beyond the abilities of the reference system to decide $P(x)$ even if $x \in A$ is known. Setting $\rho(A, P, x) \preceq b$ in some conditions below we tacitly assume that $\rho(A, P, x)$ is defined.

Definition 1. Given $A \subset X$, $P \in$ *Pred*, and $b \in B$, *b-rough subset of A generated by P and* ρ is the pair $\langle V_0(A, P, \rho, b), V^0(A, P, \rho, b) \rangle$ of (classical) subsets of A defined by

$$V_0(A, P, \rho, b) = \{x : x \in A, P(x)\,(\text{holds}), \rho(A, P, x) \preceq b\}, \tag{4}$$

$$V^0(A, P, \rho, b) = A - \{x : x \in A, \underline{\text{non}}\, P(x)\,(\text{holds}), \rho(A, \underline{\text{non}}\, P, x) \preceq b\}, \tag{5}$$

here $\underline{\text{non}}\, P$ is the predicate from *Pred* syntactically uniquely defined by putting the negation functor from $\mathcal{L}$ before P.

Worth an explicit mentioning is that if $b_1 \preceq b_2$, then

$$V_0(A, P, \rho, b_1) \subset V_0(A, P, \rho, b_2) \subset A \,\|\, P = \{x : x \in A, P(x)\,(\text{holds})\} \subset \tag{6}$$

$$\subset \; V^0(A,P,\rho,b_2) \subset V^0(A,P,\rho,b_1).$$

However, the set $A \parallel P$ does not play any role in the definition of $\langle V_0,\, V^0\rangle$ and, in a sense, it need not even to exists, i. e., the attribute of existence can be ascribed just to sets with easily decidable membership predicates like the sets $V_0(A,P,\rho,b)$ and $V^0(A,P,\rho,b)$ for sufficiently low b's. This is an important ontological difference from (1) and (2), when the set V occurs in the definition of V_* and V^*.

Moreover, Definition 1 is strictly intensional (i. e. not extensional) in the sense that V_0 and V^0 are defined unambiguously w. r. to the *predicate* P, but not w. r. to the *set* $A \parallel P$ as in the classical case. So, it may happen that two predicates P_1, P_2 are logically equivalent on A, i. e.,

$$(\forall\, x \in A)\ \ (P_1(x) \iff P_2(x)), \tag{7}$$

but $V_0(A,P_1,\rho,b) \neq V(A,P_2,\rho,b)$ and/or $V^0(A,P_1,\rho,b) \neq V^0(A,P_2,\rho,b)$. E. g., only a testing oracle for $P_1(x)$ is at hand and $P_2(x)$ can be tested just after having been converted into $P_1(x)$. The time and other costs of this conversion must be also taken into account, so that it is possible, at least for some x, that $\rho(A,P_1,x) \preceq b$ holds but $\rho(A,P_2,x) \preceq b$ does not (and similarly for non P).

A slightly different approach to b-rough sets may be as follows.

Definition 2. Given $A \subset X$ and $b \in B$, a subset $C \subset A$ is called a *b-subset of* A, if there exists $P \in$ *Pred* such that $C = A \parallel P$ and if $\rho(A,P,x) \preceq b$ holds for all $x \in A$. *Generalized b-rough set* in A is a pair $\langle V_0,\, V^0\rangle$ of b-subsets of A such that $V_0 \subset V^0$. Generalized b-rough set $\langle V_0,\, V^0\rangle$ is *consistent* w. r. to a subset $V \subset A$, if $V_0 \subset V \subset V^0$, and it is *optimal* w. r. to V, if for each generalized b-rough set $\langle V_1,\, V^1\rangle$ consistent w. r. to V both the inclusions $V_1 \subset V_0$ and $V^0 \subset V^1$ hold.

A number of results concerning special reference systems (monotonous, Boolean-valued, numerical) and b-rough sets defined by them can be found in [2].

3. FUZZY SETS DEFINED BY b-ROUGH SETS

The most simple way from a (no matter how defined) rough subset $\langle V_*,\, V^*\rangle$ of $A \subset X$ to a fuzzy set $\overline{V}$ is to define the membership function $\mu_{\overline{V}}$ by $\mu_{\overline{V}}(x) = 1$, if $x \in V_*$, $\mu_{\overline{V}}(x) = 0$ if $x \in A - V^*$, and $\mu_{\overline{V}}(x) = 1/2$, if $x \in V^* - V_*$. However, as proved by Pawlak [7] and quoted by Wygralak [8], this transformation is not one-to-one in the sense that some information is lost when going from $\langle V_*,\, V^*\rangle$ to $\overline{V}$.

Take the partially ordered set $\langle B,\, \preceq\rangle$ introduced above and denote by $\mathcal{P}(B)$ the set of all subsets of B partially ordered by inclusion. Given $b \in B$, set

$$S_1(b) = \{c:\ c \in B,\ b \preceq c\}\,, \qquad S_2(b) = \{c:\ c \in B,\ \text{not } b \preceq c\}\,. \tag{8}$$

For $A \subset X$, $P \in$ *Pred*, $x \in X$, and a reference system ρ two fuzzy subsets $\overline{A \parallel P}^1$, $\overline{A \parallel P}^2$ of A may be defined with membership functions taking their values in $\mathcal{P}(B)$. Namely, for $i = 1, 2$,

$$\mu\left(\overline{A \parallel B}^{i}\right)(x) = S\left(\rho(A, P, X)\right), \tag{9}$$

if $x \in A$ and $\rho(A, P, X)$ is defined, $\mu\left(\overline{A \parallel P}^{i}\right)(x) = \emptyset$ (the empty set) otherwise.

Both the membership functions $\mu\left(\overline{A \parallel P}^{i}\right)$, $i = 1, 2$ can be easily proved to be monotonous in an intuitive sense, cf. [2] for the corresponding proofs.

Fact 1. If $\rho(A, P, x_1) \preceq \rho(A, P, x_2)$, or if $\rho(A, P, x_2)$ is not defined, then $\mu\left(\overline{A \parallel P}^{i}\right)(x_1) \supseteq \mu\left(\overline{A \parallel P}^{i}\right)(x_2)$ for both $i = 1, 2$.

Fact 2. Let X, $A \subset X$, and *Pred* be as above, let ρ be a monotonous reference system, i.e., for each $x \in A$ and $P, Q \in$ *Pred* such that the expressions below are defined,

$$\rho(A, P \ \underline{\text{et}}\ Q, x) = \rho(A, P, x) \vee \rho(A, Q, x), \tag{10}$$

$$\rho(A, P \ \underline{\text{vel}}\ Q, x) = \rho(A, P, x) \wedge \rho(A, Q, x), \tag{11}$$

supposing that there exist the supremum ($\vee$) and minimum ($\wedge$) operations in B; <u>et</u> and <u>vel</u> are the classical propositional functors in $\mathcal{L}$. Let P_1, $P_2 \in$ *Pred*, let $A_i = A \parallel P_i$, $i = 1, 2$, let $\rho(A, P_1, x)$, $\rho(A, P_2, x)$ be defined, then

$$\mu\left(\overline{A_1 \cup A_2}^{1}\right) \supseteq \sup\left\{\mu\left(\overline{A}_1^{1}\right)(x), \mu\left(\overline{A}_2^{1}\right)(x)\right\} = \mu\left(\overline{A}_1^{1}\right)(x) \cup \mu\left(\overline{A}_2^{1}\right)(x), \tag{12}$$

$$\mu\left(\overline{A_1 \cup A_2}^{2}\right) = \sup\left\{\mu\left(\overline{A}_1^{2}\right)(x), \mu\left(\overline{A}_2^{2}\right)(x)\right\} = \mu\left(\overline{A}_1^{2}\right)(x) \cup \mu\left(\overline{A}_2^{2}\right)(x), \tag{13}$$

$$\mu\left(\overline{A_1 \cap A_2}^{1}\right) = \inf\left\{\mu\left(\overline{A}_1^{1}\right)(x), \mu\left(\overline{A}_2^{1}\right)(x)\right\} = \mu\left(\overline{A}_1^{1}\right)(x) \cap \mu\left(\overline{A}_2^{1}\right)(x), \tag{14}$$

$$\mu\left(\overline{A_1 \cup A_2}^{2}\right) \subseteq \inf\left\{\mu\left(\overline{A}_1^{2}\right)(x), \mu\left(\overline{A}_2^{2}\right)(x)\right\} = \mu\left(\overline{A}_1^{2}\right)(x) \cap \mu\left(\overline{A}_2^{2}\right)(x). \tag{15}$$

Because of a very limited extent of this contribution we have decided to concentrate our attention to basic ideas, referring to [2] for technical details, results and proofs.

REFERENCES

[1] A. A. Fraenkel and Y. Bar-Hillel: Foundations of Set Theory. North-Holland, Amsterdam 1959 (Russian translation: Mir, Moscow 1966).

[2] I. Kramosil: An alternative approach to rough sets. Kybernetika *28* (1991), 1, 1 – 25.

[3] P. Maddy: Believing the axioms. J. Symbolic Logic *53* (1988), No. 2, pp. 481 – 511, No. 3, pp. 736 – 764.

[4] E. Orlowska: Algebraic approach to database constraints. Fundamenta Informaticae *10* (1987), 1, 57 – 68.

[5] E. Orlowska: Representation of vague information. Information Systems *13* (1988), 2, 167 – 174.

[6] Z. Pawlak: Rough sets. Internat. J. Comput. Inform. Sci. *11* (1982), 3, 341 – 356.

[7] Z. Pawlak: Rough Sets and Fuzzy Sets. Institute of Computer Sciences, Polish Academy of Sciences, Report No. 540, Warsaw 1984.

[8] M. Wygralak: Some remarks on rough and fuzzy sets. Busefal *21* (1986), 1, 42 – 48.

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