Kybernetika

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Power spectrum of the quasiperiodic and the aperiodic group pulse process

POWER SPECTRUM OF THE QUASIPERIODIC AND THE APERIODIC GROUP PULSE PROCESS

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In the present paper the expression for the power spectrum of the quasiperiodic and the aperiodic group pulse random process with independent pulse occurrence times in groups is derived. Four types of the random pulse processes are distinguished according to the form of the power spectrum. These are the periodic, the quasiperiodic, the aperiodic and the homogeneous pulse processes. It is shown that the periodic group pulse process may be obtained in the limit from the quasiperiodic one and that the homogeneous process represents the special case of the aperiodic one. The derived expressions are illustrated by examples of the calculated power spectra and several possible applications of the presented models in the theory of noises are mentioned.

1. GROUP PULSE RANDOM PROCESS

In the previous work [1] we have introduced the periodic group pulse random process with independent pulse occurrence times in groups and we have found an expression for the power spectrum of this process. In the present paper we shall be dealing with a somewhat more general case, namely with the quasiperiodic and the aperiodic group pulse random process with independent pulse occurrence times in groups.

We shall assume a pulse process $\xi(t)$ which is formed of randomly occurring groups of random pulses. An example of a possible realization of this process is schematically shown in Fig. 1.

Let us introduce the group reference point that defines the position of the pulse group with respect to the selected time axis origin.[*] The position of the single pulses inside the group will be also related to this point.

Let the random number of pulses in the k-th group be N_k and let the reference

[*] For noises formed of large pulse groups it is possible to identify the group reference point with the "centre of gravity" of each group.

point of the k-th group occur in time τ_k. An interval between two successive groups is then $\vartheta_k = \tau_{k+1} - \tau_k$, where $\vartheta_k \geqq 0$. The time of the occurrence of the n-th pulse in the k-th group with respect to the reference point of this group let be φ_{kn}. Then the process $\xi(t)$ may be written in the form

$$(1) \qquad \xi(t) = \sum_{k=-\infty}^{\infty} \sum_{n=1}^{N_k} f\big(t - \tau_k - \varphi_{kn}, \boldsymbol{a}_{kn}\big) ,$$

where $f(t, \boldsymbol{a})$ is a time function determining the shape of a single pulse and $\boldsymbol{a}_{kn}$ is
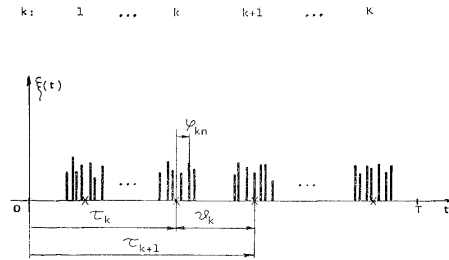

Fig. 1. The group pulse process with independent pulse occurrence times in groups.

an m-dimensional random vector of m random parameters of the n-th pulse in the k-th group.

Here we shall consider such processes only, for which the random variables τ_k, φ_{kn}, N_k and the random vector $\boldsymbol{a}_{kn}$ are mutually independent and the random variables ϑ_k, φ_{kn}, N_k and $\boldsymbol{a}_{kn}$ are also mutually independent for different k and n. We shall also assume, that the distribution functions of these random variables do not depend on n and k. The process $\xi(t)$ is then fully determined by specifying the function $f(t, \boldsymbol{a})$ and the probability densities $w_1(\varphi)$, $w_1(\vartheta)$, $w_1(N)$ and $w_m(\boldsymbol{a})$. As we shall see later, the form of the probability density $w_1(\vartheta)$ determines whether the process is a quasi-periodic, a periodic, an aperiodic or a homogeneous one.

In the following section we shall attempt to derive a general expression for the power spectrum of the process $\xi(t)$.

2. POWER SPECTRUM

Let us first consider a realization of the process (1) truncated in a principal interval $(0, T)$. We shall denote this truncated realization $\xi_{TK}(t)$. Let us assume that T is much

greater than the duration of the typical pulse group and that the interval $(0, T)$ fully includes just K groups of pulses. Hence we may write

$$(2) \qquad \xi_{TK}(t) = \sum_{k=1}^{K} \sum_{n=1}^{N_k} f(t - \tau_k - \varphi_{kn}, \boldsymbol{a}_{kn}) \, .$$

The spectrum of this truncated realization is given by the Fourier transform of (2), hence

$$(3) \qquad S_{TK}(\omega) = \sum_{k=1}^{K} \sum_{n=1}^{N_k} s(\omega, \boldsymbol{a}_{kn}) \, e^{-j\omega(\tau_k - \varphi_{kn})} \, ,$$

where $s(\omega, \boldsymbol{a}_{kn})$ is the Fourier transform of $f(t, \boldsymbol{a}_{kn})$.

The power spectrum of the process (1) will be determined from the definition formula

$$(4) \qquad \mathscr{W}(\omega) = \lim_{T \to \infty} \frac{1}{T} \overline{|S_T(\omega)|^2} \, ,$$

where the bar denotes the ensemble average.

Let us find the magnitude squared of the normalized spectrum $|S_{TK}(\omega)|^2$ first. Denoting the complex conjugate by the star we may write

$$(5) \qquad |S_{TK}(\omega)|^2 = S_{TK}(\omega) \, S_{TK}^{*}(\omega) =$$

$$= \sum_{k=1}^{K} \sum_{n=1}^{N_k} \sum_{l=1}^{K} \sum_{m=1}^{N_l} s(\omega, \boldsymbol{a}_{kn}) \, e^{-j\omega(\tau_k + \varphi_{kn})} \, s^{*}(\omega, \boldsymbol{a}_{lm}) \, e^{j\omega(\tau_l + \varphi_{lm})} \, .$$

From the four-fold sum we may take out the terms for which $k = l$ and $n = m$ simultaneously and the terms for which $k = l$ but $n \neq m$. We have thus

$$(6) \qquad |S_{TK}(\omega)|^2 = \sum_{k=1}^{K} \sum_{n=1}^{N_k} |s(\omega, \boldsymbol{a}_{kn})|^2 +$$

$$+ \sum_{k=1}^{K} \sum_{\substack{n=1 \\ n \neq m}}^{N_k} \sum_{m=1}^{N_k} s(\omega, \boldsymbol{a}_{kn}) \, e^{-j\omega\varphi_{kn}} \, s^{*}(\omega, \boldsymbol{a}_{km}) \, e^{j\omega\varphi_{km}} +$$

$$+ \sum_{k=1}^{K} \sum_{n=1}^{N_k} \sum_{\substack{l=1 \\ k \neq l}}^{K} \sum_{m=1}^{N_l} s(\omega, \boldsymbol{a}_{kn}) \, e^{-j\omega(\tau_k + \varphi_{kn})} \, s^{*}(\omega, \boldsymbol{a}_{lm}) \, e^{j\omega(\tau_l + \varphi_{lm})} \, .$$

Now we shall find the expectetion of (6) on the set of all truncated realizations that include just K pulse groups. With respect to the earlier postulated independence and stationarity of the random variables $\boldsymbol{a}_{kn}$, φ_{kn}, τ_k and N_k, the two-fold sum in (6) will contain $K\bar{N}_k$ identical terms of the type

$$(7) \qquad \overline{|s(\omega, \boldsymbol{a})|^2} \, ,$$

and the three-fold sum will contain $K\overline{(N_k^2 - N_k)}$ identical terms of the type

$$(8) \qquad \overline{|s(\omega, \boldsymbol{a})|}^2 \, |\chi_\varphi(\omega)|^2 \, ,$$

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where $\chi_\varphi(\omega)$ is the characteristic function of the random variable φ. From the four-fold sum we may take out the $\overline{N_k N_j}$ terms of the type (8), so that we finally get

$$(9) \qquad \overline{|S_{TK}(\omega)|^2} = K\overline{N}\overline{|s(\omega, a)|^2} + K\overline{N(N-1)}\,\overline{|s(\omega, a)|}^2\,|\chi_\varphi(\omega)|^2 +$$

$$+ \overline{N}^2\overline{|s(\omega, a)|}^2\,|\chi_\varphi(\omega)|^2 \sum_{\substack{k=1 \\ k \neq l}}^{K} \sum_{l=1}^{K} \overline{e^{-j\omega(\tau_k - \tau_l)}}.$$

Since the distance between two neighbouring groups is ϑ_k and the random variables ϑ_k are mutually independent by the definition for different k, we may arrange the double sum in the last term on the right hand side of (9). If $k > l$, then

$$\tau_k - \tau_l = \vartheta_{l+1} + \ldots + \vartheta_k \,,$$

and if $k < l$, then

$$\tau_l - \tau_k = -(\tau_k - \tau_l) = -(\vartheta_{l+1} + \ldots + \vartheta_k).$$

Hence

$$(10) \qquad \sum_{\substack{k=1 \\ k \neq l}}^{K} \sum_{l=1}^{K} \overline{e^{-j\omega(\tau_k - \tau_l)}} = \sum_{k=1}^{K-1} (K-k)\left[\chi_\vartheta^k(\omega) + \chi_\vartheta^{*k}(\omega)\right],$$

where $\chi_\vartheta(\omega)$ is the characteristic function of the random variable ϑ. Let us assume that for $\omega \neq 0$ is $|\chi_\vartheta(\omega)| < 1$ $(|\chi_\vartheta(\omega)| \leq 1$ for all values of $\omega)$. With the exception of $\omega = 0$ the sum of the arithmetic-geometrical series (10) equals [2]

$$(11) \qquad \sum_{k=1}^{K-1} (K-k)\left[\chi_\vartheta^k(\omega) + \chi_\vartheta^{*k}(\omega)\right] =$$

$$= 2\,\mathrm{Re}\left\{\frac{\chi_\vartheta(\omega)}{1 - \chi_\vartheta(\omega)}\left[K - \frac{1 - \chi_\vartheta^K(\omega)}{1 - \chi_\vartheta(\omega)}\right]\right\}.$$

The second term in the square brackets on the right hand side of (11) represents the sum of the finite geometrical series having K terms. Denoting this sum as

$$(12) \qquad B_\vartheta(\omega, K) = \frac{1 - \chi_\vartheta^K(\omega)}{1 - \chi_\vartheta(\omega)}$$

we may rewrite (9) to the form

$$(13) \qquad \overline{|S_{TK}(\omega)|^2} = K\overline{N}\overline{|s(\omega, a)|^2} + K\overline{N(N-1)}\,\overline{|s(\omega, a)|}^2\,|\chi_\varphi(\omega)|^2 +$$

$$+ \overline{N}^2\overline{|s(\omega, a)|}^2\,|\chi_\varphi(\omega)|^2\,2\,\mathrm{Re}\left\{\frac{\chi_\vartheta(\omega)}{1 - \chi_\vartheta(\omega)}\left[K - B_\vartheta(\omega, K)\right]\right\}.$$

So far we have found the expectation of (6) on the set of all truncated realizations that have contained just K pulse groups. The expectation of (13) on the ensemble of all truncated realizations is then

$$(14) \qquad \overline{|S_T(\omega)|^2} = \sum_{K=0}^{\infty} P(K)\,\overline{|S_{TK}(\omega)|^2}\,,$$

where $P(K)$ is a probability that there are exactly K pulse groups in the interval $(0, T)$. Substituting (13) into (14) we have

$$(15) \qquad \overline{|S_T(\omega)|^2} = \bar{K}N\overline{|s(\omega, \boldsymbol{a})|^2} + \bar{K}N(N-1)\left|\overline{s(\omega, \boldsymbol{a})}\right|^2 |\chi_\varphi(\omega)|^2 +$$

$$+ \bar{N}^2\left|\overline{s(\omega, \boldsymbol{a})}\right|^2 |\chi_\varphi(\omega)|^2 \, 2 \, \mathrm{Re} \left\{ \frac{\chi_\varphi(\omega)}{1 - \chi_\vartheta(\omega)} \left[\bar{K} - \overline{B_\vartheta(\omega, K)} \right] \right\}.$$

Now, if we increase the length T of the interval $(0, T)$, the number K of the groups included in the interval will also increase. According to the elementary theorem for the renewal processes [3] it will hold, that

$$(16) \qquad \lim_{T \to \infty} \frac{\overline{K(T)}}{T} = \frac{1}{\bar{\vartheta}} = \bar{v}.$$

Here $\bar{v}$ is the mean group density. The geometrical series (12) will converge for $T \to \infty$ (and hence for $K \to \infty$) and $\omega \neq 0$, so that it will also hold, that

$$(17) \qquad \lim_{T \to \infty} \frac{1}{T} \overline{B_\vartheta(\omega, K)} = 0, \quad \omega \neq 0.$$

For $\omega = 0$ the series (11) will diverge, which can be expressed by the delta function $\delta(\omega)$. Now we may substitute (15) to (4) and take the limit for $T \to \infty$. With the use of (16) and (17) and after rearrangement of the terms we finally get the desired expression for the power spectrum in the form

$$(18) \qquad \mathscr{W}(\omega) = \bar{v}N\overline{|s(\omega, \boldsymbol{a})|^2} + \bar{v}(\sigma_N^2 - \bar{N})\left|\overline{s(\omega, \boldsymbol{a})}\right|^2 |\chi_\varphi(\omega)|^2 +$$

$$+ \bar{v}\bar{N}^2\left|\overline{s(\omega, \boldsymbol{a})}\right|^2 |\chi_\varphi(\omega)|^2 \, Z_\vartheta(\omega) + \bar{v}^2\bar{N}^2\left|\overline{s(0, \boldsymbol{a})}\right|^2 \, 2\pi \, \delta(\omega).$$

Here σ_N^2 is the variance of the random variable N and we have further denoted

$$(19) \qquad Z_\vartheta(\omega) = 1 + 2 \, \mathrm{Re} \, \frac{\chi_\vartheta(\omega)}{1 - \chi_\vartheta(\omega)}.$$

The expression (18) is the general formula for the power spectrum of the random group pulse process with independent pulse occurrence times in groups and with independent intervals among the groups.

The function $Z_\vartheta(\omega)$ defined by the relation (19) is of a special significance as according to the form of $Z_\vartheta(\omega)$ the random pulse processes may be classified as quasiperiodic, periodic, aperiodic and homogeneous.

If the maxima of the function $Z_\vartheta(\omega)$ are finite and occur in regular points $\omega = k\omega_0$, $k = \pm 1, \pm 2, \ldots, \omega_0 = 2\pi/\bar{\vartheta}$, we call the corresponding process *quasiperiodic*. This happens to be the case when the random variable ϑ has, for example, the normal, the Laplace or the uniform distribution in an interval (a, b), where $a, b > 0$.

In the case of the *periodic* process the function $Z_\vartheta(\omega)$ is constituted by the infinite

sum of the delta functions $\delta(\omega - k\omega_0)$, where $k = \pm 1, \pm 2, \ldots, \omega_0 = 2\pi/T_0$ and T_0 is a constant distance between two neighbouring group reference points. As we shall see in the next section, the periodic process represents the limit case of the quasiperiodic process.

With the *aperiodic* process the maxima of the function $Z_\vartheta(\omega)$ occur entirely irregularily. The aperiodic process corresponds, for example, to the gamma and the Weilbull distribution of the random variable ϑ.

The special case of the aperiodic process is the *homogeneous* process for which the function $Z_\vartheta(\omega)$ equals unity.

3. EXAMPLES OF THE FUNCTION $Z_\vartheta(\omega)$

Let us deal with *the quasiperiodic process* first and let us suppose that the random variable ϑ is normally distributed with the probability density

$$(20) \qquad w_1(\vartheta) = \frac{1}{\sigma_\vartheta \sqrt{(2\pi)}} \, e^{-\frac{(\vartheta - \bar\vartheta)^2}{2\sigma_\vartheta^2}} .$$

In this case the function $Z_\vartheta(\omega) = Z_G(\omega)$ has three arguments, namely the variable ω and the parameters σ_ϑ and $\bar\vartheta$. Introducing the nondimensional variables $V = \sigma_\vartheta/\bar\vartheta$ (the variation quotient) and $\omega' = \omega\bar\vartheta$ we may reduce the number of arguments by one. We obtain

$$(21) \qquad Z_G(\omega', V) = \frac{1 - e^{-(\omega' V)^2}}{1 - 2\,e^{-1/2(\omega' V)^2} \cos(\omega') + e^{-(\omega' V)^2}} .$$

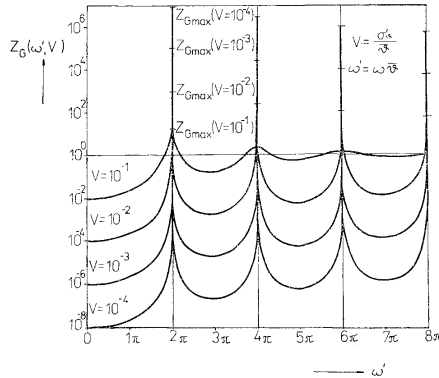The function $Z_G(\omega', V)$ is shown for several values of the parameter V in Fig. 2. The

Fig. 2. The function $Z_G(\omega', V)$.

function $Z_G(\omega', V)$ has local maxima at $\omega' = 2k\pi$, $k = \pm 1, \pm 2, \dots$ If the variance σ_ϑ^2 is decreased, the values of the function $Z_G(\omega', V)$ increase in maxima and its close vicinity and decrease in other points. The validity of the following limits may be readily verified (in this section it will be always assumed that $k = \pm 1, \pm 2, \dots$)

$$(22) \qquad \omega' = 2k\pi \qquad \lim_{V \to 0} Z_G(\omega', V) = c\,\delta(\omega'),$$

$$(23) \qquad \omega' \neq 2k\pi \qquad \lim_{V \to 0} Z_G(\omega', V) = 0,$$

$$(24) \qquad\qquad\qquad \lim_{\omega' \to 0} Z_G(\omega', V) = V^2,$$

$$(25) \qquad V > 0 \qquad \lim_{\omega' \to \infty} Z_G(\omega', V) = 1.$$

The unknown constant c in (22) may be determined by integrating the function $Z_G(\omega', V)$ in the limits from $a = (2k - 1)\pi$ to $b = (2k + 1)\pi$. We find, that

$$(26) \qquad\qquad \int_a^b Z_G(\omega', V)\,d\omega' = 2\pi = c.$$

For $\omega' V \ll 1$ the following approximate relations hold

$$(27) \qquad \omega' \neq 2k\pi \qquad Z_G(\omega', V) \doteq \frac{V^2}{2} \frac{\omega'^2}{(1 - \cos \omega')},$$

specially for

$$(28) \qquad \omega' = (2k + 1)\pi \qquad Z_{G\min} \doteq V^2 \left[(2k + 1)\frac{\pi}{2}\right]^2,$$

$$(29) \qquad \omega' = 2k\pi \qquad Z_{G\max} \doteq 10^{-p_k} V^{-2}.$$

The powers p_k of the first five maxima have the approximate values

$$p_1 = 1\,; \quad p_2 = 1{,}6\,; \quad p_3 = 1{,}95\,; \quad p_4 = 2{,}2\,; \quad p_5 = 2{,}4\,.$$

The accuracy of the approximate relation (27) is the highest in the surroundings of the minima where it is quite satisfactory, on the contrary close by the maxima its accuracy rapidly decreases.

In the surroundings of the k-th maximum it holds further ($\omega' = 2k\pi + \Delta\omega'$, $|\Delta\omega'| \ll 1$, $\Delta\omega' \neq 0$, $\omega'V \ll 1$)

$$(30) \qquad\qquad Z_G(\Delta\omega', V) \doteq \left(\frac{2k\pi V}{\Delta\omega'}\right)^2.$$

The accuracy of the relation (30) is the highest in the region of the values ω', for which $Z_G(\omega', V)$ approximately equals unity (that is for the values of $\Delta\omega'$ being in the range from about $0{\cdot}01$ to $0{\cdot}05$). The accuracy of the approximation (30) rapidly worsens for considerably small or large $\Delta\omega'$.

The expressions (29) and (30) may be used for a rough estimation of the unknown variation quotient V from the experimentally found spectral lines height and breadth.

Now let us pay heed to the limit case when $\sigma_\vartheta \to 0$, $\bar{\vartheta} = T_0$ and hence $V \to 0$. In this case the group reference points will occur periodically with the period T_0 and the considered process will be *the periodic group pulse process*. With respect to $(22)-(24)$ and (26) it will hold, that

$$(31) \qquad \lim_{\sigma_\vartheta \to 0} Z_G(\omega) = Z_D(\omega) = \omega_0 \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \delta(\omega - k\omega_0) \, ,$$

where $\omega_0 = 2\pi/T_0$. It is obvious that this result may be obtained not only with the normal distribution of random variable ϑ, but with a number of other ones as for example with the Laplace or the uniform one. Thus the periodic pulse process is the limit case of the quasiperiodic process, that is of the process with independent intervals.

As a further example let us consider *the aperiodic process*, namely when the random variable ϑ has the gamma distribution with the probability density

$$(32) \qquad w_1(\vartheta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \, \vartheta^{\alpha-1} \, e^{-\beta\vartheta} \, .$$

In this case the function $Z_\vartheta(\omega) = Z_\Gamma(\omega)$ has three arguments again, namely the variable ω and the parameters α and β. Introducing the nondimensional variable $\omega'' = \omega/\beta$ we may reduce the number of arguments by one. We obtain

$$(33) \qquad Z_\Gamma(\omega'', \alpha) = \frac{\left(1 + \omega''^2\right)^\alpha - 1}{\left(1 + \omega''^2\right)^\alpha - 2\left(1 + \omega''^2\right)^{\alpha/2} \cos\left(\alpha \operatorname{arct} \omega''\right) + 1} \, .$$

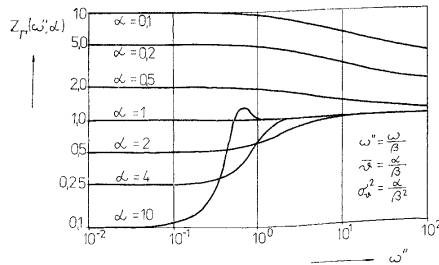**Fig. 3.** The function $Z_\Gamma(\omega'', \alpha)$.

The function $Z_\Gamma(\omega'', \alpha)$ is shown for several values of the parameter α in Fig. 3. The case when $0 < \alpha < 1$ corresponds to an attractive correlation and the case when $\alpha > 1$ to a repulsive correlation [4]. It may be seen from Fig. 3, that at the attractive correlation the low frequency portion of the spectra is raised and on the

contrary at the repulsive correlation the low frequency portion of the spectra is lowered.

The particular case when $\alpha = 1$ corresponds to uncorrelated group reference points. Now the gamma distribution passes to the exponential one with the probability density

$$(34) \qquad\qquad w_1(\vartheta) = \frac{1}{\bar{\vartheta}}\, e^{-\vartheta/\bar{\vartheta}},$$

and the function $Z_\vartheta(\omega) = Z_E(\omega)$ equals

$$(35) \qquad\qquad Z_E(\omega) = 1 .$$

This case corresponds to *the homogeneous process.*

4. EXAMPLES OF THE POWER SPECTRA

In this section we shall give some examples of the power spectra of the quasiperiodic, the periodic and the homogeneous group pulse processes. Similarly as in the earlier paper [1] we shall limit ourselves to the two important distributions of the random number of pulses in groups. First it will be the Poisson distribution, for which the probability of the occurrence of just N pulses in a group equals

$$(36) \qquad\qquad P(N) = \frac{\bar{N}^N}{N!}\, e^{-N}, \quad N = 0, 1, 2, \dots .$$

In this case the variance is $\sigma_N^2 = \bar{N}$. Next we shall consider the case, when the number of pulses in every group equals N_0. Then

$$(37) \qquad\qquad w_1(N) = \delta(N - N_0),$$

and $\sigma_N = 0, \bar{N} = N_0$.

The examples of the power spectra were calculated for pulses having the form of a two-sided symmetrical exponential and for normally distributed times of the pulses occurrence in groups. The computation was performed by an approximate method described elsewhere [1].

a) *Quasiperiodic process*

Examples of the power spectra calculated from the expression (18) under assumption that random variable ϑ is normally distributed and hence $Z_\vartheta(\omega) = Z_G(\omega)$ are given in Fig. 4. Fig. 4.a corresponds to the Poisson distribution of the number of pulses in groups (36) and Fig. 4.b corresponds to the constant number of pulses in groups (37).
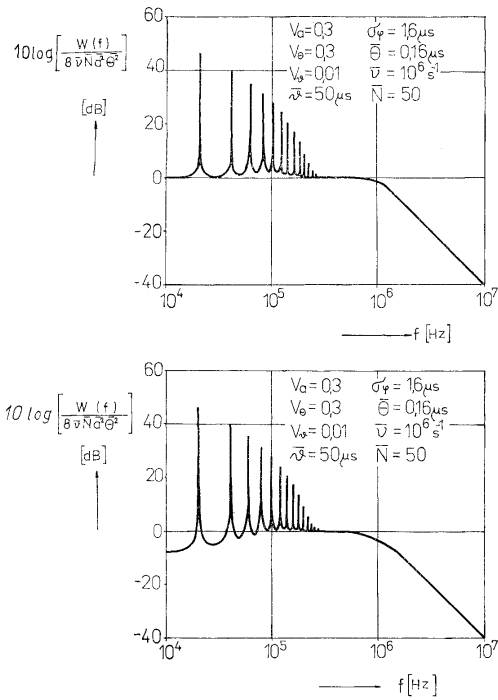
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Fig. 4. The calculated power spectra of the quasiperiodic group pulse process. a) The number of pulses in groups is Poisson distributed. b) The number of pulses in groups is constant.

b) *Periodic process*

In this case $Z_\vartheta(\omega) = Z_D(\omega)$. Using (31) the expression for the power spectrum (18) will take the form

$$(38) \qquad \mathcal{W}(\omega) = v_0 \overline{N} \overline{|s(\omega, \boldsymbol{a})|^2} + v_0 (\sigma_N^2 - \overline{N}) \overline{|s(\omega, \boldsymbol{a})|}^2 |\chi_\varphi(\omega)|^2 +$$
$$+ v_0^2 \overline{N}^2 \overline{|s(\omega, \boldsymbol{a})|}^2 |\chi_\varphi(\omega)|^2 \, 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) \, .$$

We have studied this process in the earlier work [1], however, there we have derived the expression for the power spectrum (38) by the direct procedure. Here we have obtained it as the limit case of the quasiperiodic process. Examples of the

calculated power spectra of the periodic process were given in the mentioned paper [1].

c) *Homogeneous process*

In this case $Z_\vartheta(\omega) = Z_E(\omega) = 1$. If the number of pulses in groups is Poisson distributed we shall obtain from (18) readily

$$(39) \qquad \mathscr{W}(\omega) = \bar{\nu}\bar{N}\overline{|s(\omega, \boldsymbol{a})|^2} + \bar{\nu}\bar{N}^2\left|\overline{s(\omega, \boldsymbol{a})}\right|^2 |\chi_\varphi(\omega)|^2 + $$
$$+ \bar{\nu}^2\bar{N}^2\left|\overline{s(0, \boldsymbol{a})}\right|^2 2\pi\,\delta(\omega)\,.$$

An example of the calculated power spectrum (39) is shown in Fig. 5 by the full line.

$$10\log\left[\frac{W(f)}{8\bar{\nu}\bar{N}\,\overline{a^2}\bar{\theta}^2}\right]$$

[dB]

V$_a$=0,3 $\overline{\vartheta}$=50μs
V$_\theta$=0,3   σ$_\varphi$=1,6μs
$\bar{\nu}$ = 10^{6}s^{-1} $\overline{\theta}$=0,16μs
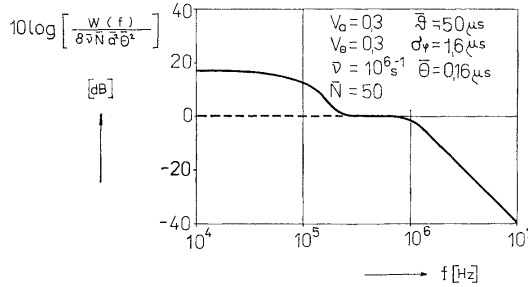$\bar{N}$ = 50

f [Hz]

Fig. 5. The calculated power spectra of the homogeneous group pulse process.

In the case of the constant number of pulses in groups we shall give heed to the process with $N_0 = 1$ only. Then we have from (18)

$$(40) \qquad \mathscr{W}(\omega) = \bar{\nu}\overline{|s(\omega, \boldsymbol{a})|^2} + \bar{\nu}^2\left|\overline{s(0, \boldsymbol{a})}\right|^2 2\pi\,\delta(\omega)\,.$$

As we may see, the form of the power spectrum of the homogeneous one-pulse process does not depend on the distribution of the random variable φ and it conforms with the expression for the power spectrum of the homogeneous Poisson process [1]. An example of the power spectrum (40) is shown in Fig. 5 by the dashed line.

5. CONCLUSION

In the paper the quasiperiodic and the aperiodic group pulse process with independent pulse occurrence times in groups has been described and the general expression for the power spectrum of this process has been derived. The results have been illustrated by examples of the calculated power spectra of several particular processes.

It was shown that the periodic process represents the limit case of the quasiperiodic process and that the homogeneous process is the special case of the aperiodic

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one. The periodic and the homogeneous processes are of theoretical value first of all as they consider strict periodicity or complete independence of the groups respectively. Even if such strictly periodic and strictly independent processes never occur in real environment, these theoretical models approach many practically important signals and noises considerably. Thanks to their relative simplicity the periodic and the homogeneous processes are therefore often used to approximate such real signals and noises.

Due to the strict periodicity the spectral lines of the periodic process are infinitely narrow and infinitely high (the delta functions). On the contrary the quasiperiodic process has non-zero variance σ_ϑ^2 and thus takes into account possible nonstability of the generating mechanism. As a result the spectral lines of this process have finite breadth and height and thus this process approaches the truth signals and noises better then the periodic one. However, this better approximation is paid for by greater complexity. Using the expressions (29) and (30) it is possible to estimate an approximate value of the variance σ_ϑ^2 from the experimentally determined spectral lines height and breadth.

It may be expected, that some results of the Section 3., e.g. formula (30) and (31), that have been derived under assumption of the normal distribution, could be also obtained for a number of other distributions.

The quasiperiodic and the aperiodic processes may be used as theoretical models of several noises and signals. Thus the quasiperiodic process with the number of pulses in groups Poisson distributed represents a more realistic version of the Poisson periodic process. Hence it may be used, for example, as a more exact model of the cavitation noise at acoustical cavitation.

Among the processes with the constant number of pulses in groups the case when $N_0 = 1$ assumes the greatest importance. This is the quasiperiodic or the periodic one-pulse process used to approximate the pulse-code signals.

The homogeneous group pulse process may be regarded as an analogy to the process used by Bittel and Storm [5] as a first approximation of Barkhausen noise in ferromagnetics and of flicker noise in valves.

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