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FENCHEL-LAGRANGE DUALITY IN VECTOR FRACTIONAL PROGRAMMING VIA ABSTRACT DUALITY SCHEME

TRAN QUOC CHIEN

This paper deals with a generalization of both Fenchel and Lagrange duality in vector fractional programming. In the first section the concepts of maximum and supremal are introduced and discussed. In Section 2 a simple abstract duality scheme is presented. The last section is devoted to the so-called Fenchel-Lagrange duality in vector fractional programming which is built up on the basis of the abstract duality scheme and a set separation theorem.

0. INTRODUCTION

The duality questions in one-objective fractional programming have extensively been studied by many authors. Most of dual programs are established via a transformation to a convex program, see Schaible [10, 11]. In Cambini and Martein [2] a duality of Fenchel type via a separation theorem of two convex sets is introduced. In Tran Quoc Chien [5, 6, 7] a duality theory of Lagrange type is built up on the basis of the so-called abstract duality scheme. In this work we introduce a more general abstract duality scheme which seems to be able to unify all known non-differentiable duality theories as Lagrange duality, Fenchel duality and perturbation duality theory. One of the applications of the abstract duality scheme is just the Fenchel-Lagrange duality in vector fractional programming given in Section 3 of this work. Another application of this scheme will be presented in the following paper where the perturbation theory of duality in vector optimization is built up on its basis.

1. OPTIMALITY CONCEPTS

1.1. Basic assumptions. Throughout this work we suppose that all spaces are real and Y is an ordered linear space if other requirements are not added. All elementary notions as *linear hull*, *convex hull*, *affine subspace*, *dual space*, *affine function*, *convex* resp. *concave function* or *core* (cor A), *intrinsic core* (icr A), *algebraic closure* (lin A) etc. can be found in Holmes [9]. Basic notions concerning partial ordering and ordered spaces can be foun in Grätzen [8]. The positive cone Y_+ of Y is supposed to have nonempty core.

1.2. Notations. For elements $a, b \in Y$ we denote

$$a \geqq b \quad \text{iff} \quad a - b \in Y_+$$

$$a > b \quad \text{iff} \quad a - b \in Y_{++} = Y_+ \setminus \{0\}$$

$$a \gg b \quad \text{iff} \quad a - b \in \operatorname{cor} Y_+$$

$$a \gtreqless b \quad \text{iff} \quad a \nless b\,.$$

For two subsets A and B in Y we define

$$A \succ B \quad \text{iff} \quad \forall a \in A\, \forall b \in B\colon a \succ b$$

where $\succ$ may be any relation of $\geqq$, $>$, $\gg$ and $\gtreqless$.

A is said to be *bounded* (resp. *weakly bounded*) *from above* if there exists a point $a \in Y$ such that

$$A \leqq a \quad (\text{resp. } A \lesseqgtr a)\,.$$

Analogously, the *boundedness* (resp. *the weak boundedness*) *from below* is defined.

1.3. Definition. A nonempty set $\Omega \subset Y$ is called a Y_+-*quasiinterval* if

$$\Omega = (\Omega - Y_+) \cap (\Omega + Y_+)\,.$$

The cone Y_+ is said to be *reproducing* if $Y_+ - Y_+ = Y$.

Given a subset A and a Y_+-quasiinterval Ω in Y, an element $a \in \Omega$ is called a *supremal* of A with respect to Ω if

$$a \in \operatorname{lin}(A - Y_+) \quad \text{and} \quad (a + Y_{++}) \cap \Omega \cap \operatorname{lin}(A - Y_+) = \emptyset\,.$$

A point $a \in A$ is called a *maximum* of A if $A \lesseqgtr a$. The set of all supremals with respect to Ω resp. all maxima of A are denoted by $\operatorname{Sup}_\Omega A$ resp. Max A. Analogously, an *infimal* with respect to Ω, a *minimum*, Min A and $\operatorname{Inf}_\Omega A$ are defined. If $\Omega = Y$ then the letter Ω is omitted.

1.4. Remark. (i) The notion Y_+-quasiinterval is more general than the traditional definition of interval (a set of the form $\{y \in Y\colon a \leqq y \leqq b\}$). For example, let $Y = R^2$ and $Y_+ = R^2_+$, then the parallelogram in Fig. 1 is a Y_+-quasiinterval, but it is not an interval.

(ii) Generally, the inclusion Max $A \subset$ Sup A does not hold. For example, let $Y = R^2$, $Y_+ = R^2_+$ and (see Fig. 2)

$$A = \{(y_1, y_2) \in R^2 : (0, 0) \leqq (y_1, y_2) \ll (1, 1)\} \cup \{(0, 1), (1, 0)\}\,.$$

Then

$$\operatorname{Max} A = \{(0, 1), (1, 0)\}, \quad \text{whereas} \quad \operatorname{Sup} A = \{(1, 1)\}.$$

In case that dim Y is finite and Y_+, A are closed, then $\operatorname{Max} A \subset \operatorname{Sup} A$. Nevertheless, we have $A \cap \operatorname{Sup} A \subset \operatorname{Max} A$.

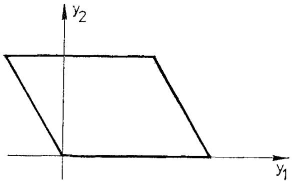

Fig. 1.

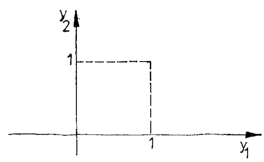

Fig. 2.

1.5. Lemma. (i) $\operatorname{Max} A = \operatorname{Max}(A - Y_+)$, $\operatorname{Min} A = \operatorname{Min}(A + Y_+)$.
(ii) $\operatorname{Sup}_\Omega A = \operatorname{Sup}_\Omega(\operatorname{lin} A) = \operatorname{Sup}_\Omega(A - \operatorname{lin} Y_+)$.
(iii) $\operatorname{Inf}_\Omega A = \operatorname{Inf}_\Omega(\operatorname{lin} A) = \operatorname{Inf}_\Omega(A + \operatorname{lin} Y_+)$.

Proof. Since assertion (i) is evident and assertion (iii) is analogous to (ii), it suffices to prove (ii). Obviously

$$\operatorname{lin}(A - Y_+) \subset \operatorname{lin}(\operatorname{lin}(A) - Y_+)$$

and the inverse inclusion follows from $\operatorname{lin}(A) - Y_+ \subset \operatorname{lin}(A - Y_+)$. We have thus proved $\operatorname{Sup}_\Omega A = \operatorname{Sup}_\Omega(\operatorname{lin} A)$. Further, we have

$$\operatorname{lin}(A - Y_+) \supset \operatorname{lin}((A - \operatorname{lin} Y_+) - Y_+)$$

for $(A - Y_+) \supset (A - \operatorname{lin} Y_+) - Y_+$. Since the inverse inclusion is obvious we obtain $\operatorname{Sup}_\Omega A = \operatorname{Sup}_\Omega(A - \operatorname{lin} Y_+)$.

1.6. Lemma. Suppose that Y is a linear topological space with $\operatorname{int} Y_+ = \operatorname{cor} Y_+$ and A is an arbitrary subset in Y. Then

$$\operatorname{lin}(A - Y_+) = \overline{A - Y_+}.$$

Proof. Obviously $\operatorname{lin}(A - Y_+) \subset \overline{A - Y_+}$. So it remains to prove the inverse inclusion. Let $a \in \overline{A - Y_+}$. Choose a point $e \in \operatorname{int} Y_+$ and let e' be an arbitrary point on the segment $[a - e, a[$. Since $a \in \overline{A - Y_+}$ and $e' \ll a$ there exists a point $a' \in A - Y_+$ such that $e' \ll a'$. Hence $e' \in A - Y_+$ and consequently, by definition, $a \in \operatorname{lin}(A - Y_+)$.

1.7. Definition. A linear functional $\Phi \in Y_+^*$ is *strictly positive* if $x \in Y_{++} \Rightarrow \Phi(x) > 0$. A *base* for Y_+ is a nonempty convex subset B of Y_+ with $0 \notin B$ such that every $x \in Y_{++}$ has a unique representation of the form λb where $b \in B$ and $\lambda > 0$.

If $\Phi \in Y_+^*$ is strictly positive and we set $B = [\Phi, 1] \cap Y_+$, where $[\Phi, \alpha] = \{x \in Y: \Phi(x) = \alpha\}$, then B is a base for Y_+.

From [9] (§ 6, C) it follows immediately

1.8. Lemma. If Y is a linear topological space of finite dimension and Y_+ is closed, then Y_+ has a compact base and every set of the form $\{y \in Y : a \leqq y \leqq b\}$ is compact.

1.9. Lemma. Suppose that Y is a linear topological space of finite dimension, Y_+ is closed and A is such a subset in Y that $\text{lin}\, A = \bar{A}$. Then

$$\text{Sup}\, A \subset \text{lin}\, A\,.$$

Proof. Let $a \in \text{Sup}\, A$. Fix arbitrary $\bar{a} \gg a$ and $\bar{\bar{a}} \ll a$. Then the set $C = (\bar{a} + Y_+) \cap (\bar{\bar{a}} - Y_+)$ is compact (by Lemma 1.8) and $a \in \text{int}\, C$. Suppose, on the contrary, that $a \notin \text{lin}\, A = \bar{A}$. There exists a neighbourhood U of a such that $U \cap \bar{A} = \emptyset$ and $U \subset C$ (for $a \in \text{int}\, C$). Since $a \in \text{lin}\,(A - Y_+) = \overline{A - Y_+}$ (see Lemma 1.6), there exist $y_\lambda \in A - Y_+$ with $y_\lambda \to a$. For each y_λ there exists an $y_\lambda^+ \in Y_{++}$ such that $y_\lambda + y_\lambda^+ \in A$. Since $\bar{A} \cap U = \emptyset$ and $U \subset C$ one can choose $0 < t_\lambda \leqq 1$ such that $a_\lambda = y_\lambda + t_\lambda y_\lambda^+ \in C$ and $a_\lambda \notin U$. If C is compact one can, without loss of generality, admit that $a_\lambda \to a'$. We have then $a' \in A - Y_+$ and $a' > a$ which contradicts $a \in \text{Sup}\, A$.

1.10. Definition. A set $C \subset$ is said to be *inside stable* if

$$(C - C) \cap Y_{++} = \emptyset\,,$$

sup-stable with respect to $A \subset Y$ if

$$\forall a \in A\ \exists c \in C : a \leqq c\,,$$

inf-stable with respect to $A \subset Y$ if

$$\forall a \in A\ \exists c \in C : c \leqq a\,.$$

Obviously

1.11. Proposition. The sets $\text{Max}\, A$, $\text{Min}\, A$, $\text{Sup}_\Omega\, A$ and $\text{Inf}_\Omega\, A$ are inside stable.

If the concerned sets are, moreover, sup-stable resp. inf-stable with respect to A then we have the same concept of solution as the von Neumann core in game theory. We try now to find out some sufficient conditions for $\text{Sup}_\Omega\, A$ and $\text{Max}\, A$ resp. $\text{Inf}_\Omega\, A$ and $\text{Min}\, A$ to be sup-stable resp. inf-stable with respect to A.

1.12. Lemma. Suppose that Y is a linear topological space of finite dimension, Y_+ is closed and $A \subset Y$ is bounded from above. Then $\overline{A - Y_+} = \bar{A} - Y_+$.

Proof. Let $a \in \overline{A - Y_+}$. There exist $a_\lambda \in A$ and $y_\lambda \in Y_+$ such that $a_\lambda - y_\lambda \to a$ or $a'_\lambda = a_\lambda - y_\lambda - a \to 0$. Since A is bounded from above there is a point $u \in Y$ such that $a_\lambda = y_\lambda + a + a'_\lambda \leqq u$ for all a_λ. For $a'_\lambda \to 0$ there exist $e \in \text{int}\, Y_+$ and λ_0 such that $y_\lambda < u - a + e$ for all $\lambda > \lambda_0$, which, considering the compactness of the set $\{y \in Y_+ : y < u - a + e\}$, implies the existence of a subnet $(y_{\lambda'}) \subset (y_\lambda)$

with $y_{\lambda'} \to y_+ \in Y_+$. Then $a_{\lambda'} \to y_+ + a \in \bar{A}$. We have thus $a \in \bar{A} - Y_+$. The inverse inclusion is obvious.

1.13. Proposition. Suppose that Y is a linear ordered topological space such that

(i) $$Y_+ = \bar{Y}_+ \quad \text{or} \quad Y_{++} = \operatorname{cor} Y_+$$

and A is a nonempty compact set in Y. Then Max A and Min A are nonempty.

Proof. Let C be a chain in A (i.e. $\forall c, d \in C: c \geqq d$ or $d \geqq c$). If C has a greatest element then C is obviously bounded from above. If C has no greatest element then C can be regarded as a net in A because one may write $C = (x)_{x \in C}$ and C is an oriented direction. Then since A is compact there exists a limit point $a \in A$ of C. Condition (i) then guarantees $a \geqq x$ for all $x \in C$. Our assertion follows now from the Zorn Lemma. The proof that Min $A \neq \emptyset$ is analogous.

1.14. Proposition. Suppose that all conditions of Proposition 1.13 remain valid. Then Max A resp. Min A is sup-stable resp. inf-stable with respect to A.

Proof. It suffices to prove that Max A is sup-stable with respect to A because the proof that Min A is inf-stable with respect to A is similar. The proof is divided into two parts. Let $y \in A$.

(i) $Y_+ = \bar{Y}_+$: Put $B = A \cap (y + Y_+)$. Since B is nonempty and compact, Max B is nonempty by Proposition 1.13. Obviously Max $B \subset$ Max A and for any point $a \in$ Max B we have $a \geqq y$.

(ii) $Y_{++} = \operatorname{cor} Y_+$: If there is no $y' \in A$ with $y' - y \in \operatorname{cor} Y_+$ then y is evidently a maximum of A. If there exists an $y' \in A$ with $y' \gg y$ then consider the set $C = A \cap (y' + \bar{Y}_+)$. C is nonempty and compact, hence, by Proposition 1.13, Max $C \neq \emptyset$. Obviously Max $C \subset$ Max A and for any $a \in$ Max C we have $a \geqq y$ (for $a - y' \in \bar{Y}_+$ and $y' - y \in \operatorname{cor} Y_+$).

1.15. Proposition. Suppose that Y is a linear topological space of finite dimension, Y_+ is closed, $A \subset Y$ and Ω is a Y_+-quasiinterval with $A \subset \Omega$. If A is bounded from above resp. below, then $\operatorname{Sup}_\Omega A$ resp. $\operatorname{Inf}_\Omega A$ is sup-stable resp. inf-stable with respect to A.

Proof. Suppose that A is bounded from above. Let $u \in Y$ be such that $A \subset u - Y_+$ and $\Phi \in Y_+^*$ be such that $\{y \in Y_+ : \Phi(y) = 1\}$ is a compact base of Y_+. Given $y \in A$ then the set $S = \{x \in u - Y_+ : \Phi(x) \geqq \Phi(y)\}$ is compact (see Holmes [9]). Consequently $B = (y + Y_+) \cap \bar{A}$ is compact. Hence, by Proposition 1.13, Max $B \neq \emptyset$. Since A is bounded from above, $\overline{A - Y_+} = \bar{A} - Y_+$ by Lemma 1.12. We have then Max $B \subset \operatorname{Sup}_\Omega A$ and every $a \in$ Max B satisfies $a \geqq y$.

In the same way one proves the inf-stability of Inf A with respect to A if A is bounded from below.

1.16. Proposition. Suppose that $Y_{++} = \operatorname{cor} Y_+ = \operatorname{int} Y_+$, Ω is a closed Y_+-quasi-interval, $A \subset \Omega$ and one of the following conditions holds

(i) A is a bounded from above,
(ii) A is weakly bounded from above and Y_+ is reproducing.
Then $\mathrm{Sup}_\Omega A$ is sup-stable with respect to A.

Proof. Let $y \in A$. If (i) holds there exists $s \in Y$ with $s \geqq A$. If (ii) holds then there exist $u \in Y$ with $u \notin A - Y_{++}$ and $c, d \in Y_{++}$ such that $c - d = u - y$ (note that Y_+ is reproducing) and in this case we take $s = c + y$. Set $M = \{y + t(s - y): t \geqq 0\} \cap \Omega \cap \mathrm{lin}(A - Y_+)$ and $t_0 = \sup\{t: y + t(s - y) \in M\}$. Since $s \notin \mathrm{cor}(A - Y_+)$ and M is closed, $t_0 \leqq 1$ and $a_0 = y + t_0(s - y) \in M$. If there exists an $a > a_0$ with $a \in \Omega \cap \mathrm{lin}(A - Y_+)$ then $a_0 \in \mathrm{cor}(a - Y_+)$ (for $Y_{++} = \mathrm{cor}\, Y_+$) and it follows that there exists $t > t_0$ with $y + t(s - y) \in M$ which contradicts definition of t_0. Hence $a_0 \in \mathrm{Sup}_\Omega A$ and $a_0 \geqq y$.

In the same way we obtain

1.17. Proposition. Suppose that $Y_{++} = \mathrm{cor}\, Y_+ = \mathrm{int}\, Y_+$, Ω is a closed Y_+-quasi-interval, $A \subset \Omega$ and one of the following conditions holds
(i) A is bounded from below,
(ii) A is weakly bounded from below and Y_+ is reproducing.
Then $\mathrm{Inf}_\Omega A$ is inf-stable with respect to A.

1.18. Lemma. Suppose that $Y_{++} = \mathrm{cor}\, Y_+$, Ω is a Y_+-quasiinterval, $A \subset \Omega$ and $B = Y \setminus (A - Y)$. Then $\mathrm{Sup}_\Omega A \subset \mathrm{Inf}_\Omega B$.

Proof. Let $a \in \mathrm{Sup}_\Omega A$ then, by definition,
(i) $(a + Y_{++}) \cap \Omega \cap \mathrm{lin}(A - Y_+) = \emptyset$
and
(ii) $a \in \Omega \cap \mathrm{lin}(A - Y_+)$.
Given $y_+ \in Y_{++}$ we prove that
(iii) $a + y_+ \notin \mathrm{lin}(A - Y_+)$.
Indeed, if $a + y_+ \in \mathrm{lin}(A - Y_+)$ then $a \in \mathrm{cor}(A - Y_+)$ which contradicts (i) (note that Ω is a Y_+-quasiinterval). We have then $a + y_+ \in B = B + Y_+$ and $a + ty_+ \to a \in \mathrm{lin}(B + Y_+)$ as $t \to 0$. If $y \in \Omega$ such that $y < a$ then, by (ii), $y \in \mathrm{cor}(A - Y_+)$ and since $\mathrm{cor}(A - Y_+) \cap \mathrm{lin}(B + Y_+) = \emptyset$ we have $y \notin \mathrm{lin}(B + Y_+)$. Hence $a \in \mathrm{Inf}_\Omega B$.

1.19. Corollary. Suppose that $Y_{++} = \mathrm{cor}\, Y_+$, $A \subset Y$ and $B = Y \setminus (A - Y_+)$. Then $\mathrm{Sup}\, A = \mathrm{Inf}\, B$.

Proof. The statement follows from Lemma 1.14 and the fact that $A \subset \Omega = Y$ and $B \subset \Omega = Y$.

1.20. Proposition. Let $(A_\lambda)_{\lambda \in \Lambda}$ be a family of subsets in Y such that $\mathrm{Max}\, A_\lambda$ resp. $\mathrm{Min}\, A_\lambda$ are sup-stable resp. inf-stable with respect to A_λ for all $\lambda \in \Lambda$. Then

(i) $$\mathrm{Max}\,(\bigcup_{\lambda \in \Lambda} A_\lambda) = \mathrm{Max}\,(\bigcup_{\lambda \in \Lambda} \mathrm{Max}\, A_\lambda)$$

resp.

(ii) $$\operatorname{Min}\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right) = \operatorname{Min}\left(\bigcup_{\lambda\in\Lambda} \operatorname{Min} A_\lambda\right).$$

Proof. The statement follows immediately from definitions.

1.21. Proposition. Let $(A_\lambda)_{\lambda\in\Lambda}$ be a family of subsets in Y and Ω be a Y_+-quasi-interval in Y such that $\operatorname{Sup}_\Omega A_\lambda$ resp. $\operatorname{Inf}_\Omega A_\lambda$ are sup-stable resp. inf-stable with respect to A_λ. Then

(i) $$\operatorname{Sup}_\Omega\left(\bigcup_{\lambda\in\Lambda} \operatorname{Sup}_\Omega A_\lambda\right) = \operatorname{Sup}_\Omega\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)$$

resp.

(ii) $$\operatorname{Inf}_\Omega\left(\bigcup_{\lambda\in\Lambda} \operatorname{Inf}_\Omega A_\lambda\right) = \operatorname{Inf}_\Omega\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right).$$

Proof. From the sup-stability of $\operatorname{Sup}_\Omega A_\lambda$ with respect to A_λ for all λ we have

$$\bigcup_{\lambda\in\Lambda} A_\lambda - Y_+ \subset \bigcup_{\lambda\in\Lambda} \operatorname{Sup}_\Omega A_\lambda - Y_+$$

which implies

(iii) $$\operatorname{lin}\left(\bigcup_\lambda A_\lambda - Y_+\right) \subset \operatorname{lin}\left(\bigcup_\lambda \operatorname{Sup}_\Omega A_\lambda - Y_+\right).$$

Further, we have

$$\operatorname{Sup}_\Omega A_\lambda \subset \operatorname{lin}(A_\lambda - Y_+) \Rightarrow \bigcup_\lambda \operatorname{Sup}_\Omega A_\lambda \subset \bigcup_\lambda \operatorname{lin}(A_\lambda - Y_+) \Rightarrow$$

$$\Rightarrow \bigcup_\lambda \operatorname{Sup}_\Omega A_\lambda - Y_+ \subset \bigcup_\lambda \operatorname{lin}(A_\lambda - Y_+) \subset \operatorname{lin}\left(\bigcup_\lambda A_\lambda - Y_+\right) \Rightarrow$$

$$\Rightarrow \operatorname{lin}\left(\bigcup_\lambda \operatorname{Sup}_\Omega A_\lambda - Y_+\right) \subset \operatorname{lin}\left(\bigcup_\lambda A_\lambda - Y_+\right),$$

which, together with (iii), gives

$$\operatorname{lin}\left(\bigcup_\Lambda A_\lambda - Y_+\right) = \operatorname{lin}\left(\bigcup_\Lambda \operatorname{Sup}_\Omega A_\lambda - Y_+\right).$$

From the last equality we obtain then the equality (i). Equality (ii) is proved similarly.

2. ABSTRACT DUALITY SCHEME

2.1. Basic assumptions. In the sequel suppose that Ω is a Y_+-quasiinterval in Y and $\mathscr{P}$ and $\mathscr{D}$ are arbitrary fixed sets. Further, suppose that $P: \Omega \to \mathscr{P}$ and $D: \Omega \to \mathscr{D}$ are multivalued maps fulfilling the following conditions

(i) Primal Availability:

$$y_1 < y_2 \Rightarrow P(y_1) \supset P(y_2),$$

(ii) Dual Availability:

$$y_1 < y_2 \Rightarrow D(y_1) \subset D(y_2).$$

2.2. Definition. Put

$$\mathscr{P}_0 = \{p \in \mathscr{P} \mid \exists y \in \Omega: p \in P(y)\} = \bigcup_{y\in\Omega} P(y),$$

$$\mathscr{D}_0 = \bigcup_{y \in \Omega} D(y)$$

$$\mu(p) = \{y \in \Omega \mid p \in P(y)\} \quad p \in \mathscr{P}_0$$

and

$$\nu(d) = \{y \in \Omega \mid d \in D(y)\} \quad d \in \mathscr{D}_0 .$$

Problems

(2.2.1)
$$\mathrm{Sup}_\Omega\, \mu(\mathscr{P}_0)$$

resp.

(2.2.2)
$$\mathrm{Inf}_\Omega\, \nu(\mathscr{D}_0)$$

are called *abstract primal* resp. *abstract dual*. The points $p^* \in \mathscr{P}_0$ resp. $d^* \in \mathscr{D}_0$ are called *optimal solutions* of the primal resp. the dual if $\mu(p^*) \cap \mathrm{Sup}_\Omega\, \mu(\mathscr{P}_0) \neq \emptyset$ resp. $\nu(d^*) \cap \mathrm{Inf}_\Omega\, \nu(\mathscr{D}_0) \neq \emptyset$.

Analogously are defined the *abstract max-primal*

$$\mathrm{Max}\, \mu(\mathscr{P}_0)$$

and the *abstract min-dual*

$$\mathrm{Min}\, \nu(\mathscr{D}_0)$$

and their optimal solutions.

2.3. Theorem. (Weak Duality.) If the condition

(iii) Weak Duality Condition:

$$D(y) \neq \emptyset \Rightarrow P(y') = \emptyset \quad \forall y' > y .$$

holds, then

$$\mu(\mathscr{P}_0) \gtreqless \nu(\mathscr{D}_0) .$$

Proof. Let $y' \in \mu(\mathscr{P}_0)$ and $y'' \in \nu(\mathscr{D}_0)$. We have then $P(y') \neq \emptyset$ and $D(y'') \neq \emptyset$. Hence, by condition (iii) y' cannot be greater than y''.

2.4. Corollary. (Max-Min Strong Duality.) If the weak duality condition holds then

$$\mathrm{Max}\, \mu(\mathscr{P}_0) \cap \mathrm{Min}\, \nu(\mathscr{D}_0) = \mu(\mathscr{P}_0) \cap \nu(\mathscr{D}_0) .$$

2.5. Theorem.(Sup-Inf Strong Duality.) Suppose that $Y_{++} = \mathrm{cor}\, Y_+$ and the weak duality condition and the following one

(iv) Sup-Inf Strong Duality Condition:

$$\forall y' \in \mathrm{cor}\, \Omega (P(y) = \emptyset \quad \forall y > y' \Rightarrow D(y) \neq \emptyset \ \forall y > y')$$

hold then

$$\mathrm{cor}\,(\Omega) \cap \mathrm{Sup}_\Omega\, \mu(\mathscr{P}_0) = \mathrm{cor}\,(\Omega) \cap \mathrm{Inf}_\Omega\, \nu(\mathscr{D}_0) .$$

Proof. If $\mathscr{P}_0 = \emptyset$ or $\mathscr{D}_0 = \emptyset$ then, by condition (iv),

$$\mathrm{cor}\,(\Omega) \cap \mathrm{Sup}_\Omega\, \mu(\mathscr{P}_0) = \emptyset = \mathrm{cor}\,(\Omega) \cap \mathrm{Inf}_\Omega\, \nu(\mathscr{D}_0).$$

Suppose hence that $\mathscr{P}_0 \neq \emptyset$ and $\mathscr{D}_0 \neq \emptyset$. Let $y^* \in \mathrm{cor}\,(\Omega) \cap \mathrm{Sup}_\Omega\, \mu(\mathscr{P}_0)$ which means

$$y^* \in \Omega \cap \mathrm{lin}\,(\mu(\mathscr{P}_0) - Y_+) \quad \text{and} \quad (y^* + Y_{++}) \cap \mathrm{lin}\,(\mu(\mathscr{P}_0) - Y_+) \cap \Omega = \emptyset.$$

We have then

$$P(y^* + y_+) = \emptyset \quad \forall y_+ \in Y_{++}: y^* + y_+ \in \Omega ,$$

which, by condition (iv), implies that

$$D(y^* + y_+) \neq \emptyset \quad \forall y_+ \in Y_{++} .$$

Consequently, $y^* \in \Omega \cap \text{lin}\,(\nu(\mathscr{D}_0) + Y_+)$.

For any $y_+ \in Y_{++}$ such that $y^* - y_+ \in \Omega$ we have

$$y^* - y_+ \in \text{cor}\,(\mu(\mathscr{P}_0) - Y_+) .$$

So, since (because of the weak duality condition)

$$\text{cor}\,(\mu(\mathscr{P}_0) - Y_+) \cap \text{cor}\,(\nu(\mathscr{D}_0) + Y_+) = \emptyset ,$$

one has

$$y^* - y_+ \notin \text{lin}\,(\nu(\mathscr{D}_0) + Y_+) .$$

We have thus proved that

$$y^* \in \text{cor}\,(\Omega) \cap \text{Inf}_\Omega\, \nu(\mathscr{D}_0) .$$

Conversely, let

$$y^* \in \text{cor}\,(\Omega) \cap \text{Inf}_\Omega\, \nu(\mathscr{D}_0) ,$$

which means

$$y^* \in \Omega \cap \text{lin}\,(\nu(\mathscr{D}_0) + Y_+) \quad \text{and} \quad (y^* - Y_{++}) \cap \text{lin}\,(\nu(\mathscr{D}_0) + Y_+) \cap \Omega = \emptyset .$$

Analogously, by condition (iv), we have

$$y^* \in \Omega \cap \text{lin}\,(\mu(\mathscr{P}_0) - Y_+)$$

and

$$y^* + y_+ \in \text{cor}\,(\nu(\mathscr{D}_0) + Y_+)$$

for all $y_+ \in Y_{++}$ with $y^* + y_+ \in \Omega$. Hence

$$y^* \in \text{cor}\,(\Omega) \cap \text{Sup}_\Omega\, \mu(\mathscr{P}_0) .$$

3. FENCHEL-LAGRANGE DUALITY FOR VECTOR FRACTIONAL PROGRAMMING

3.1. Definition. Suppose that X is a linear space, U_{nk} and V_{nk} are nonempty subsets in X, $u_{nk}: U_{nk} \to R$ and $v_{nk}: V_{nk} \to R$ are real functions for $n = 1, \ldots, N$ and $k = 1, \ldots \ldots, k(n)$. Further, let Z be a linear ordered space, S be a nonempty subset in X and $g: S \to Z$. Let Ω be a nonempty R^N_+-quasiinterval in $R^N =: Y$. Here in this section, in order to avoid misunderstanding we accept the following notations:

If $Y_+ = \text{cor}\, R^N_+$ then the Max, Sup_Ω, Min and Inf_Ω notations, which are introduced in Section 1 will be replaced by Max^w, Sup^w_Ω, Min^w, and Inf^w_Ω respectively. If $Y_+ = R^N_+$ we will use the notations Max^s, Sup^s_Ω, Min^s and Inf^s_Ω respectively. If there is no assumption about Y_+ we will use the standard notations without the letters w and s.

Now put

$$\mathscr{F} = \{x \in S \cap (\bigcap_{n=1}^{N} \bigcap_{k=1}^{K(n)} U_{nk} \cap V_{nk}) : g(x) \in Z_+\}$$

and, under the assumption that all occurring denominators do not vanish on $\mathscr{F}$,

$$f(x) = (\sum_{k=1}^{K(n)} u_{nk}(x) / \sum_{k=1}^{K(n)} v_{nk}(x))_{n=1,\ldots,N} .$$

We shall consider the problem

(FP) $$\text{Max-Sup}_\Omega f(\mathscr{F})$$

where the notation Max-Sup$_\Omega$ means that both maximum and supremal concepts will be studied. Problem (FP) is called the *vector fractional program.*

3.2. Remark. The duality questions of scalar fractional programming have been investigated by Schaible [10, 11] and other authors. In Cambini [2] a version of Fenchel duality in scalar fractional programming was introduced. As far as the vector fractional programming is concerned, in Tran Quoc Chien [5, 6, 7] a duality theory of Lagrange type is constructed via the abstract duality scheme. In this section, on the basis of the abstract duality scheme introduced in Section 2, a unified duality theory will be built up, which contains all old Lagrange and Fenchel duality and gives a considerable possibility of numerical applications.

3.3. Abstract primal problem

Put

$$\mathscr{P} = X \times \prod_{n=1}^{N} (R \times R^{K(n)}) \times Z$$

$$P_0(y) = P_0 = X \times \prod_{n=1}^{N} (R_+ \times R^{K(n)}) \times Z_+ \quad \forall y \in \Omega ,$$

$$P_{nk}(y) = \{(x, (r_{n0}, \ldots, r_{nK(n)})_{n=1\ldots N}, z) \in \mathscr{P} : x \in S \cap U_{nk} \cap V_{nk} \,\&$$

$$\& \, r_{nk} \leqq u_{nk}(x) - y_n v_{nk}(x) \,\&\, r_{n0} \leqq \sum_{i=1}^{K(n)} r_{ni} \,\&\, z \leqq g(x)\}$$

and

$$P(y) = P_0 \cap (\bigcap_{n=1}^{N} \bigcap_{k=1}^{K(n)} P_{nk}(y)) \quad \text{for} \quad y = (y_1, \ldots, y_n) \in \Omega .$$

From definitions it follows immediately

3.3.1. Lemma. If $v_{nk}(x) \geqq 0$ for all feasible n, k and x then the multivalued map P: $\Omega \to \mathscr{P}$, that was just defined above, satisfies the primal availability.

Now following the approach and notations in Section 2 we obtain the abstract primal problem

(P) $$\text{Max-Sup}_\Omega \, \mu(\mathscr{P}) .$$

It is easy to verify the following

3.3.2. $$f(\mathscr{F}) - R^N_+ = \mu(\mathscr{P}_0) - R^N_+ .$$

Hence, as a consequence of Proposition 1.5, we obtain

3.3.3. Proposition. Problems (FP) and (P) are equivalent in the sense

$$\mathrm{Max}^s f(\mathscr{F}) = \mathrm{Max}^s \mu(\mathscr{P}_0)$$

and

$$\mathrm{Sup}_\Omega f(\mathscr{F}) = \mathrm{Sup}_\Omega \mu(\mathscr{P}_0) .$$

Now in order to establish a reasonable dual to problem (P) we recall first the following concept of set separation.

3.4. Set separation

3.4.1. Definition. Given a family $\{A_i: i \in I\}$ of subsets in X, a family of linear functionals on $X\{\Phi_i: i \in I\}$, not all zero, is said to *separate* $\{A_i: i \in I\}$ (in Vlach's sense) if

(i) $\sum_{i \in I} \Phi_i = 0$

(ii) $\sum_{i \in I} \sup_{x \in A_i} \langle \Phi_i; x \rangle \leqq 0$

(iii) there exists at least one $j \in I$ such that

$$\inf_{x \in A_j} \langle \Phi_j, x \rangle < \sup_{x \in A_j} \langle \Phi_j, x \rangle .$$

A family $\{A_i: i \in I\}$ is said to be *separated* if there exists a family $\{\Phi_i: i \in I\}$ of linear functionals on X which separates $\{A_i: i \in I\}$.

3.4.2. Theorem. A finite family $\{A_i: i \in I\}$ of convex subsets in X is separated

(a) if and only if $\bigcap_{i \in I} \mathrm{icr}\, A_i = \emptyset$, when $\mathrm{icr}\, A_i \neq \emptyset$ for all $i \in I$.

(b) if and only if $A_j \cap \bigcap_{i \in I \setminus \{j\}} \mathrm{cor}\, A_i = \emptyset$, when $\mathrm{cor}\, A_i \neq \emptyset$ $\forall i \in I \setminus \{j\}$.

(c) if $A_j \cap \bigcap_{i \in I \setminus \{j\}} \mathrm{icr}\, A_i = \emptyset$, when $\mathrm{icr}\, A_i \neq \emptyset$ $\forall i \in I \setminus \{j\}$ and codim $(\mathrm{aff}\, A_i) < +\infty$ $\forall i \in I \setminus \{j\}$.

Proof. See Bair [1], Theorem 2.1 of Chapter 6.

3.5. Abstract dual problem

Let $\mathscr{P}^*$ be the dual space to $\mathscr{P}$. Every element $\bar{p} \in \mathscr{P}^*$ has the form

$$\bar{p} = (\bar{x}, (\bar{r}_{n0}, \bar{r}_{n1}, \ldots, \bar{r}_{nK(n)})_{n=1\ldots N}, \bar{z})$$

where $\bar{x} \in X^*$, $\bar{r}_{nk} \in R$ $\forall n = 1 \ldots N$ and $k = 0, 1 \ldots K(n)$, $\bar{z} \in Z^*$.

Let $\{\bar{p}^0, \bar{p}^{nk}: n = 1 \ldots N, k = 1 \ldots K(n)\}$ be a family of $1 + \sum_{n=1}^{N} K(n)$ functionals in $\mathscr{P}^*$ of the following form

3.5.1.
$$\bar{p}^0 = (\bar{x}^0, (\bar{r}^0_{n0}, \bar{r}^0_{n1}, \ldots, \bar{r}^0_{nK(n)})_{n=1\ldots N}, \bar{z}^0)$$
$$\bar{p}^{nk} = (\bar{x}^{nk}, (\bar{r}^{nk}_{n0}, \bar{r}^{nk}_{n1}, \ldots, \bar{r}^{nk}_{nK(n)})_{n=1\ldots N}, \bar{z}^{nk})$$

3.5.2. Lemma. If $\sup\limits_{p\in P_0} \langle \bar{p}^0, p\rangle < +\infty$, then

(i) $\bar{x}^0 = 0, \; -\bar{z}^0 \in Z^*_+$

(ii) $\bar{r}^0_{n0} \leqq 0$ and $\bar{r}^0_{nk} = 0 \; \forall n = 1 \ldots N \; \forall k = 1 \ldots K(n)$.

Proof. The statement follows easily from the structure of P_0.

3.5.3. Lemma. If $s_{nk} = \sup\limits_{p\in P_{nk}(y)} \langle \bar{p}^{nk}, p\rangle < +\infty$, then

(i) $\bar{r}^{nk}_{mi} = 0 \; \forall m \neq n \; \forall i = 0, 1, \ldots, K(m)$ and $\bar{z}^{nk} \in Z^*_+$

(ii) $\bar{r}^{nk}_{ni} = -\bar{r}^{nk}_{n0} \; \forall i \in \{1, \ldots, K(n)\} \setminus \{k\}$.

Proof. The statement (i) follows immediately from the structure of $P_{nk}(y)$. Since $\bar{r}^{nk}_{nk}$ and $\bar{r}^{nk}_{n0}$ are nonnegative (for r_{nk} and r_{n0} may be arbitrarily negative) one can write

$$s_{nk} = \sup_{\substack{x\in S\cap U_{nk}\cap V_{nk} \\ r_{ni}\in R}} [\langle \bar{x}^{nk}, x\rangle + \sum_{i\neq k,0} (\bar{r}^{nk}_{n0} + \bar{r}^{nk}_{ni}) \,.\, r_{ni} +$$
$$+ (\bar{r}^{nk}_{n0} + \bar{r}^{nk}_{nk}) \,.\, (u_{nk}(x) - y_n v_{nk}(x)) + \langle \bar{z}^{nk}, g(x)\rangle] \,.$$

Since r_{ni} are arbitrary for $i \neq k, 0$, we have

$$\bar{r}^{nk}_{n0} + \bar{r}^{nk}_{ni} = 0 \quad \forall i \neq k, 0$$

or

$$\bar{r}^{nk}_{ni} = -\bar{r}^{nk}_{n0} \qquad \forall i \neq k, 0\,,$$

From Lemma 3.5.3 it follows immediately

3.5.4. Lemma. If $\sup\limits_{p\in P_{nk}(y)} \langle \bar{p}^{nk}, p\rangle < +\infty$, then

$$\sup\{\langle \bar{p}^{nk}, p\rangle : p \in P_{nk}(y)\} = \sup\{\langle \bar{x}^{nk}, x\rangle + (\bar{r}^{nk}_{n0} + \bar{r}^{nk}_{nk}) \,.\, (u_{nk}(x) -$$
$$- y_n v_{nk}(x)) + \langle \bar{z}^{nk}, g(x)\rangle : x \in S \cap U_{nk} \cap V_{nk}\}\,.$$

3.5.5. Lemma. If the family (3.5.1) separates the family $\{P_0, P_{nk}(y) : n = 1 \ldots \ldots N, k = 1 \ldots K(n)\}$, then

(i) $\sum\limits_{n=1}^{N} \sum\limits_{k=1}^{K(n)} \bar{x}^{nk} = 0$

(ii) For each $n = 1, \ldots, N$ $\bar{r}^{nk}_{n0} + \bar{r}^{nk}_{nk} = \bar{r}^n$ is constant for all $k = 1, \ldots, K(n)$

(iii) $s_{nk}(y) = \sup\{\langle \bar{p}^{nk}, p\rangle : p \in P_{nk}(y)\} = \sup\{\langle \bar{x}^{nk}, x\rangle + \bar{r}^n(u_{nk}(x) - y_n v_{nk}(x)) + \langle \bar{z}^{nk}, g(x)\rangle : x \in S \cap U_{nk} \cap V_{nk}\}$

(iv) $\sum\limits_{n=1}^{N} \sum\limits_{k=1}^{K(n)} s_{nk}(y) \leqq 0.$

Proof. From definition of set separation and Lemma 3.5.2(i) we have immediately assertions (i) and (iv). Assertion (iii) follows then from Lemma 3.5.4 and assertion (ii).

So it remains to prove (ii). Fix an $n \in \{1, \ldots, N\}$. From Lemma 3.5.2(ii), Lemma 3.5.3(i) and condition $\bar{p}^0 + \sum_{n=1}^{N} \sum_{k=1}^{K(n)} \bar{p}^{nk} = 0$ we obtain

$$\sum_{k=1}^{K(n)} \bar{r}_{n0}^{nk} = -\bar{r}_{n0}^{0} \geqq 0$$

and

$$\sum_{i=1}^{K(n)} \bar{r}_{nk}^{ni} = 0 \quad \forall k = 1, \ldots, K(n) .$$

Now combining Lemma 3.5.3(ii) and the last equality we have

$$\bar{r}_{n0}^{nk} + \bar{r}_{nk}^{nk} = \bar{r}_{n0}^{nk} - \sum_{i \neq k, 0} \bar{r}_{nk}^{ni} = \sum_{i=1}^{K(n)} \bar{r}_{n0}^{ni} = \bar{r}^n \geqq 0 \quad \text{for all} \quad k = 1, \ldots, K(n) .$$

The lemma is thus proved.

3.5.6. Definition. We introduce now the following set

$$\mathscr{D} = \{d = (\bar{x}^{nk}, \bar{z}^{nk}, \bar{r}^n)_{\substack{n=1\ldots N \\ k=1\ldots K(n)}} \mid \bar{x}^{nk} \in X^*, \bar{z}^{nk} \in Z_+^*, \forall n, k, (\bar{r}^1, \ldots, \bar{r}^N) \in R_{++}^N \; \& \; \& \sum_{n=1}^{N} \sum_{k=1}^{K(n)} \bar{x}^{nk} = 0\}$$

and the multivalued maps

$$D^w(y) = \{d \in \mathscr{D}: \sum_{n=1}^{N} \sum_{k=1}^{K(n)} s_{nk}(d, y) \leqq 0\}$$

and

$$D^s(y) = \{d \in D^w(y): \bar{r}^n > 0 \quad \forall n = 1 \ldots N\} ,$$

where

$$s_{nk}(d, y) = \sup \{\langle \bar{x}^{nk}, x \rangle + \bar{r}^n(u_{nk}(x) - y_n v_{nk}(x)) + + \langle \bar{z}^{nk}, g(x) \rangle : x \in S \cap U_{nk} \cap V_{nk}\} .$$

3.5.7. Lemma. If $v_{nk}(x) \geqq 0$ for all feasible n, k and x then both maps $D^w(y)$ and $D^s(y)$ satisfy the dual availability.

Proof. The statement follows immediately from definition.

3.5.8. Assumption. In the sequel we suppose that

$$v_{nk}(x) \geqq 0 \quad \text{for all feasible } n, k \text{ and } x .$$

3.5.9. Lemma. If $Y_+ = \{(y_1, \ldots, y_N) \in R^N: y_n > 0 \;\; \forall n = 1 \ldots N\}$, then the maps $P(y)$ and $D^w(y)$ satisfy the weak duality condition.

Proof. Let $y^* \in \Omega$ such that $D^w(y^*) \neq \emptyset$. Let $y > y^*$ which means $y_n > y_n^*$ for all $n = 1, \ldots, N$. Suppose, on the contrary, that $P(y) \neq \emptyset$. Choose $p' = (x', (r'_{n0}, \ldots, r'_{nK(n)})_{n=1\ldots N}, g(x')) \in P(y)$. We have

$$\sum_{k=1}^{K(n)} (u_{nk}(x') - y_n v_{nk}(x')) \geqq 0 \quad \text{for all feasible } n, \text{ and}$$

$$g(x') \in Z_+ .$$

Since $v_{nk}(x) \geqq 0$ for all feasible n, k and x and $\sum_{k=1}^{K(n)} v_{nk}(x) \neq 0$ for all n, there exists, for each n, a $k \in \{1, \ldots, K(n)\}$ with $v_{nk}(x') > 0$. So we have, for all n,

$$\sum_{k=1}^{K(n)} (u_{nk}(x') - y_n^* v_{nk}(x')) > \sum_{k=1}^{K(n)} (u_{nk}(x') - y_n v_{nk}(x')) \geqq 0 .$$

Hence, for a $d \in D^w(y^*)$ we have

$$0 \geqq \sum_{n,k} s_{nk}(d, y) \geqq \sum_{n,k} [\langle \bar{x}^{nk}, x' \rangle + \bar{r}^n(u_{nk}(x') - y_n^* v_{nk}(x')) + \langle \bar{z}^{nk}, g(x') \rangle] >$$
$$> \sum_{n,k} \bar{r}^n(u_{nk}(x') - y_n v_{nk}(x')) \geqq 0$$

which is absurd. We have thus proved $P(y) = \emptyset$.

Analogously we have

3.5.10. Lemma. The maps $P(y)$ and $D^s(y)$ satisfy the weak duality condition

3.5.11. Definition. Denote

(i) $\mathscr{D}^w = \bigcup_{y \in \Omega} D^w(y)$ with $v^w(d) = \{y \in \Omega : d \in D^w(y)\}$

and

(ii) $\mathscr{D}^s = \bigcup_{y \in \Omega} D^s(y)$ with $v^s(d) = \{y \in \Omega : d \in D^s(y)\}$.

According to Definition 2.2 we will have the following dual problems to the fractional program (FP):

$$\mathrm{Min}^w\text{-}\mathrm{Inf}_\Omega^w\, v^w(\mathscr{D}^w)$$

and

$$\mathrm{Min}^s\text{-}\mathrm{Inf}_\Omega^s\, \gamma^s(\mathscr{D}^s) ,$$

which are called *Fenchel-Lagrange duals* to program (FP).

Now we shall prove some duality principles for this duality.

3.6. Theorem. (Weak Duality.) We have

$$f(\mathscr{F}) \gtreqless v^w(\mathscr{D}^w)$$

and

$$f(\mathscr{F}) \gtreqless v^s(\mathscr{D}^s) .$$

Proof. The statement is a consequence of Theorem 2.3, Lemmas 3.5.9 and 3.5.10 and relation 3.3.2.

3.7. Theorem. (Max^s-Min^s Strong Duality.)

$$\mathrm{Max}^s f(\mathscr{F}) \cap \mathrm{Min}^s v^s(\mathscr{D}^s) = f(\mathscr{F}) \cap v^s(\mathscr{D}^s)$$

Proof. The assertion follows from Corollary 2.4, Proposition 3.3.3, relation 3.3.2 and Lemma 3.5.10.

3.8. Fenchel-Slater constraint qualification

(i) *Fenchel constraint qualification*:

$$\bigcap_{n,k} \operatorname{icr} [S \cap U_{nk} \cap V_{nk}] \neq \emptyset \quad \text{and} \quad \operatorname{icr} Z_+ \neq \emptyset$$

(ii) *Slater constraint qualification*:

$$\forall \bar{z} \in Z^*_{++} \ \exists x \in S \cap \left(\bigcap_{n,k} U_{nk} \cap V_{nk} \right): \langle \bar{z}, g(x) \rangle > 0 \, .$$

It is easy to prove the following

3.9. Lemma. If the Fenchel constraint qualification holds then

$$\operatorname{icr} P_0 \neq \emptyset \quad \text{and} \quad \operatorname{icr} P_{nk}(y) \neq \emptyset \quad \text{for all feasible } n, k \text{ and } y \, .$$

3.10. Lemma. Suppose that $S \cap U_{nk} \cap V_{nk}$ are convex and $u_k(x) - y_n v_{nk}(x)$ are concave for all feasible n, k and y and $g(x)$ is concave on S. Then the Fenchel-Slater constraint qualification implies the Sup^w-Inf^w strong duality condition.

Proof. The sets P_0 and $P_{nk}(y)$ are convex and have nonempty relative core for all feasible n, k and y by Lemma 3.9. If $P(y) = \emptyset$ then by Theorem 3.4.2(a) there exists a family $\{\bar{p}^0, \bar{p}^{nk}: n = 1 \dots N, k = 1 \dots K(n)\} \subset \mathscr{P}^*$ which separates the family $\{P_0, P_{nk}(y): n = 1 \dots N, k = 1 \dots K(n)\}$. Put

$$\bar{r}^n = \sum_{k=1}^{K(n)} \bar{r}^{nk}_{n0} \quad \forall n = 1 \dots N$$

and

$$d = (\bar{x}^{nk}, \bar{z}^{nk}, \bar{r}^n)_{\substack{n=1\dots N \\ k=1\dots K(n)}} \, .$$

Then by Lemma 3.5.5

$$\sum_{n,k} s_{nk}(d, y) \leqq 0 \, .$$

Suppose, on the contrary, that $d \notin \mathscr{D}$ which means $\bar{r}^n = 0$ for all n. Then, in virtue of Lemma 3.5.3 and the fact that $\bar{r}^{nk}_{n0} \geqq 0$ for all k, we have $\bar{r}^{nk}_{ni} = 0$ for all feasible n, k and i. If there is $\bar{z}^{nk} \in Z^*_{++}$ then by the Slater constraint qualification there exists $x \in \bigcap_{n,k} (S \cap U_{nk} \cap V_{nk})$ with $\langle \bar{z}^{nk}, g(x) \rangle > 0$ that leads to the following absurdity

$$0 \geqq \sum_{n,k} s_{nk}(d, y) \geqq \sum_{n,k} \langle \bar{z}^{nk}, g(x) \rangle > 0 \, .$$

So $\bar{z}^{nk} = 0$ for all feasible n and k. In this case there exists, at least one $\bar{x}^{nk} \neq 0$. Then the family $\{\bar{x}^{nk}: n = 1 \dots N, k = 1 \dots K(n)\}$ separates the family $\{S \cap U_{nk} \cap V_{nk}: n = 1 \dots N, k = 1 \dots K(n)\}$ which, in virtue of Theorem 3.4.2(a), contradicts the Fenchel constraint qualification. The lemma is thus proved.

3.11. Theorem. (Sup^w-Inf^w Strong Duality.) Suppose that the following conditions hold:

(i) $g(x)$ is concave on convex set S,

(ii) $u_{nk}(x) - y_n v_{nk}(x)$ are concave on convex set $S \cap U_{nk} \cap V_{nk}$ for all feasible n, k and y,

(iii) the Fenchel-Slater constraint qualification.

Then

$$\mathrm{Sup}_\Omega^w (f(\mathscr{F})) \cap \mathrm{cor}\, \Omega = \mathrm{Inf}_\Omega^w (v^w(\mathscr{D}^w)) \cap \mathrm{cor}\, \Omega .$$

Proof. The statement follows from Lemma 3.10, Proposition 3.3.3 and Theorem 2.5.

In the sequel we will transform the Fenchel-Lagrange duals to the so-called canonical Fenchel-Lagrange duals which are more suitable for the numerical calculation.

3.12. Assumption. In the sequel we suppose that $v_{nk}(x)$ are positive for all feasible n, k and x and Ω is of the following form

$$\Omega = [a_1, b_1] \times \ldots \times [a_n, b_n] \times \ldots \times [a_N, b_N]$$

where a_n, $b_n \in R \cup \{-\infty, +\infty\}$ and the notations $[-\infty$ resp. $+\infty]$ are equivalent to $(-\infty$ resp. $+\infty)$.

3.13. Definition. Putting

$$\mathscr{L} = \{\bar{d} = (\bar{x}^{nk}, \bar{z}^{nk}, \bar{r}^n, \bar{s}^{nk})_{\substack{n=1\ldots N\\ k=1\ldots K(n)}} : \bar{x}^{nk} \in X^*, \bar{z}^{nk} \in Z_+^*, \bar{r}^n > 0,$$
$$\bar{s}^{nk} \in R, \forall n, k, \sum_{n,k} \bar{x}^{nk} = 0 \;\&\; \sum_{n,k} \bar{s}^{nk} = 0\} ,$$

we define the following function $L\colon \mathscr{L} \to \Omega$

$$L(\bar{d}) = (y_1, \ldots, y_N)$$

where

$$y_n = \max\left\{a_n, \sup_{k=1\ldots K(n)} \sup_{x \in S \cap U_{nk} \cap V_{nk}} \frac{\langle \bar{x}^{nk}, x\rangle + \bar{r}^n u_{nk}(x) + \langle \bar{z}^{nk}, g(x)\rangle - \bar{s}^{nk}}{\bar{r}^n v_{nk}(x)}\right\}.$$

The problem

$$\text{Min-Inf}_\Omega\, L(\mathscr{L})$$

will be called the *canonical Fenchel-Lagrange dual* to program (FP).

3.14. Lemma.

$$L(\mathscr{L}) + R_+^N = v^s(\mathscr{D}^s) + R_+^N .$$

Proof. Obviously $L(\mathscr{L}) \subset v^s(\mathscr{D}^s)$, hence $L(\mathscr{L}) + R_+^N \subset v^s(\mathscr{D}^s) + R_+^N$. Conversely, let $d = (\bar{x}^{nk}, \bar{z}^{nk}, \bar{r}^n) \in \mathscr{D}^s$ and $y \in v^s(d)$ which means

$$\sum_{n,k} s_{nk}(d, y) \leq 0 .$$

Then one can choose $\bar{s}^{nk} \geqq s_{nk}(d, y)$ for all n, k such that $\sum_{n,k} \bar{s}^{nk} = 0$ and $L(\bar{x}^{nk}, \bar{z}^{nk}, \bar{r}^n, \bar{s}^{nk}) \leqq y$. So we have $y \in L(\mathscr{L}) + R^N_+$.

3.15. Lemma.

$$\text{Min}^s\text{-Inf}_\Omega\, L(\mathscr{L}) = \text{Min}^s\text{-Inf}_\Omega\, v^s(\mathscr{D}^s)\,.$$

Proof. The statement follows from Lemmas 1.5 and 3.14.

Combining Theorem 3.7 and Lemma 3.15 we obtain

3.16. Theorem. (Max^s-Min^s Strong Duality.)

$$\text{Max}^s f(\mathscr{F}) \cap \text{Min}^s L(\mathscr{L}) = f(\mathscr{F}) \cap L(\mathscr{L})\,.$$

The following two lemmas are evident

3.17. Lemma. If $v^w(\mathscr{D}^w) \cap \text{cor}\,\Omega \subset \text{lin}\, v^s(\mathscr{D}^s)$ then

$$\text{Inf}^w_\Omega\, v^w(\mathscr{D}^w) \cap \text{cor}\,\Omega = \text{Inf}^w_\Omega\, v^s(\mathscr{D}^s) \cap \text{cor}\,\Omega\,.$$

3.18. Lemma. If $N = 1$ then $\mathscr{D}^w = \mathscr{D}^s$ and

$$v^w(\mathscr{D}^w) = v^s(\mathscr{D}^s)\,.$$

3.19. Lemma. Suppose that the following conditions hold:

(i) $\forall y \in \Omega\ \exists M\ \forall n\ \forall k\ \forall x \in S \cap U_{nk} \cap V_{nk}\colon u_{nk}(x) - y_n\, v_{nk}(x) < M$,

(ii) $\forall n\ \exists k\ \exists c > 0\ \forall x \in S \cap U_{nk} \cap V_{nk}\colon v_{nk}(x) \geqq c$.

Then

$$v^w(\mathscr{D}^w) \cap \text{cor}\,\Omega \subset \text{lin}\, v^s(\mathscr{D}^s)\,.$$

Proof. Let $y \in v^w(\mathscr{D}^w) \cap \text{cor}\,\Omega$. There exists $d = (\bar{x}^{nk}, \bar{z}^{nk}, \bar{r}^n) \in \mathscr{D}^w$ with

$$\sum_{n,k} s_{nk}(d, y) \leqq 0\,.$$

Without loss of generality we can suppose that $\bar{r}^1 > 0$ and $v_{11}(x) \geqq c$ for all $x \in S \cap U_{11} \cap V_{11}$. Fix an arbitrary $\delta > 0$ and choose an $\varepsilon > 0$ such that

$$\sum_{n=1}^{N} K(n)\,.\,\varepsilon\,.\,M < \bar{r}^1\,.\,\delta\,.\,c\,.$$

Now it is easy to verify that $d_0 = (\bar{x}^{nk}, \bar{z}^{nk}, \bar{r}^n_0)$, where

$$\bar{r}^1_0 = \bar{r}^1 \quad \text{and} \quad \bar{r}^n_0 = \bar{r}^n + \varepsilon \quad \forall n \neq 1\,,$$

belongs to $\mathscr{D}^s$ for $(y_1 + \delta, y_2, \ldots, y_n) \in v^s(d_0)$.

So, by a limit passage ($\delta \downarrow 0$), we obtain $y \in \text{lin}\, v^s(\mathscr{D}^s)$.

Summarizing Theorem 3.11, Lemma 3.15 and Lemma 3.17 we obtain

3.20. Theorem. (Sup^w-Inf^w Strong Duality.) Suppose that the following conditions hold:

(i) $g(x)$ is concave on the convex set S,

(ii) $u_{nk}(x) - y_n v_{nk}(x)$ are concave on convex set $S \cap U_{nk} \cap V_{nk}$ for all feasible n, k and y,
(iii) The Fenchel-Slater constraint qualification,
(iv) $v^w(\mathscr{D}^w) \cap \operatorname{cor} \Omega \subset \operatorname{lin} v^s(\mathscr{D}^s)$.
Then

$$\operatorname{Sup}_\Omega^w (f(\mathscr{F})) \cap \operatorname{cor} \Omega = \operatorname{Inf}_\Omega^w (L(\mathscr{L})) \cap \operatorname{cor} \Omega .$$

3.21. Proposition. Suppose that $u_{nk}(x)$ are concave on U_{nk} for all n and k. If for each $n = 1, \ldots, N$ one of the following cases holds:
(i) $v_{nk}(x)$ are affine for all $k = 1, \ldots, K(n)$ and $[a_n, b_n] = (-\infty, +\infty)$
(ii) $v_{nk}(x)$ are convex for all $k = 1, \ldots, K(n)$ and $[a_n, b_n] = [0, +\infty)$
(iii) $v_{nk}(x)$ are concave for all $k = 1, \ldots, K(n)$ and $[a_n, b_n] = (-\infty, 0]$,
then condition (ii) of Theorem 3.20 is fulfilled.

Proof. The statement follows easily from the properties of concave functions.

Now we shall consider some special cases where the Fenchel-Lagrange duality can be considerably simplified.

3.22. Fenchel duality. If the constraint $g(x) \in Z_+$ does not occur in program (FP), then $g(x)$ may be regarded as $g(x) = +\infty$ for all $x \in X$. The dual to program (FP), defined in Definition 3.13, is called now the *Fenchel dual.* Put

$$\mathscr{L}_1 = \{(\bar{x}^{nk}, \bar{r}^n, \bar{s}^{nk})_{\substack{n=1\ldots N\\ k=1\ldots K(n)}} : \bar{x}^{nk} \in X^*, \bar{r}^n > 0, \bar{s}^{nk} \in R, \forall n, k, \sum_{n,k} \bar{x}^{nk} = 0, \sum_{n,k} \bar{s}^{nk} = 0\}$$

and

$$L(\bar{x}^{nk}, \bar{r}^n, \bar{s}^{nk}) = (y_1, \ldots, y_N)$$

where

$$y_n = \max \left\{ a_n, \sup_{k=1\ldots K(n)} \sup_{x \in U_{nk} \cap V_{nk}} \frac{\langle \bar{x}^{nk}, x \rangle + \bar{r}^n u_{nk}(x) - \bar{s}^{nk}}{\bar{r}^n v_{nk}(x)} \right\}.$$

The Fenchel dual has then the form

$$\text{Min-Inf}_\Omega L(\mathscr{L}_1) .$$

Theorem 3.16 and Theorem 3.20, where the Slater constraint qualification is automatically fulfilled, hold for this duality.

3.23. Lagrange duality. If in program (FP) all function are assumed to be defined on all space X, then the canonical Fenchel-Lagrange dual attains the form

$$\text{Min-Inf}_\Omega L(\mathscr{L}_2)$$

where

$$\mathscr{L}_2 = \{(\bar{z}^{nk}, \bar{r}^n, \bar{s}^{nk}) : \bar{z}^{nk} \in Z_+^*, \bar{r}^n > 0, \bar{s}^{nk} \in R, \sum_{n,k} \bar{s}^{nk} = 0\}$$

and

$$L(\bar{z}^{nk}, \bar{r}^n, \bar{s}^{nk}) = (y_1, \ldots, y_N)$$

with

$$y_n = \max \left\{ a_n, \sup_{k=1\ldots K(n)} \sup_{x \in X} \frac{\bar{r}^n u_{nk}(x) + \langle \bar{z}^{nk}, g(x) \rangle - \bar{s}^{nk}}{\bar{r}^n v_{nk}(x)} \right\}.$$

This dual is called the *Lagrange dual* to program (FP). Theorem 3.16 and Theorem 3.20, where the Fenchel constraint qualification is reduced to cor $X \neq \emptyset$, also hold for this duality.

3.24. Scalar Fractional Programming

Let us consider the program

$$(\alpha) \qquad \alpha = \sup_{\Omega} \{ \sum_{k=1}^{K} u_k(x) / \sum_{k=1}^{K} v_k(x) : x \in S \cap (\bigcap_{k=1}^{K} U_k \cap V_k) \,\&\, g(x) \in Z_+ \} .$$

For program (α) we suppose the following assumptions

(i) $g(x)$ is concave on convex set S,

(ii) $u_k(x)$ are concave on U_k for all $k = 1, \ldots, K$,

(iii) $v_k(x)$ are all affine on V_k and $\Omega = (-\infty, +\infty)$ or $v_k(x)$ are all convex on V_k and $\Omega = [0, +\infty)$ and $\alpha > 0$ or $v_k(x)$ are all concave on V_k and $\Omega = (-\infty, 0]$.

3.24.1. Lemma. The Fenchel-Lagrange dual to program (α) is

$$(\beta) \qquad \beta = \inf_{\mathscr{L}_0} \sup_{k=1\ldots K} \sup_{x \in S \cap U_k \cap V_k} \frac{\langle \bar{x}^k, x \rangle + u_k(x) + \langle \bar{z}^k, g(x) \rangle - \bar{s}^k}{v_k(x)}$$

where

$$\mathscr{L}_0 = \{(\bar{x}^k, \bar{z}^k, \bar{s}^k) : \bar{x}^k \in X^*, \bar{z}^k \in Z_+^*, \bar{s}^k \in R, \ k = 1, \ldots, K, \ \sum_{k=1}^{K} \bar{x}^k = 0, \ \sum_{k=1}^{K} \bar{s}^k = 0\} .$$

Proof. According to Definition 3.13 it suffices to prove for $\Omega = [0, +\infty)$

$$y = \sup_{k=1\ldots K} \sup_{x \in S \cap U_k \cap V_k} \frac{\langle \bar{x}^k, x \rangle + u_k(x) + \langle \bar{z}^k, g(x) \rangle - \bar{s}^k}{v_k(x)} \geqq 0 \quad \forall (\bar{x}^k, \bar{z}^k, \bar{s}^k) \in \mathscr{L}_0 .$$

Indeed, let x' be a feasible solution of program (α) with $\sum_{k=1}^{K} u_k(x') > 0$, then we have

$$y \, v_k(x') \geqq \langle \bar{x}^k, x' \rangle + u_k(x') + \langle \bar{z}^k, g(x') \rangle - \bar{s}^k \quad \forall k = 1 \ldots K \Rightarrow$$

$$\Rightarrow y \sum_{k=1}^{K} v_k(x') \geqq \sum_{k=1}^{K} u_k(x') + \sum_{k=1}^{K} \langle \bar{z}^k, g(x') \rangle > 0 \Rightarrow y > 0 .$$

Summarizing Lemmas 3.18, 3.21, 3.24.1 and Theorem 3.20 we obtain

3.24.2. Theorem. (Strong Duality.) If the Fenchel-Slater constraint qualification holds then $\alpha = \beta$.

Finally we shall consider a more special case of program (α):

$$(\alpha_0) \qquad \alpha = \sup_{\Omega} \{ \sum_{k=1}^{K} u_k(x)/v(x) : x \in S \cap V \cap (\bigcap_{k=1}^{K} U_k) \,\&\, g(x) \in Z_+ \} .$$

Put $v_k(x) = v(x)/K$ and $V_k = V$ for all $k = 1, \ldots, K$. Then the Fenchel-Lagrange dual to program (α_0) is (by Lemma 3.24.1)

$$(\beta_0) \qquad \beta = \inf_{\mathscr{L}_0} \psi(\bar{x}^k, \bar{z}^k, \bar{s}^k)$$

where

$$\psi(\bar{x}^k, \bar{z}^k, \bar{s}^k) = \sup_{k=1\ldots K} \sup_{x \in S \cap V \cap Uk} K \frac{\langle \bar{x}^k, x\rangle + u_k(x) + \langle \bar{z}^k, g(x)\rangle - \bar{s}^k}{v(x)}$$

and

$$\mathscr{L}_0 = \{(\bar{x}^k, \bar{z}^k, \bar{s}^k) : \bar{x}^k \in X^*, \bar{z}^k \in Z_+^*, \bar{s}^k \in R, \sum_{k=1}^K \bar{x}^k = 0, \sum_{k=1}^K \bar{s}^k = 0\}.$$

Put

$$\varphi(\bar{x}^k, \bar{z}^k, \bar{s}^k) = \sum_{k=1}^K \sup_{x \in S \cap V \cap U_k} \frac{\langle \bar{x}^k, x\rangle + u_k(x) + \langle \bar{z}^k, g(x)\rangle - \bar{s}^k}{v(x)}.$$

Obviously

3.24.3. $$\varphi(\bar{x}^k, \bar{z}^k, \bar{s}^k) \leqq \psi(\bar{x}^k, \bar{z}^k, \bar{s}^k) \quad \forall (\bar{x}^k, \bar{z}^e, \bar{s}^k) \in \mathscr{L}_0.$$

3.24.4. Lemma. For any feasible solution x of program (α_0) and any $(\bar{x}^k, \bar{z}^k, \bar{s}^k) \in \mathscr{L}_0$ we have

$$\varphi(\bar{x}^k, \bar{z}^k, \bar{s}^k) \geqq \sum_{k=1}^K u_k(x)/v(x).$$

Proof. Indeed, one has

$$\varphi(\bar{x}^k, \bar{z}^k, \bar{s}^k) \geqq \sum_{k=1}^K \frac{\langle \bar{x}^k, x\rangle + u_k(x) + \langle \bar{z}^k, g(x)\rangle - \bar{s}^k}{v(x)} \geqq \frac{\sum_{k=1}^K u_k(x)}{v(x)}.$$

Program

$$(\beta') \qquad \beta' = \inf_{\mathscr{L}_0} \varphi(\bar{x}^k, \bar{z}^k, \bar{s}^k)$$

is called the *revised Fenchel-Lagrange dual* to program (α_0).

From Theorem 3.24.2 and the relation 3.24.3 it follows

3.24.5. Theorem. (Strong Duality.) If the Fenchel-Slater constraint qualification holds, then $\alpha = \beta'$.

3.25. Remark. In Cambini, Martein [2] a dual similar to (β') has been established for program (α_0), where $K = 2$.

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