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Jiří Anděl; María Gómez

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TWO-DIMENSIONAL LONG MEMORY MODELS

JIŘÍ ANDĚL, MARÍA GÓMEZ

Two-dimensional long memory time series models are defined and investigated. Their covariance function and matrix of spectral densities are derived and the corresponding AR(∞) and MA(∞) representations are given.

1. INTRODUCTION

Let X_t be a one-dimensional stationary time series with a covariance function $R(t)$ and a spectral density $f(\lambda)$. In some applications, especially in hydrology, $f(\lambda)$ exhibits a high peak at $\lambda = 0$. To describe properly this phenomenon, some models were proposed in which $f(\lambda) \to \infty$ as $\lambda \to 0$. One of the simplest models of this kind is

$$(1 - \mathsf{B})^{\delta} X_t = \varepsilon_t \,, \tag{1.1}$$

where ε_t is a white noise, B is the back-shift operator and $\delta \in (0, \frac{1}{2})$. This process X_t is called the fractionally differenced white noise (FDWN). If $\mathsf{E}\varepsilon_t = 0$ and $\operatorname{var} \varepsilon_t = \sigma^2$, then the spectral density of X_t is

$$f(\lambda) = (2\pi)^{-1} \sigma^2 [4 \sin^2(\lambda/2)]^{-\delta}$$

and the covariance function is

$$R(t) = (-1)^t \sigma^2 \Gamma(1 - 2\delta)/[\Gamma(t + 1 - \delta) \Gamma(-t + 1 - \delta)] \,.$$

Since $\sum |R(t)| = \infty$, X_t is called a process with long memory. This definition was proposed by McLeod and Hipel [7]. On the other hand, it is known that the covariance function $R(t)$ of any stationary ARMA process satisfies $\sum |R(t)| < \infty$ and thus stationary ARMA processes are processes with short memory.

An introduction to long memory time series models is published by Granger and Joyeux [5]. Hosking [6] derives formulas for the spectral density and the covariance function. Geweke and Porter-Hudak [4] deal also with the problem of estimating parameters in long memory models. Anděl [1] presents a survey of results and methods which concern the long memory models. In his paper also other relevant references to this subject can be found.

The model (1.1) can be generalized to p dimensions as follows. Let ε_t be a p-dimensional white noise with $\mathsf{E}\varepsilon_t = \mathbf{0}$, $\operatorname{var} \varepsilon_t = \boldsymbol{V}$. Let $\boldsymbol{A}$ be a $p \times p$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_p$. Assume that $|\lambda_j| \leqq 1$ for all j and that the equality holds for at least one j. Then a p-dimensional process $\mathbf{X}_t$ satisfying

$$(1.2) \qquad (I - \boldsymbol{A}\mathsf{B})^{\delta} \mathbf{X}_t = \varepsilon_t\,, \quad 0 < \delta < \tfrac{1}{2}\,,$$

can be considered as a p-dimensional generalization of the FDWN. However, a deeper analysis of the model is necessary because the conditions mentioned above do not guarantee the existence of the process $\mathbf{X}_t$ in (1.2). This will be shown in Sections 3 and 4.

Generally, a p-dimensional stationary process $\mathbf{X}_t$ with a covariance function $\mathbf{R}(t) = (R_{jk}(t))_{j,k=1}^{p}$ will be called a long memory process, if

$$\sum_t \sum_{j=1}^{p} \sum_{k=1}^{p} |R_{jk}(t)| = \infty\,.$$

If the sum is finite, $\mathbf{X}_t$ will be called a process with short memory.

In this paper we investigate the model (1.2) when $p = 2$. The eigenvalues λ_1, λ_2 of the matrix

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

are the roots of the equation

$$(1.3) \qquad \lambda^2 - (a_{11} + a_{22})\,\lambda + a_{11}a_{22} - a_{12}a_{21} = 0\,.$$

The following cases will be considered:

a. $\lambda_1 = 1$, $\lambda_2 \in [-1, 1)$.
b. $\lambda_1 = \lambda_2 = 1$.
c. $\lambda_1 = \lambda_2 = -1$.
d. $\lambda_1 = \mathrm{e}^{\mathrm{i}\omega}$, $\lambda_2 = \mathrm{e}^{-\mathrm{i}\omega}$, $\omega \in (0, \pi)$.

In the paper we do not study the problem of estimating parameters. Since the elements of the matrix of spectral densities are not bounded, the procedure proposed by Dunsmuir and Hannan [2] is not applicable. This point would need a special research.

2. CASE $\lambda_1 = 1$, $\lambda_2 \in [-1, 1)$

In this section we denote $w = \lambda_2$.

Lemma 2.1. Let $z \neq 1$. If $w \neq 0$ then let also $z \neq 1/w$. Then for any real n

$$(\boldsymbol{I} - z\boldsymbol{A})^n = (1 - w)^{-1}\left[(1 - z)^n (\boldsymbol{A} - w\boldsymbol{I}) + (1 - wz)^n (\boldsymbol{I} - \boldsymbol{A})\right].$$

Proof. If $z = 0$, then the assertion clearly holds. Assume that $z \neq 0$. Let μ_1, μ_2 be the eigenvalues of the matrix $\mathbf{I} - z\mathbf{A}$. Since

$$\left|\mathbf{I} - z\mathbf{A} - \mu\mathbf{I}\right| = (-z)^2 \left|\mathbf{A} - [(1 - \mu)/z]\,\mathbf{I}\right|$$

and the eigenvalues of $\mathbf{A}$ are 1 and w, we have $(1 - \mu_1)/z = 1$, $(1 - \mu_2)/z = w$. From here $\mu_1 = 1 - z$, $\mu_2 = 1 - wz$. Since $\mu_1 \neq \mu_2$, it holds

$$(\mathbf{I} - z\mathbf{A})^n = \mu_1^n \mathbf{Z}_1 + \mu_2^n \mathbf{Z}_2 \,, \tag{2.1}$$

where $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are the components of the matrix $\mathbf{I} - z\mathbf{A}$. Choosing $n = 0$ and $n = 1$ we get the equations

$$\mathbf{Z}_1 + \mathbf{Z}_2 = \mathbf{I} \,,$$

$$(1 - z)\,\mathbf{Z}_1 + (1 - wz)\,\mathbf{Z}_2 = \mathbf{I} - z\mathbf{A} \,.$$

Thus

$$\mathbf{Z}_1 = (1 - w)^{-1}(\mathbf{A} - w\mathbf{I}) \,, \quad \mathbf{Z}_2 = (1 - w)^{-1}(\mathbf{I} - \mathbf{A})$$

and the assertion follows from (2.1). □

Lemma 2.2. Let $|z| < 1$ and $\delta \in (0, \frac{1}{2})$. Then

$$(\mathbf{I} - z\mathbf{A})^{-\delta} = \sum_{j=0}^{\infty} b_j \mathbf{C}_j z^j \,, \quad (\mathbf{I} - z\mathbf{A})^{\delta} = \sum_{j=0}^{\infty} a_j \mathbf{C}_j z^j \,,$$

where

$$\mathbf{C}_j = (1 - w)^{-1}\left[\mathbf{A} - w\mathbf{I} + (\mathbf{I} - \mathbf{A})\,w^j\right], \quad j = 0, 1, \ldots$$

and the coefficients a_j, b_j are defined in Lemma 6.1.

Proof. The assertion follows from Theorem 2.1 and Lemma 6.1. □

Theorem 2.3. The MA(∞) representation of the process $\mathbf{X}_t$ is

$$\mathbf{X}_t = \sum_{j=0}^{\infty} b_j \mathbf{C}_j \varepsilon_{t-j} \tag{2.2}$$

and the AR(∞) representation of $\mathbf{X}_t$ is

$$\sum_{j=0}^{\infty} a_j \mathbf{C}_j \mathbf{X}_{t-j} = \varepsilon_t \,. \tag{2.3}$$

Proof. From (1.2) we have

$$\mathbf{X}_t = (\mathbf{I} - \mathbf{A}\mathrm{B})^{-\delta}\,\varepsilon_t = \sum_{j=0}^{\infty} b_j \mathbf{C}_j \varepsilon_{t-j} \,.$$

It can be shown that there exists a constant $M > 0$ such that $\mathrm{Tr}\ \mathbf{C}_j' \mathbf{C}_j \mathbf{V} < M$ for all j (the symbol $'$ denotes the transposition). Since $\sum b_j^2 < \infty$, it follows from Lemma 6.2 that the series $\sum b_j \mathbf{C}_j \varepsilon_{t-j}$ converges in the quadratic mean.

Formula (2.3) follows immediately from (1.2). The convergence in the quadratic mean of the series (2.3) is ensured by Lemma 6.3. □

Theorem 2.4. The process $\boldsymbol{X}_t$ possesses the matrix of spectral densities

$$\boldsymbol{f}(\lambda) = (2\pi)^{-1} (1 - w)^{-2} \left[|1 - e^{-i\lambda}|^{-2\delta} \boldsymbol{G}_1 + |1 - w\, e^{-i\lambda}|^{-2\delta} \boldsymbol{G}_2 + \right. \tag{2.4}$$

$$\left. + (1 - e^{-i\lambda})^{-\delta} (1 - w\, e^{i\lambda})^{-\delta} \boldsymbol{G}_3 + (1 - w\, e^{-i\lambda})^{-\delta} (1 - e^{i\lambda})^{-\delta} \boldsymbol{G}_3' \right],$$

where

$$\boldsymbol{G}_1 = (\boldsymbol{A} - w\boldsymbol{I})\, \boldsymbol{V}(\boldsymbol{A}' - w\boldsymbol{I}), \quad \boldsymbol{G}_2 = (\boldsymbol{I} - \boldsymbol{A})\, \boldsymbol{V}(\boldsymbol{I} - \boldsymbol{A}'),$$

$$\boldsymbol{G}_3 = (\boldsymbol{A} - w\boldsymbol{I})\, \boldsymbol{V}(\boldsymbol{I} - \boldsymbol{A}').$$

Proof. The white noise ε_t has the spectral decomposition

$$\varepsilon_t = \int_{-\pi}^{\pi} e^{it\lambda}\, dZ(\lambda),$$

where $Z(\cdot)$ is a random measure satisfying

$$\mathsf{E}\, dZ(\lambda)\, dZ^*(\lambda) = (2\pi)^{-1}\, \boldsymbol{V}\, d\lambda$$

($*$ denotes transposition and complex conjugation). From (2.2) we have

$$\boldsymbol{X}_t = \underset{N\to\infty}{\text{l.i.m.}} \sum_{j=0}^{N} b_j \boldsymbol{C}_j \varepsilon_{t-j} = \underset{N\to\infty}{\text{l.i.m.}} \int_{-\pi}^{\pi} \sum_{j=0}^{N} b_j \boldsymbol{C}_j\, e^{i(t-j)\lambda}\, dZ(\lambda).$$

Since $\sum b_j \boldsymbol{C}_j \varepsilon_{t-j}$ converges in the quadratic mean, the series $\sum b_j \boldsymbol{C}_j\, e^{i(t-j)\lambda}$ converges in the quadratic mean with respect to $(2\pi)^{-1}\, \boldsymbol{V}$. If $\lambda \neq 0$, $\lambda \neq \pm\pi$, then Lemmas 6.5, 6.6 and 6.8 ensure that

$$\lim_{N\to\infty} \sum_{j=0}^{N} b_j \boldsymbol{C}_j\, e^{-ij\lambda} = (\boldsymbol{I} - e^{-i\lambda}\boldsymbol{A})^{-\delta}.$$

From Lemma 6.7 we get

$$\boldsymbol{X}_t = \int_{-\pi}^{\pi} e^{it\lambda} (\boldsymbol{I} - e^{-i\lambda}\boldsymbol{A})^{-\delta}\, dZ(\lambda).$$

The covariance function of the process $\boldsymbol{X}_t$ is

$$\boldsymbol{R}(t) = \mathsf{E}\boldsymbol{X}_t\boldsymbol{X}_0^* = (2\pi)^{-1} \int_{-\pi}^{\pi} e^{it\lambda} (\boldsymbol{I} - e^{-i\lambda}\boldsymbol{A})^{-\delta}\, \boldsymbol{V}[(\boldsymbol{I} - e^{-i\lambda}\boldsymbol{A})^{-\delta}]^*\, d\lambda.$$

It implies that

$$\boldsymbol{f}(\lambda) = (2\pi)^{-1} (\boldsymbol{I} - e^{-i\lambda}\boldsymbol{A})^{-\delta}\, \boldsymbol{V}[(\boldsymbol{I} - e^{-i\lambda}\boldsymbol{A})^{-\delta}]^*.$$

Using Lemma 2.1 we get the assertion. □

Consider the covariance function $\boldsymbol{R}(t)$. Since $\boldsymbol{R}(-t) = \boldsymbol{R}^*(t)$, it suffices to assume $t \geqq 0$.

Theorem 2.5. If $t \geqq 0$, then

$$\boldsymbol{R}(t) = (1 - w)^{-2} \left[\left(\sum_{k=0}^{\infty} b_k b_{t+k}\right) \boldsymbol{G}_1 + \left(w^t \sum_{k=0}^{\infty} b_k b_{t+k} w^{2k}\right) \boldsymbol{G}_2 + \right.$$

$$\left. + \left(\sum_{k=0}^{\infty} b_k b_{t+k} w^k\right) \boldsymbol{G}_3 + \left(w^t \sum_{k=0}^{\infty} b_k b_{t+k} w^k\right) \boldsymbol{G}_3' \right].$$

Proof. The covariance function $\boldsymbol{R}(t)$ can be calculated from

$$\boldsymbol{R}(t) = \int_{-\pi}^{\pi} e^{it\lambda}\, \boldsymbol{f}(\lambda)\, d\lambda.$$

Theorem 2.4 yields

$$\boldsymbol{R}(t) = (2\pi)^{-1}(1 - w)^{-2}(J_1\boldsymbol{G}_1 + J_2\boldsymbol{G}_2 + J_3\boldsymbol{G}_3 + J_4\boldsymbol{G}_3'),$$

where

$$J_1 = \int_{-\pi}^{\pi} e^{it\lambda}|1 - e^{-i\lambda}|^{-2\delta}\, d\lambda, \quad J_2 = \int_{-\pi}^{\pi} e^{it\lambda}|1 - w\, e^{-i\lambda}|^{-2\delta}\, d\lambda,$$
$$J_3 = \int_{-\pi}^{\pi} e^{it\lambda}(1 - e^{-i\lambda})^{-\delta}(1 - w\, e^{i\lambda})^{-\delta}\, d\lambda,$$
$$J_4 = \int_{-\pi}^{\pi} e^{it\lambda}(1 - w\, e^{-i\lambda})^{-\delta}(1 - e^{i\lambda})^{-\delta}\, d\lambda.$$

For example, consider J_2. From Lemmas 6.1, 6.5, 6.6 and 6.8 (see the Appendix) we have

$$(2.5) \qquad (1 - w\, e^{-i\lambda})^{-\delta} = \sum_{j=0}^{\infty} b_j w^j e^{-ij\lambda},$$

$$(2.6) \qquad (1 - w\, e^{i\lambda})^{-\delta} = \sum_{k=0}^{\infty} b_k w^k e^{ik\lambda}.$$

The series converge for all $\lambda \in (-\pi, \pi)$. Since the functions $(1 - w\, e^{-i\lambda})^{-\delta}$ and $(1 - w\, e^{i\lambda})^{-\delta}$ are absolutely integrable, the series (2.5) and (2.6) are their Fourier series (see Lemma 6.9). Using Lemma 6.11 we obtain the Fourier series of the function

$$(1 - w\, e^{-i\lambda})^{-\delta}(1 - w\, e^{i\lambda})^{-\delta} = |1 - w\, e^{-i\lambda}|^{-2\delta}$$

in the form

$$\sum_{j=0}^{\infty} b_j w^j e^{-ij\lambda} \sum_{k=0}^{\infty} b_k w^k e^{ik\lambda} = \sum_{s=-\infty}^{\infty} \sum_{u=\max(0,-s)}^{\infty} b_u b_{s+u} w^{s+2u} e^{is\lambda}.$$

According to Lemma 6.10 we have for $t \geqq 0$

$$\int_{-\pi}^{\pi} e^{it\lambda}|1 - w\, e^{-i\lambda}|^{-2\delta}\, d\lambda = 2\pi w^t \sum_{k=0}^{\infty} b_k b_{t+k} w^{2k}.$$

All the other integrals J_1, J_3 and J_4 can be calculated in the same way. □

The authors know an explicit formula only for J_1. Using the method described by Anděl [1], pp. 106–107 one gets

$$J_1 = 2\int_0^{\pi} \cos t\lambda\, (4\sin^2(\tfrac{1}{2}\lambda))^{-\delta}\, d\lambda = 4\int_0^{\pi/2} \cos 2tx\, (4\sin^2 x)^{-\delta}\, dx =$$
$$= (-1)^t\, 2\pi\, \Gamma(1 - 2\delta)/[\Gamma(t + 1 - \delta)\Gamma(-t + 1 - \delta)].$$

The values of J_2, J_3 and J_4 must be calculated either by numerical integration or by help of the series given in Theorem 2.5. If w is not near to ± 1, then the convergence of the series is rather fast.

Theorem 2.6. If $\boldsymbol{AV} \neq w\boldsymbol{V}$, then $\boldsymbol{X}_t$ is a long memory process.

Proof. The spectral density $\boldsymbol{f}(\lambda)$ from Theorem 2.4 can be also written in the form

$$\boldsymbol{f}(\lambda) = (2\pi)^{-1}(1 - w)^{-2}|1 - e^{-i\lambda}|^{-2\delta}[\boldsymbol{G}_1 + |1 - e^{-i\lambda}|^{2\delta}|1 - w\, e^{-i\lambda}|^{-2\delta}\boldsymbol{G}_2 +$$
$$+ (1 - e^{i\lambda})^{\delta}(1 - w\, e^{i\lambda})^{-\delta}\boldsymbol{G}_3 + (1 - e^{-i\lambda})^{\delta}(1 - w\, e^{-i\lambda})^{-\delta}\boldsymbol{G}_3'].$$

It is clear that $\boldsymbol{f}(\lambda)$ is not bounded in the neighbourhood of the origin, if $\boldsymbol{G}_1 \neq \boldsymbol{0}$. It means that $\boldsymbol{G}_1 \neq \boldsymbol{0}$ implies $\sum_t \sum_j \sum_k |R_{jk}(t)| = \infty$ otherwise $\boldsymbol{f}(\lambda)$ would be continuous and therefore bounded on $[-\pi, \pi]$. But $\boldsymbol{G}_1 = \boldsymbol{0}$ if and only if $(\boldsymbol{A} - w\boldsymbol{I})\, \boldsymbol{V} = \boldsymbol{0}$. □

Example. Let

$$\boldsymbol{A} = \frac{1}{12}\begin{pmatrix} 11 & 5 \\ 1 & 7 \end{pmatrix}, \quad \boldsymbol{V} = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}.$$

The matrix $\boldsymbol{A}$ has eigenvalues 1 and 0·5. Further,

$$\boldsymbol{Z}_1 = \frac{1}{6}\begin{pmatrix} 5 & 5 \\ 1 & 1 \end{pmatrix}, \quad \boldsymbol{Z}_2 = \frac{1}{6}\begin{pmatrix} 1 & -5 \\ -1 & 5 \end{pmatrix},$$

$$\boldsymbol{G}_1 = \frac{1}{48}\begin{pmatrix} 25 & 5 \\ 5 & 1 \end{pmatrix}, \quad \boldsymbol{G}_2 = \frac{29}{48}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \boldsymbol{G}_3 = \frac{1}{48}\begin{pmatrix} -15 & 15 \\ -3 & 3 \end{pmatrix}.$$

The process $\boldsymbol{X}_t = (\boldsymbol{I} - \boldsymbol{A}\mathrm{B})^{-0.4}\, \varepsilon_t$ has long memory. Its spectral density is given by the formula (2.4). The spectral densities $f_{11}(\lambda), f_{22}(\lambda)$, coherence and phase diagrams are plotted in Fig. 1–4, respectively. From this example we can see that the model could be suitable for such two-dimensional time series, where the spectral

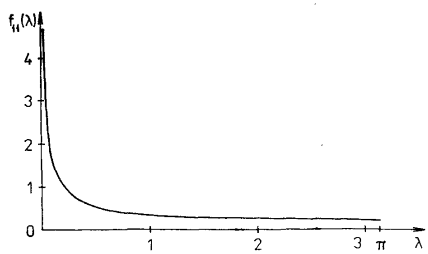

Fig. 1. Spectral density $f_{11}(\lambda)$.

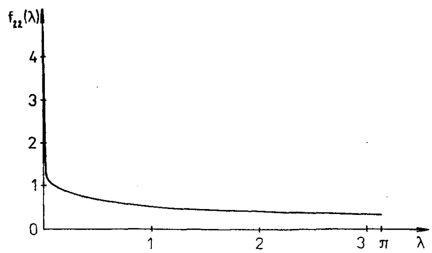

Fig. 2. Spectral density $f_{22}(\lambda)$.

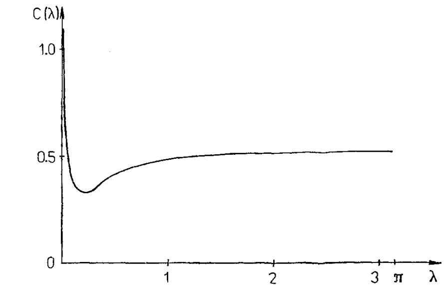

Fig. 3. Coherence diagram.

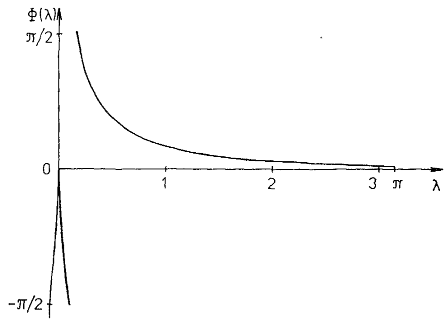

Fig. 4. Phase diagram.

densities $f_{11}(\lambda)$ and $f_{22}(\lambda)$ are not bounded in the neighbourhood of $\lambda = 0$. Such situations occur in hydrology, for example (see [4]).

3. CASE $\lambda_1 = \lambda_2 = 1$

Lemma 3.1. Let $z \neq 1$. Then for any real n

$$(\mathbf{I} - z\mathbf{A})^n = (1 - z)^n \mathbf{I} + n(1 - z)^{n-1} z(\mathbf{I} - \mathbf{A}) .$$

Proof. The matrix $\mathbf{I} - z\mathbf{A}$ has eigenvalues $\mu_1 = \mu_2 = 1 - z$. Thus according to the Perron formula

$$(\mathbf{I} - z\mathbf{A})^n = (1 - z)^n \mathbf{Z}_1 + n(1 - z)^{n-1} \mathbf{Z}_2 ,$$

where $\mathbf{Z}_1 = \mathbf{I}$, $\mathbf{Z}_2 = z(\mathbf{I} - \mathbf{A})$. □

Lemma 3.2. If $|z| < 1$ and $\delta \in (0, \frac{1}{2})$, then

$$(I - zA)^{-\delta} = \sum_{j=0}^{\infty} [I - j(I - A)]\, b_j z^j .$$

Proof. The assertion follows from Lemma 3.1, since

$$(1 - z)^{-\delta} = \sum b_j z^j , \quad \delta(1 - z)^{-\delta-1} = \sum j b_j z^{j-1} . \qquad \square$$

Theorem 3.3. The process $X_t = (I - AB)^{-\delta} \varepsilon_t$ exists if and only if

$$\text{(3.1)} \qquad \operatorname{Tr}(2I - A - A')\, V = 0 , \quad \operatorname{Tr}(I - A')(I - A)\, V = 0 .$$

If the conditions (3.1) are fulfilled, then

$$\text{(3.2)} \qquad X_t = \sum_{j=0}^{\infty} [I - j(I - A)]\, b_j \varepsilon_{t-j} .$$

Proof. In view of Lemma 3.2 the process X_t must have form (3.2). But according to Lemma 6.2 the series (3.2) converges in the quadratic mean if and only if

$$\text{(3.3)} \qquad \sum_{j=0}^{\infty} b_j^2 \operatorname{Tr} [I - j(I - A')]\, [I - j(I - A)]\, V < \infty .$$

Since

$$b_j^2 \sim [1/\Gamma(\delta)]\, j^{2\delta-2} ,$$

the series (3.3) converges if and only if the coefficients by j and by j^2 are zeros. $\square$

The general form of the matrix A is given in Lemma 6.12. Thus it is possible to investigate in which cases (3.1) holds. Because it is clear that the existence of X_t is ensured only in very special cases, we do not describe further details here.

4. CASE $\lambda_1 = \lambda_2 = -1$

All the derivations are quite analogous to those given in Section 3 and so we present only the results.

Lemma 4.1. Let $z \neq -1$. Then for any real n

$$(I - zA)^n = (1 + z)^n I - n(1 + z)^{n-1} z(I + A) .$$

Lemma 4.2. If $|z| < 1$ and $\delta \in (0, \frac{1}{2})$, then

$$(I - zA)^{-\delta} = \sum_{j=0}^{\infty} (-1)^j [I - j(I + A)]\, b_j z^j .$$

Theorem 4.3. The process $X_t = (I - AB)^{-\delta} \varepsilon_t$ exists if and only if

$$\operatorname{Tr}(2I + A + A')\, V = 0 , \quad \operatorname{Tr}(I + A')(I + A)\, V = 0 .$$

If these conditions are fulfilled, then

$$X_t = \sum_{j=0}^{\infty} (-1)^j [I - j(I + A)]\, b_j \varepsilon_{t-j} .$$

Let us remark that the general form of the matrix $\boldsymbol{A}$ with eigenvalues $\lambda_1 = \lambda_2 = -1$ is given in Lemma 6.13.

5. CASE $\lambda_1 = e^{i\omega}$, $\lambda_2 = e^{-i\omega}$, $\omega \in (0, \pi)$

This case is analogous to that investigated in Section 2 and so we do not introduce all the details in the proofs.

Lemma 5.1. If $z \neq e^{i\omega}$, $z \neq e^{-i\omega}$, then for any real n

$$(\boldsymbol{I} - z\boldsymbol{A})^n = i(2\sin\omega)^{-1}\left[(1 - z e^{i\omega})^n (e^{-i\omega}\boldsymbol{I} - \boldsymbol{A}) + (1 - z e^{-i\omega})^n (\boldsymbol{A} - e^{i\omega}\boldsymbol{I})\right].$$

Proof. The matrix $\boldsymbol{I} - z\boldsymbol{A}$ has eigenvalues $1 - z e^{i\omega}$ and $1 - z e^{-i\omega}$. The components of $\boldsymbol{I} - z\boldsymbol{A}$ are

$$\boldsymbol{Z}_1 = i(2\sin\omega)^{-1}(e^{-i\omega}\boldsymbol{I} - \boldsymbol{A}), \quad \boldsymbol{Z}_2 = i(2\sin\omega)^{-1}(\boldsymbol{A} - e^{i\omega}\boldsymbol{I}). \qquad \square$$

Lemma 5.2. Let $|z| < 1$, $\delta \in (0, \frac{1}{2})$. Then

$$(\boldsymbol{I} - z\boldsymbol{A})^{-\delta} = (\sin\omega)^{-1}\sum_{j=0}^{\infty}\left[\sin j\omega\boldsymbol{A} - \sin(j-1)\,\omega\boldsymbol{I}\right] b_j z^j.$$

Proof. From Lemma 5.1 and from Lemma 6.1 we have

$$(\boldsymbol{I} - z\boldsymbol{A})^{-\delta} = i(2\sin\omega)^{-1}\sum_{j=0}^{\infty} b_j \boldsymbol{C}_j z^j,$$

where

$$\boldsymbol{C}_j = e^{ij\omega}(e^{-i\omega}\boldsymbol{I} - \boldsymbol{A}) + e^{-ij\omega}(\boldsymbol{A} - e^{i\omega}\boldsymbol{I}).$$

Rearranging the terms we obtain the assertion. $\square$

Theorem 5.3. The MA(∞) representation of the process $\boldsymbol{X}_t$ is

$$\boldsymbol{X}_t = (\sin\omega)^{-1}\sum_{j=0}^{\infty}\left[\sin j\omega\boldsymbol{A} - \sin(j-1)\,\omega\boldsymbol{I}\right] b_j \varepsilon_{t-j}.$$

The AR(∞) representation of $\boldsymbol{X}_t$ is

$$(\sin\omega)^{-1}\sum_{j=0}^{\infty}\left[\sin j\omega\boldsymbol{A} - \sin(j-1)\,\omega\boldsymbol{I}\right] a_j \boldsymbol{X}_{t-j} = \varepsilon_t.$$

The proof is analogous to that of Theorem 2.3. $\square$

Theorem 5.4. The process $\boldsymbol{X}_t$ possesses the matrix of spectral densities

$$\boldsymbol{f}(\lambda) = (8\pi\sin^2\omega)^{-1}\{[2 - 2\cos(\lambda - \omega)]^{-\delta}\boldsymbol{G}_1 + [2 - 2\cos(\lambda + \omega)]^{-\delta}\boldsymbol{G}_2 + [1 - 2e^{i\omega}\cos\lambda + e^{2i\omega}]^{-\delta}\boldsymbol{G}_3 + [1 - 2e^{-i\omega}\cos\lambda + e^{-2i\omega}]^{-\delta}\boldsymbol{G}_3^*\},$$

where

$$\boldsymbol{G}_1 = (e^{-i\omega}\boldsymbol{I} - \boldsymbol{A})\boldsymbol{V}(e^{i\omega}\boldsymbol{I} - \boldsymbol{A}'), \quad \boldsymbol{G}_2 = (\boldsymbol{A} - e^{i\omega}\boldsymbol{I})\boldsymbol{V}(\boldsymbol{A}' - e^{-i\omega}\boldsymbol{I}),$$

$$\boldsymbol{G}_3 = (e^{-i\omega}\boldsymbol{I} - \boldsymbol{A})\boldsymbol{V}(\boldsymbol{A}' - e^{-i\omega}\boldsymbol{I}).$$

Proof. In the same way as in the proof of Theorem 2.4 we obtain

$$f(\lambda) = (8\pi \sin^2 \omega)^{-1} \left[(1 - e^{i(\omega-\lambda)})^{-\delta} (1 - e^{i(\lambda-\omega)})^{-\delta} \boldsymbol{G}_1 + \right.$$
$$+ (1 - e^{-i(\lambda+\omega)})^{-\delta} (1 - e^{i(\lambda+\omega)})^{-\delta} \boldsymbol{G}_2 +$$
$$+ (1 - e^{i(\omega-\lambda)})^{-\delta} (1 - e^{i(\lambda+\omega)})^{-\delta} \boldsymbol{G}_3 +$$
$$\left. + (1 - e^{-i(\lambda+\omega)})^{-\delta} (1 - e^{i(\lambda-\omega)})^{-\delta} \boldsymbol{G}_3^*\right].$$

After a simple calculation we get the formula introduced in Theorem 5.4. □

Let us remark that applying the procedure introduced in the proof of Theorem 2.4 we can derive an alternative expression

$$f(\lambda) = (2\pi \sin^2 \omega)^{-1} \sum_{s=-\infty}^{\infty} \sum_{k=\max(0,-s)}^{\infty} b_k b_{k+s} \times$$
$$\times \left[\sin (k + s)\, \omega \boldsymbol{A} - \sin (k + s - 1)\, \omega \boldsymbol{I}\right] \boldsymbol{V}\left[\sin k\omega \boldsymbol{A}' - \sin (k - 1)\, \omega \boldsymbol{I}\right] e^{-is\lambda}.$$

Theorem 5.5. If $t \geqq 0$ then the covariance function is

$$\boldsymbol{R}(t) = (4 \sin^2 \omega)^{-1} (-1)^t \left[e^{it\omega} \sum_{k=0}^{\infty} (-1)^k b_k b_{t+k} \boldsymbol{G}_1 + \right.$$
$$+ e^{-it\omega} \sum_{k=0}^{\infty} (-1)^k b_k b_{t+k} \boldsymbol{G}_2 + e^{it\omega} \sum_{k=0}^{\infty} (-1)^k b_k b_{t+k} e^{2ik\omega} \boldsymbol{G}_3 +$$
$$\left. + e^{-it\omega} \sum_{k=0}^{\infty} (-1)^k b_k b_{t+k} e^{-2ik\omega} \boldsymbol{G}_3^*\right].$$

Proof. The formula can be proved in the same way as Theorem 2.5. □

Theorem 5.6. If $\boldsymbol{G}_1 \neq \boldsymbol{0}$, then $\boldsymbol{X}_t$ is a long memory process.

The proof is analogous to that of Theorem 2.6. □

Example. Let $\boldsymbol{V} = \boldsymbol{I}$ and

$$\boldsymbol{A} = \begin{pmatrix} \frac{1}{2} & 1 \\ -\frac{3}{4} & \frac{1}{2} \end{pmatrix}.$$

The eigenvalues of $\boldsymbol{A}$ are $\lambda_1 = e^{i\omega}$, $\lambda_2 = e^{-i\omega}$, where $\omega = \pi/3$. After some computations we get

$$\boldsymbol{G}_1 = \begin{pmatrix} \frac{7}{4} - \cos \omega & \frac{1}{8} - \frac{1}{4} \cos \omega - i \frac{7}{4} \sin \omega \\ \frac{1}{8} - \frac{1}{4} \cos \omega + i \frac{7}{4} \sin \omega & \frac{29}{16} - \cos \omega \end{pmatrix},$$

$$\boldsymbol{G}_2 = \boldsymbol{G}_1',$$

$$\boldsymbol{G}_3 = \begin{pmatrix} -\frac{3}{4} + \cos \omega - \cos 2\omega + i(\sin 2\omega - \sin \omega) & -\frac{1}{8} + \frac{1}{4} \cos \omega - i \frac{1}{4} \sin \omega \\ -\frac{1}{8} + \frac{1}{4} \cos \omega - i \frac{1}{4} \sin \omega & -\frac{13}{16} + \cos \omega - \cos 2\omega + i(\sin 2\omega - \sin \omega) \end{pmatrix}.$$

The spectral densities $f_{11}(\lambda)$, $f_{22}(\lambda)$, coherence and phase diagrams for $\delta = 0{\cdot}4$

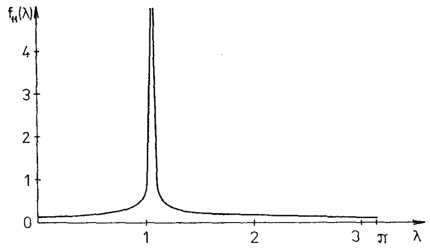

Fig. 5. Spectral density $f_{11}(\lambda)$.

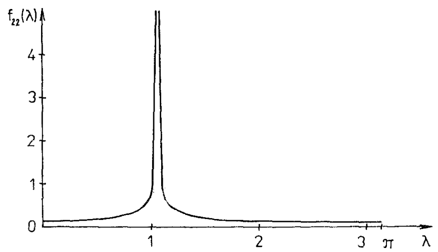

Fig. 6. Spectral density $f_{22}(\lambda)$.

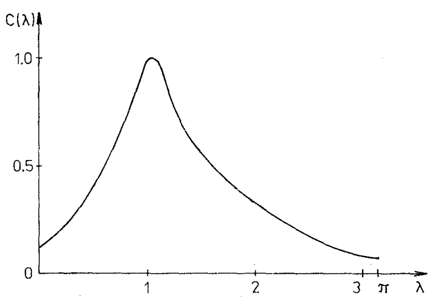

Fig. 7. Coherence diagram.

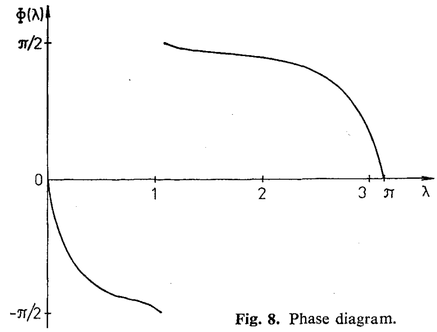

Fig. 8. Phase diagram.

are plotted in Figs. 5 – 8, respectively. The spectral densities $f_{11}(\lambda)$ and $f_{22}(\lambda)$ are very similar to the spectral density of a one-dimensional seasonal persistent process

$$[(1 - e^{i\omega} B)(1 - e^{-i\omega} B)]^{\delta} \boldsymbol{X}_t = \varepsilon_t$$

(cf. Anděl [1], p. 114). It seems that the models of this kind are suitable for describing the time series which exhibit some seasonal behaviour corresponding to the frequency ω.

6. APPENDIX

Lemma 6.1. Let $\delta \in (-\frac{1}{2}, \frac{1}{2})$ and $|z| < 1$. Then

$$(1 - z)^{\delta} = \sum_{j=0}^{\infty} a_j z^j , \quad (1 - z)^{-\delta} = \sum_{j=0}^{\infty} b_j z^j ,$$

where

$$a_j = \Gamma(j - \delta)/[\Gamma(-\delta)\Gamma(j + 1)] = (j - 1 - \delta)(j - 2 - \delta)\dots(1 - \delta)(-\delta)/j! ,$$

$$b_j = \Gamma(j + \delta)/[\Gamma(\delta)\Gamma(j + 1)] = (j - 1 + \delta)(j - 2 + \delta)\dots(1 + \delta)\delta/j! .$$

If $j \to \infty$, then

$$j^{1+\delta} a_j \to 1/\Gamma(-\delta) , \quad j^{1-\delta} b_j \to 1/\Gamma(\delta) .$$

Proof. The assertions follow from the Maclaurin formula and from the Stirling formula. □

Lemma 6.2. Let $\boldsymbol{\varepsilon}_t$ be uncorrelated p-dimensional random vectors such that $\mathsf{E}\, \boldsymbol{\varepsilon}_t = \mathbf{0}$, $\operatorname{var} \boldsymbol{\varepsilon}_t = \boldsymbol{V}$. Let $\boldsymbol{D}_j$ be $p \times p$ matrices. Then the series $\sum_{j=0}^{\infty} \boldsymbol{D}_j \boldsymbol{\varepsilon}_{t-j}$ converges in the quadratic mean if and only if

$$\sum_{j=0}^{\infty} \operatorname{Tr} \boldsymbol{D}_j' \boldsymbol{D}_j \boldsymbol{V} < \infty . \tag{6.1}$$

Proof. Let

$$\mathbf{S}_N = \sum_{j=0}^{N} \mathbf{D}_j \varepsilon_{t-j}$$

and let $m \geqq 1$. Then

$$\mathrm{E}(\mathbf{S}_{N+m} - \mathbf{S}_N)' (\mathbf{S}_{N+m} - \mathbf{S}_N) = \sum_{j=N+1}^{N+m} \sum_{k=N+1}^{N+m} \mathrm{E}\varepsilon'_{t-j} \mathbf{D}'_j \mathbf{D}_k \varepsilon_{t-k} =$$

$$= \sum_{j=N+1}^{N+m} \sum_{k=N+1}^{N+m} \mathrm{Tr}\, \mathbf{D}'_j \mathbf{D}_k\, \mathrm{E}\varepsilon_{t-k} \varepsilon'_{t-j} = \sum_{j=N+1}^{N+m} \mathrm{Tr}\, \mathbf{D}'_j \mathbf{D}_j \mathbf{V}.$$

From here it is clear that $\mathbf{S}_N$ has a limit in the quadratic mean if and only if (6.1) holds. □

Lemma 6.3. Let $\mathbf{X}_t$ be a p-dimensional stationary process with $\mathrm{E}\mathbf{X}_t = \mathbf{0}$, var $\mathbf{D}_t = \mathbf{W}$. Let $\mathbf{F}_j$ be $p \times p$ matrices. If

$$\sum_{j=0}^{\infty} (\mathrm{Tr}\, \mathbf{F}'_j \mathbf{F}_j \mathbf{W})^{1/2} < \infty, \tag{6.2}$$

then the series

$$\sum_{j=0}^{\infty} \mathbf{F}_j \mathbf{X}_{t-j}$$

converges in the quadratic mean.

Proof. Let

$$\mathbf{S}_N = \sum_{j=0}^{N} \mathbf{F}_j \mathbf{X}_{t-j}$$

and let $m \geqq 1$. Then

$$\mathrm{E}(\mathbf{S}_{N+m} - \mathbf{S}_N)' (\mathbf{S}_{N+m} - \mathbf{S}_N) = \sum_{j=N+1}^{N+m} \sum_{k=N+1}^{N+m} \mathrm{E}\mathbf{X}'_{t-j} \mathbf{F}'_j \mathbf{F}_k \mathbf{X}_{t-k}.$$

Define

$$\mathbf{Y} = (Y_1, \ldots, Y_p)' = \mathbf{F}_j \mathbf{X}_{t-j}, \quad \mathbf{Z} = (Z_1, \ldots, Z_p)' = \mathbf{F}_k \mathbf{X}_{t-k}.$$

Then

$$|\mathrm{E}\mathbf{Y}'\mathbf{Z}| \leqq \sum_{i=1}^{p} \mathrm{E}|Y_i Z_i| \leqq \sum_{i=1}^{p} (\mathrm{E}Y_i^2\, \mathrm{E}Z_i^2)^{1/2} \leqq$$

$$\leqq \sum_{i=1}^{p} (\mathrm{E}\mathbf{Y}'\mathbf{Y}\, \mathrm{E}\mathbf{Z}'\mathbf{Z})^{1/2} = p(\mathrm{Tr}\, \mathbf{F}'_j \mathbf{F}_j \mathbf{W}\, \mathrm{Tr}\, \mathbf{F}'_k \mathbf{F}_k \mathbf{W})^{1/2}.$$

Thus

$$\mathrm{E}(\mathbf{S}_{N+m} - \mathbf{S}_N)' (\mathbf{S}_{N+m} - \mathbf{S}_N) \leqq p \sum_{j=N+1}^{N+m} \sum_{k=N+1}^{N+m} (\mathrm{Tr}\, \mathbf{F}'_j \mathbf{F}_j \mathbf{W}\, \mathrm{Tr} \mathbf{F}'_k \mathbf{F}_k \mathbf{W})^{1/2} =$$

$$= p[\sum_{j=N+1}^{N+m} (\mathrm{Tr}\, \mathbf{F}'_j \mathbf{F}_j \mathbf{W})^{1/2}]^2.$$

If (6.2) holds, then $\mathbf{S}_N$ is a Cauchy sequence and thus it converges in the quadratic mean. □

Lemma 6.4. Let u_n and α_n be sequences of complex numbers such that the following conditions are fulfilled:

(a) There exists a constant $M > 0$ such that $|u_1 + \ldots + u_N| \leqq M$ for all $N = 1, 2, \ldots$

(b) $\alpha_n \to 0$.

(c) $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$.

Then the series $\alpha_1 u_1 + \alpha_2 u_2 + \ldots$ is convergent.

Proof. The proof is analogous to that of the Dirichlet criterion. Denote

$$S_n = \alpha_1 u_1 + \ldots + \alpha_n u_n, \quad s_n = u_1 + \ldots + u_n.$$

Then

$$S_n = s_1(\alpha_1 - \alpha_2) + s_2(\alpha_2 - \alpha_3) + \ldots + s_{n-1}(\alpha_{n-1} - \alpha_n) + s_n \alpha_n.$$

If $m \geqq 1$, then

$$S_{n+m} - S_n = \sum_{k=n}^{n+m-1} s_k(\alpha_k - \alpha_{k+1}) + s_{n+m}\alpha_{n+m} - s_n \alpha_n,$$

$$|S_{n+m} - S_n| \leqq M\Big(\sum_{k=n}^{n+m-1} |\alpha_k - \alpha_{k+1}| + |\alpha_n| + |\alpha_{n+m}|\Big) \to 0$$

as $n \to \infty$. It implies that S_n converges. □

Lemma 6.5. Let $\alpha_n = (u + vw^n)\, b_n$, where u, v and w are complex numbers such that $|w| < 1$. Then the series $\sum_{n=0}^{\infty} \alpha_n \mathrm{e}^{in\lambda}$ is convergent for $0 \neq \lambda \in [-\pi, \pi]$.

Proof. Let $\lambda \neq 0$, $\lambda \in [-\pi, \pi]$. Since

$$\sum_{k=0}^{N} \mathrm{e}^{ik\lambda} = (\mathrm{e}^{i\lambda} - 1)^{-1} [\mathrm{e}^{i(N+1)\lambda} - 1],$$

we have

$$\Big|\sum_{k=0}^{N} \mathrm{e}^{ik\lambda}\Big| \leqq 2|\mathrm{e}^{i\lambda} - 1|^{-1} = |\sin(\lambda/2)|^{-1}.$$

Further

$$b_0 = 1, \quad b_{n+1} = (n + \delta)\, b_n/(n + 1), \quad b_n - b_{n+1} = (1 - \delta)\, b_n/(n + 1)$$

and

$$\sum b_n/n < \infty.$$

Thus

$$\sum|\alpha_n - \alpha_{n+1}| \leqq |u| \sum|b_n - b_{n+1}| + |v| \sum|b_n - w b_{n+1}|\, |w|^n \leqq$$
$$\leqq |u|\,(1 - \delta) \sum(b_n/n) + 2|v| \sum|w|^n < \infty.$$

The assertion follows from Lemma 6.4. □

Lemma 6.6. The series $\sum_{n=0}^{\infty} (-1)^n\, b_n\, \mathrm{e}^{in\lambda}$ is convergent for $\lambda \in (-\pi, \pi)$.

Proof. If $\lambda = 0$, then the assertion is obvious, since $b_n \searrow 0$. Assume $\lambda \neq 0$. We have

$$\Big|\sum_{n=0}^{N} (-1)^n \mathrm{e}^{in\lambda}\Big| = |(-\mathrm{e}^{-i\lambda} - 1)^{-1} [(-1)^{N+1} \mathrm{e}^{i(N+1)\lambda} - 1]| \leqq$$
$$\leqq 2|1 + \mathrm{e}^{i\lambda}|^{-1} = |1/\cos(\lambda/2)|.$$

Further,

$$\sum |b_n - b_{n+1}| = (1 - \delta) \sum b_n/(n + 1) \leqq (1 - \delta) \sum b_n/n < \infty .$$

Now, we apply Lemma 6.4. □

Lemma 6.7. Let μ be a finite measure. Let $f_1, f_2, \ldots$ be a sequence of measurable functions. If there exist functions f and g such that

$$f_n \to f \quad \text{a.e.} \quad [\mu] \quad \text{and} \quad \int |f_n - g|^2 \, d\mu \to 0 ,$$

then $f = g$ a.e. $[\mu]$.

Proof. The assumptions ensure that $f_n \xrightarrow{\mu} f$, $f_n \xrightarrow{\mu} g$. This implies $f = g$ a.e. $[\mu]$. □

Lemma 6.8. Let a series $\sum_{n=0}^{\infty} q_n r^n e^{inx} = \varphi(r e^{ix})$ converge for $r \in [0, 1)$. If the series $\sum_{n=0}^{\infty} q_n e^{inx}$ converges, then

$$\sum_{n=0}^{\infty} q_n e^{inx} = \lim_{r \to 1^-} \varphi(r e^{ix}) .$$

The proof follows from the Abel theorem (cf. [3], p. 470).

Lemma 6.9 (generalized P. du Bois Reymond's theorem). If f is absolutely integrable on $[-\pi, \pi]$ and if

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{6.3}$$

holds everywhere on $[-\pi, \pi]$ except, maybe, for a finite number of points x, then the series (6.3) is also the Fourier series of the function f.

Proof. See [3], Section 751, p. 626. □

Lemma 6.10. Let $f, \varphi \in L_2[-\pi, \pi]$. If each term of the Fourier series of the function f is multiplied by φ and integrated over $[-\pi, \pi]$, then the sum of these integrals is equal to $\int_{-\pi}^{\pi} f\varphi$. The assertion remains also valid in the case when f is absolutely integrable on $[-\pi, \pi]$ and φ has bounded variation.

Proof. See [3], Section 737, p. 590. □

Lemma 6.11. Let $f, \varphi \in L_2 [-\pi, \pi]$. Then the Fourier coefficients of the function $f\varphi$ are given by formal multiplication of the Fourier series of f and φ.

Proof. See [3], Section 738, p. 592. □

Lemma 6.12. A 2×2 matrix $\boldsymbol{A}$ has eigenvalues $\lambda_1 = \lambda_2 = 1$ if and only if

$$\boldsymbol{A} = \begin{pmatrix} 1 + t & b \\ c & 1 - t \end{pmatrix} \tag{6.4}$$

where t, b, c are arbitrary real numbers satisfying $t^2 + bc = 0$.

Proof. Let

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has eigenvalues $\lambda_1 = \lambda_2 = 1$. Since $|\mathbf{A}| = \lambda_1 \lambda_2$, $\operatorname{Tr} \mathbf{A} = \lambda_1 + \lambda_2$, we obtain $ad - bc = 1$, $a + d = 2$. If we write a in the form $a = 1 + t$, then $d = 1 - t$ and from the first equation we get $t^2 + bc = 0$.

On the other hand, if $\mathbf{A}$ has the form (6.4) and $t^2 + bc = 0$, then (1.3) implies $\lambda_1 = \lambda_2 = 1$. □

Lemma 6.13. A 2×2 matrix $\mathbf{A}$ has eigenvalues $\lambda_1 = \lambda_2 = -1$ if and only if

$$\mathbf{A} = \begin{pmatrix} -1 + t & b \\ c & -1 - t \end{pmatrix}$$

where t, b, c are arbitrary real numbers satisfying $t^2 + bc = 0$.

The proof is analogous to that of Lemma 6.12. □

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Prof. RNDr. Jiří Anděl, DrSc., Universita Karlova, matematicko-fyzikální fakulta (Charles University — Faculty of Mathematics and Physics), Sokolovská 83, 186 00 Praha 8. Czechoslovakia.

Professor María Gómez, Polytechnical University, Faculty of Informatics, Carretera Valencia ***Km.*** *7, **Madrid** 31. Spain.*