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DUALITY IN VECTOR OPTIMIZATION
Part II. Vector Quasiconcave Programming

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In this part of the tripaper, on the basis of the abstract theory presented in the first part, a duality theory is developed for the vector quasiconcave programming. In Section 3 some necessary concepts and assertions of (quasi)convexity are introduced. Section 4 deals with the duality theory in vector quasiconcave programming with affine constraints. Finally, in Section 5 a limit approach is proposed to define the dual problems for the vector quasiconcave programming with convex constraints.

3. QUASICONVEXITY OF OPERATORS AND RELATED CONCEPTS

In the following definitions X is a topological linear space, Y is a topological linear space ordered by a convex cone Y_+ with $\operatorname{int} Y_+ \neq \emptyset$ and $Y_+ \cap (-Y_+) = \{0\}$. Let $D \subset X$ be a convex subset, having at least two points.

Given an operator $G : D \to Y$ we say that G is *quasiconvex* in D if the lower set

$$\{x \in D \mid G(x) \leqq b\})$$

is convex for all $b \in Y$ (or equivalently: all its strict lower sets $\{x \in D \mid G(x) < b\}$ are convex).

G is *convex* in D if for all $x, y \in D$ and $\lambda \in (0, 1)$

$$G[\lambda x + (1 - \lambda) y] \leqq \lambda G(x) + (1 - \lambda) G(y).$$

G is (*quasi*) *concave* if $-G$ is (quasi) convex. G je *affine* if for all $x, y \in D$ and $\lambda \in$ $\in (0, 1)$

$$G[\lambda x + (1 - \lambda) y] = \lambda G(x) + (1 - \lambda) G(y).$$

G is *quasimonotonic* if it is both quasiconvex and quasiconcave. G is *lower* (*upper*) *semicontinuous* in D if its lower sets (upper sets) are closed with respect to D.

A subset $A \subset X$ is a *polytope* if it is an intersection of a finite number of half-spaces. Obviously, if Y is of finite dimension and G is affine, then all lower (upper) sets of G are polytopes.

Theorem 3.1. (Theorem on polytopal feasible sets.)

If Y is of finite dimension and G is a lower semicontinuous quasimonotonic operator on a polytope A, then the feasible set

$$\{x \in A \mid G(x) \leqq b\}$$

is a polytope for any $b \in Y$.

Proof. The proof is similiar to that in Martos [2] (cf. [2] page 78). The only difference is that X need not be finitely dimensional. □

Suppose that X and Y are Banach spaces, G is Fréchet-differentiable at $\bar{x} \in D$, i.e. there exists a continuous linear operator $G'(\bar{x}) : X \to Y$ such that

$$\lim_{\|\Delta x\| \to 0} \frac{\|G(\bar{x} + \Delta x) - G(\bar{x}) - \langle G'(\bar{x}), \Delta x\rangle\|}{\|\Delta x\|} = 0$$

G is said to be *locally quasiconvex* at $\bar{x}$ with respect to D if for all $x \in D$

$$G(x) \leqq G(\bar{x}) \Rightarrow \langle G'(\bar{x}), x - \bar{x}\rangle \leqq 0 .$$

G is *locally pseudoconvex* at $\bar{x}$ with respect to D if it is locally quasiconvex at $\bar{x}$ with respect to D and for all $x \in D$

$$G(x) < G(\bar{x}) \Rightarrow \langle G'(\bar{x}), x - \bar{x}\rangle < 0 .$$

G is *pseudoconvex* in D if it is locally pseudoconvex at any $x \in D$ with respect to D. G is *pseudoconcave* if $-G$ is pseudoconvex. G is *pseudomonotonic* if it is both pseudoconvex and pseudoconcave.

Theorem 3.2. (Linearization theorem for pseudomonotonic constraints.)

Let $G(x)$ be a pseudomonotonic operator in the convex set D and let $\beta \in G(D) \subset Y$. Then for any x° such that $G(x^\circ) = \beta$ we have

$$\mathscr{L}(\beta) = \{x \in D \mid G(x) \leqq \beta\} = \{\in D \mid \langle G'(x^\circ), x - x^\circ\rangle \leqq 0\} .$$

If, in addition, D is a polytope and Y is of finite dimension, then $\mathscr{L}(\beta)$ is a polytope.

Proof. The proof is analogous to that of Theorem 43 in [2]. □

We formulate finally a separation theorem of convex sets which will be used in the next section.

Theorem 3.3. (see [3].) If a nonempty, relatively open convex set M does not meet a nonempty polytope N, then there existes a hyperplane H, strictly separating M from N, i.e. there exists a continuous linear functional f and a scalar α such that

$$\langle f, x\rangle \leqq \alpha < \langle f, y\rangle \quad \forall x \in N , \quad \forall y \in M .$$

4. VECTOR QUASICONCAVE PROGRAMMING WITH AFFINE CONSTRAINTS

In this section X, Y, Z and W are topological linear spaces, Y and Z are ordered by the positive convex cones Y_+ and Z_+ (int $Y_+ \neq \emptyset$ and $Y_+ \cap (-Y_+) = \{0\}$), D is a subset of X. Further, suppose that we are given an affine operator $F_1 : D \to W$, a quasiconcave operator $F_2 : B \to Y$, where $F_1(D) \subset B \subset W$, and an affine operator $G : X \to Z$. Denote $F(x) = F_2[F_1(x)]$ $\forall x \in D$. The following problem

$$\text{find } \mathrm{Sup}^w \{F(x) \mid x \in D, G(x) = 0\} = S^w$$

is called the *vector quasiconcave programming problem with affine constraints.*

Throug hout the paper we suppose that the problem (4.1) has a feasible solution. Further in order to apply the theory developed in Section 2 we slightly modify the problem (4.1) as follows. We put

$$\mu_F(x) = \{y \in Y \mid y \leqq F(x)\}$$

for any $x \in D$. Then instead of the problem (4.1) we will study the following program

(I) $$\text{find } \mathrm{Sup}^w \bigcup_{x \in D, G(x)=0} \mu_F(x) = S_I .$$

This modification is rather formal than essential since if $y^* \in S_I$, $y^* \in \mu_F(x^*)$, $x^* \in \mathscr{D} = \{x \in D \mid G(x) = 0\}$, then obviously $F(x^*) \in S^w$ and x^* is an optimal solution of the problem (4.1).

Now we can convert problem (I) into the abstract model from Section 2 of [1]. We put

$$E = Z \times W; \quad \Lambda_* = -\infty ; \quad \Lambda^* = +\infty$$

$$P = \{(z; w) \in E \mid \exists x \in D : z = G(x) \,\&\, w = F_1(x)\}$$

$$Q_y = \{(z; w) \in E \mid z = 0 \,\&\, w \in F_1(D) \,\&\, F_2(w) \geqq y\} \quad \forall y \in Y$$

$$Q = \bigcup_{y \in Y} Q_y ; \quad P_0 = P \cap Q$$

$$\mu(a) = \{y \in Y \mid a \in Q_y\} \; \forall a \in P_0$$

We have then the problem

($\tilde{\mathrm{I}}$) $$\text{find } \mathrm{Sup}^w \bigcup_{a \in P_0} \mu(a) = S_I$$

Lemma 4.1.

$$\bigcup_{x \in \mathscr{D}} \mu_F(x) = \bigcup_{a \in P_0} \mu(a)$$

Hence the problems (I) and ($\tilde{\mathrm{I}}$) are equivalent.

Proof. Let $y \in \bigcup_{x \in \mathscr{D}} \mu_F(x)$, then there exists an $x' \in D$ such that $y \leqq F(x')$. Put $a' = (0; F_1(x'))$. Evidently $a' \in P \cap Q_y$ consequently $y \in \mu(a') \subset \bigcup_{a \in P_0} \mu(a)$.

Conversely if $y \in \bigcup_{a \in P_0} \mu(a)$ then there is an $a' \in P_0$ such that $a' \in Q_y$. It means that there is an $x' \in D$ with $a' = (0; F_1(x'))$ and $F_2[F_1(x')] = F(x') \geqq y$ which implies

$$y \in \mu_F(x') \subset \bigcup_{x \in \mathscr{D}} \mu_F(x) \qquad \square$$

Now for any $(z^*; w^*; r) \in Z^* \times W^* \times R$ we denote

$$H_{z^*,w^*,r} = \{(z; w) \in Z \times W \mid \langle z, z \rangle + \langle w^*, w \rangle \leqq r\}$$

and

$$E^* = \{H_{z^*,w^*,r} \mid z^* \in Z^* \;\&\; w^* \in W^* \;\&\; r \in R\}$$

Further put

$$P^* = \{H \in E^* \mid P \subset H\}$$

$$Q_y^* = \{H \in E^* \mid H \cap Q_{y'} = \emptyset \;\; \forall y' \gtrsim y\}$$

$$Q^* = \bigcup_{y \in Y} Q_y^*$$

$$P_0^* = P^* \cap Q^*$$

$$v(H) = \{y \in Y \mid H \in Q_y^*\} \quad \forall H \in P_0^* .$$

According to Section 2 we have the following dual problem

$$(\tilde{\text{I}}^*) \qquad \text{find} \quad \text{Inf}^{\text{w}} \bigcup_{H \in P_0^*} v(H) = I_{I^*}$$

Lemma 4.2. If P is a polytope and F_2 is lower semicontinuous, then

$$S_I = I_{I^*} \tag{4.2}$$

Proof. Using Theorem 3.3 it is easy to verify the conditions $[A_1]$, $[A_2]$. Thus according to Theorem 2.2 (4.2) holds. $\square$

Corollary. If X, Z and W are of finite dimension, D is a polytope and F_2 is lower semicontinuius, then

$$S_I = I_{I^*} .$$

Lemma 4.3. For any $H_{z^*,w^*,r} \in P_0^*$

$$\text{Inf}^{\text{w}} v(H_{z^*,w^*,r}) = \text{Sup}^{\text{w}} \bigcup_{\substack{w \in F_1(D) \\ \langle w^*, w \rangle \leqq r}} \mu_{F_2} w)$$

Proof. Let $y \in \text{Inf}^{\text{w}} v(H_{z^*,w^*,r})$. Then for any $y' < y$, $y' \notin v(H_{z^*,w^*,r})$, it means that there is $w \in F_1(D)$ such that $\langle w^*, w \rangle \leqq r$ and $y' \lneqq F_2(w)$. It follows

$$y' \in \bigcup_{\substack{w \in F_1(D) \\ \langle w^*, w \rangle \leqq r}} \mu_{F_2}(w)$$

hence

$$y \in \mathrm{Sup}^w \bigcup_{\substack{w \in F_1(D) \\ \langle w^*, w\rangle \leqq r}} \mu_{F_2}(w)$$

Now let $y \in \mathrm{Sup}^w \bigcup_{\substack{w \in F_1(D) \\ \langle w^*, w\rangle \leqq r}} \mu_{F_2}(w)$. By the same consideration we have $y \in v(H_{z^*, w^*, r})$ and hence $y \in \mathrm{Inf}^w\, v(H_{z^*, w^*, r})$. □

We have now

$$P_0^* = \{H_{z^*,w^*,r} \mid z^* \in Z^*, w^* \in W^*, r \in R : \langle z^*, z\rangle + \langle w^*, w\rangle \leqq \tag{4.3}$$
$$\leqq r\ \forall (z; w) \in P\ \&\ \mathrm{Sup}^w \{F_2(w) \mid w \in F_1(D) : \langle w^*, w\rangle \leqq r\} \neq \emptyset\}$$

Put

$$r(z^*, w^*) = \sup \{\langle z^*, z\rangle + \langle w^*, w\rangle \mid (z; w) \in P\} \tag{4.4}$$

$$\mathscr{L} = \{(z^*; w^*) \in Z^* \times W^* \mid \exists r : H_{z^*,w^*,r} \in P_0^*\} \tag{4.5}$$

then obviously

$$P_0^* = \{H_{z^*,w^*,r} \mid (z^*; w^*) \in \mathscr{L}\ \&\ r \geqq r(z^*, w^*)\} \tag{4.6}$$

Since

$$v(H_{z^*,w^*,r}) \subset v(H_{z^*,w^*,r(z^*,w^*)}) \quad \forall r \geqq r(z^*, w^*)$$

we have

$$\bigcup_{H \in P_0^*} v(H) = \bigcup_{(z^*,w^*) \in \mathscr{L}} v(H_{z^*,w^*,r(z^*,w^*)}) \tag{4.7}$$

Further

$$\mathrm{Inf}^w \bigcup_{H \in P_0^*} v(H) = \mathrm{Inf}^w \bigcup_{(z^*,w^*) \in \mathscr{L}} v(H_{z^*,w^*,r(z^*,w^*)}) \quad \text{(see (4.7))}$$

$$= \mathrm{Inf}^w \bigcup_{(z^*,w^*) \in \mathscr{L}} \mathrm{Inf}^w\, v(H_{z^*,w^*,r(z^*,w^*)}) \quad \text{(see Remark 1 of Sec. 1.)}$$

$$= \mathrm{Inf}^w \bigcup_{(z^*,w^*) \in \mathscr{L}} \mathrm{Sup}^w \bigcup_{\substack{w \in F_1(D) \\ \langle w^*, w\rangle \leqq r(z^*,w^*)}} \mu_{F_2}(w) \quad \text{(see Lemma 4.3)}$$

$$= \mathrm{Inf}^w \bigcup_{(z^*,w^*) \in \mathscr{L}} L(z^*, w^*)$$

where

$$L(z^*, w^*) = \mathrm{Sup}^w \bigcup_{\substack{w \in F_1(D) \\ \langle w^*, w\rangle \leqq r(z^*,w^*)}} \mu_{F_2}(w) \tag{4.8}$$

Thus we have proved

Lemma 4.4. Problem $(\tilde{\mathrm{I}}^*)$ is equivalent to the problem

$$\text{find } \mathrm{Inf}^w \bigcup_{(z^*,w^*) \in \mathscr{L}} L(z^*, w^*) = I_{I^*} \tag{I*}$$

where $L(z^*, w^*)$ is defined in (4.8).

The problem (I^*) is called the T_1-dual to problem (I). From Lemmas 4.1, 4.2 and 4.4 it follows

Theorem 4.1. If P is a polytope and F_2 is lower semicontinuous, then

$$S_I = I_{I^*} \tag{4.9}$$

Corollary. If X, Z and W are of finite dimension, D is a polytope and F_2 is lower semicontinuous, then (4.9) holds.

Remark 1. The problem with inequality constraints

$$(\mathrm{I}_{<(>)}) \qquad \text{find } \mathrm{Sup}^{\mathrm{w}} \{F(x) \mid x \in D : G(x) \leqq (\geqq)\, 0\}$$

where F, G and D remain as in problem (I), can be transformed to the problem with equality constraints by the traditional way as in linear programming. Then after some simple arrangements we obtain the dual of $(\mathrm{I}_{<(>)})$

$$(\mathrm{I}^*_{<(>)}) \qquad \text{find } \mathrm{Inf}^{\mathrm{w}} \bigcup_{(z^*, w^*) \in \mathscr{L}_{<(>)}} L(z^*, w^*)$$

where

$$\mathscr{L}_{<(>)} = \{(z^*; w^*) \in \mathscr{L} \mid z^* \leqq (\geqq)\, 0\} \tag{4.10}$$

and $L(z^*, w^*)$ is defined as in (4.8).

Remark 2. If the constraint operator G is not affine, but quasimonotonic or pseudomonotonic, then after a linearization of the feasible set (see Theorems 3.1 and 3.2) we can apply the theory introduced above.

We shall now apply this duality theory to some examples.

Example 1. Suppose that

$$\boldsymbol{A} = (a_{ij}) \quad \text{is an } m \times n \text{ matrix},$$
$$\boldsymbol{b} = (b_i) \quad \text{is a vector in } R^m$$
$$F(x) = \begin{pmatrix} x_1^2 \\ \vdots \\ x_n^2 \end{pmatrix} \quad \text{for} \quad x = (x_1 \ldots x_n)' \in R^n .$$

We are given then the problem

$$\text{find } \mathrm{Sup}^{\mathrm{w}} \left\{ \begin{pmatrix} x_1^2 \\ \vdots \\ x_n^2 \end{pmatrix} \middle| \ x = (x_1, \ldots, x_n) \geqq 0 \ \&\ \boldsymbol{A}x \leqq \boldsymbol{b} \right\} \tag{4.11}$$

Since $F(x)$ is quasiconcave in R^n_+, we can apply the theory introduced above. Here,

$$X = W = Y = R^n , \quad Z = R^m$$
$$G(x) = \boldsymbol{A}x - \boldsymbol{b} , \quad F_1(x) = x .$$

Further, for any $z \in R^m$ and $w \in R^n$

$$r(z, w) = \sup_{x \geqq 0} \{z'(\boldsymbol{A}x - \boldsymbol{b}) + w'x\} = \sup_{x \geqq 0} \{(z'\boldsymbol{A} + w')\,x - z'\boldsymbol{b}\} =$$

$$= \begin{cases} -z'\boldsymbol{b} & \text{if} \quad z'\boldsymbol{A} + w' \leqq 0 \\ +\infty & \text{otherwise} \end{cases}$$

and

$$\operatorname{Sup}^w \left\{ \begin{pmatrix} x_1^2 \\ \vdots \\ x_n^2 \end{pmatrix} \,\middle|\, x \geqq 0 \,\&\, w'x \leqq -z'b \right\} \neq \emptyset \Leftrightarrow w \geqq 0 \,.$$

Hence according to (4.10)

$$(4.12) \qquad \mathscr{L}_< = \{(z; w) \in R^m \times R^n \mid z \leqq 0 \,\&\, z'\boldsymbol{A} + w' \leqq 0 \,\&\, w \geqq 0\}$$

and

$$(4.13) \qquad L(z, w) = \operatorname{Sup}^w \bigcup_{\substack{x \geqq 0 \\ w'x \leqq -z'b}} \mu_F(x) \,.$$

The dual of problem (4.11) is then

$$(4.14) \qquad \text{find } \operatorname{Inf}^w \bigcup_{(z,w) \in \mathscr{L}_<} L(z, w)$$

where $\mathscr{L}_<$ and $L(z, w)$ are defined in (4.12) and (4.13).

We illustrate it by a concrete example. Let

$$\boldsymbol{A} = \begin{pmatrix} 1, & 1 \\ 1, & -1 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

then the problem (4.11) has the form

$$(4.15) \qquad \text{find } \operatorname{Sup}^w \left\{ \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix} \,\middle|\, x_1, x_2 \geqq 0 \,\&\, x_1 + x_2 \leqq 1 \,\&\, x_1 - x_2 \leqq 0 \right\}.$$

According to (4.12) we have

$$(4.16) \qquad \mathscr{L}_< = \{(z; w) \in R^2 \times R^2 \mid z \leqq 0 \,\&\, w \geqq 0 \,\&\, z_1 + z_2 + w_1 \leqq$$
$$\leqq 0 \,\&\, z_1 - z_2 + w_2 \leqq 0\}$$

and

$$r(z, w) = -z_1 \quad \forall (z; w) \in \mathscr{L}_< \,.$$

It is easy to see that

$$(4.17) \qquad z_1 < 0 \quad \text{and} \quad z_1 \leqq z_2 \leqq 0$$

and

$$(4.18) \qquad \bigcup_{\substack{x \geqq 0 \\ w_1x_1 + w_2x_2 \leqq -z_1}} \mu_F(x) = \bigcup_{\substack{x \geqq 0 \\ \left(-\frac{w_1}{z_1}\right)x_1 + \left(-\frac{w_2}{z_1}\right)x_2 \leqq 1}} \mu_F(x)$$

So the T_1-dual problem of (4.15) is

(4.19) $$\text{find } \operatorname{Inf}^{w} \bigcup_{(z,w)\in\mathscr{L}_1} L(z, w)$$

where

(4.20) $$\mathscr{L}_1 = \{(z; w_1; w_2) \in R^3 \mid -1 \leqq z \leqq 0 \;\&\; w_1, w_2 \geqq 0 \;\&$$
$$\& -1 + z + w_1 \leqq 0 \;\& -1 - z + w_2 \leqq 0\}$$

and

(4.21) $$L(z, w) = \operatorname{Sup}^{w} \bigcup_{\substack{x \geq 0 \\ w_1x_1 + w_2x_2 \leq 1}} \mu_F(x) .$$

The set $L(z, w)$ in (4.21) is illustrated in Fig. 2 where $z = -\frac{1}{2}$, $w_1 = \frac{3}{2}$ and $w_2 = \frac{1}{2}$.

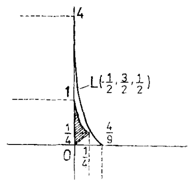

Fig. 2.

Example 2. Given a matrix
$$\mathbf{A} = [a_{ij}]_{\substack{i=1,\ldots,m+p \\ j=0,1,\ldots,n}}$$
we define

(4.22) $$\mathscr{D} = \{x = (x_1, \ldots, x_n)' \in R^n \mid x \geqq 0 \& \; a_{i,0} + \sum_{j=1}^{n} a_{i,j}x_j = 0 \,,$$
$$i = 1, \ldots, m \;\&\; a_{m+k,0} + \sum_{j=1}^{n} a_{m+k,j}x_j > 0 \,, \quad k = 1, \ldots, p\}$$

(4.23) $$f_1(x) = \prod_{k=1}^{p} (a_{m+k,0} + \sum_{j=1}^{n} a_{m+k,j}x_j)$$

(4.24) $$f_2(x) = \sum_{k=1}^{p} e^{a_{m+k,0}} + \sum_{j=1}^{n} a_{m+k,j}x_j \,.$$

The problem in question is

(4.25) $$\text{find } \operatorname{Sup}^{w} \left\{ \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \mid x \in \mathscr{D} \right\} .$$

Put

$$X = R^n \times R^p$$
$$D = \{(x; w) \in R^n \times R^p \mid x \geqq 0 \;\&\; w > 0\}$$
$$g_i(x, w) = a_{i,0} + \sum_{j=1}^{n} a_{i,j}x_j, \quad i = 1, \ldots, m$$

$$g_{m+k}(x, w) = a_{m+k,0} + \sum_{j=1}^{n} a_{m+k,j} x_j - w_k, \quad k = 1, \ldots, p$$

$$G = (g_1, \ldots, g_{m+p})$$

$$F = \begin{pmatrix} \prod_{k=1}^{p} w_k \\ \sum_{k=1}^{p} e^{w_k} \end{pmatrix}$$

then the problem (4.25) can be rewritten as follows:

(4.26) $$\text{find } \mathrm{Sup}^w \{F(x, w) \mid (x; w) \in D \,\&\, G(x; w) = 0\}$$

For any $z \in R^{m+p}$ and $v \in R^p$ we have

(4.27) $$r(z, v) = \sup_{(x,w)\in D} \Big(\sum_{i=1}^{m+p} z_i g_i(x, w) + \sum_{k=1}^{p} v_k w_k\Big) =$$

$$= \sup_{(x,w)\in D} \Big(\sum_{i=1}^{m+p} z_i a_{i,0} + \sum_{j=1}^{n}\Big(\sum_{i=1}^{m+p} z_i a_{ij}\Big) x_j + \sum_{k=1}^{p} (v_k - z_{m+k})\, w_k\Big) =$$

$$= \begin{cases} \sum_{i=1}^{m+p} z_i a_{i,0} & \text{if } \sum_{i=1}^{m+p} z_i a_{ij} \leqq 0 \;\forall j = 1, \ldots, n \,\&\, v_k - z_{m+k} \leqq 0 \;\forall k = 1, \ldots, p \\ +\infty & \text{otherwise} \end{cases}$$

and

(4.28) $$\mathrm{Sup}^w \left\{ \begin{pmatrix} \prod_{k=1}^{p} w_k \\ \sum_{k=1}^{p} e^{w_k} \end{pmatrix} \,\middle|\, w > 0 \,\&\, \sum_{k=1}^{p} v_k w_k \leqq \sum_{i=1}^{m+p} z_i a_{i,0} \right\} \neq \emptyset$$

$$\Leftrightarrow \sum_{i=1}^{m+p} z_i a_{i,0} > 0 \,\&\, v_k > 0 \;\forall k = 1, \ldots p$$

Summarizing (4.27) and (4.28) we have

(4.29) $$\mathscr{L} = \Big\{(z; v) \,\Big|\, z \in R^{m+p} \,\&\, v \in R^p \,\&\, \sum_{i=1}^{m+p} z_i a_{i,0} > 0 \,\&\, \sum_{i=1}^{m+p} z_i a_{i,j} \leqq 0 \;\; \forall j = 1, \ldots, n \,\&\, 0 < v_k \leqq z_{m+k} \;\forall k = 1, \ldots, p\Big\}$$

and

(4.30) $$L(z, v) = \mathrm{Sup}^w \bigcup_{\substack{w>0 \\ \sum_{k=1}^{p} v_k w_k \leqq \sum_{i=1}^{m+p} z_i a_{i,0}}} \mu_F(w) \;\forall (z; v) \in \mathscr{L}\,.$$

Notice that if $0 < u_k \leqq v_k \leqq z_{m+k} \;\forall k = 1, \ldots, p$ then

(4.31) $$\forall y \in \bigcup_{\substack{w>0 \\ \sum_{k=1}^{p} v_k w_k \leqq \sum_{i=1}^{m+p} z_i a_{i,0}}} \mu_F(w) \;\exists y' \in \bigcup_{\substack{w>0 \\ \sum_{k=1}^{p} \mu_k w_k \leqq \sum_{i=1}^{m+p} z_i a_{i,0}}} \mu_F(w) : y' > y$$

From (4.31) it follows that

(4.32)
$$\mathrm{Inf}^{w} \bigcup_{(z,v)\in\mathscr{L}} L(z, v) = \mathrm{Inf}^{w} \bigcup_{z\in\mathscr{L}_0} L(z)$$

where

(4.33)
$$\mathscr{L}_0 = \{z \in R^{m+p} \mid \sum_{i=1}^{m+p} z_i a_{i,0} > 0 \,\&\, \sum_{i=1}^{m+p} z_i a_{i,j} \leqq 0 \quad \forall j = 1, \ldots, n$$
$$\& \; z_{m+k} > 0 \;\forall k = 1, \ldots, p\}$$

and

(4.34)
$$L(z) = \mathrm{Sup}^{w} \bigcup_{\substack{w>0 \\ \sum_{k=1}^{p} z_{m+k} w_k \leqq \sum_{i=1}^{m+p} z_i a_{i,0}}} \mu_F(w) .$$

Hence the problem

(4.35)
$$\text{find } \mathrm{Inf}^{w} \bigcup_{z\in\mathscr{L}_0} L(z)$$

where $\mathscr{L}_0$ and $L(z)$ are defined in (4.33) and (4.34) is the dual of problem (4.26), thus of the initial problem (4.25).

5. VECTOR QUASICONCAVE PROGRAMMING WITH CONVEX CONSTRAINTS

In this section a duality theory is developed for the vector quasiconcave programming the constraint operator of which is not affine, but convex or, equivalently, the feasible set of which is not polytopal but convex. In the foregoing section we have seen that Theorem 3.3 plays a crucial role in proving the strong duality principle (see Lemma 4.2) and it is easy to verify that it may fail if N is not a polytope. Therefore we cannot use Theorem 3.3 directly if the constraint operator is not affine. Our main idea is to approximate the convex feasible set by a sequence of polytopal sets. After that, by a limit passage, we obtain the dual problem.

At first let us define some basic distance notions. In this section X and Y are Banach spaces. For any subsets A and B of X we define

(5.1)
$$\varrho(A, B) = \sup_{w\in A} \inf_{v\in B} \|v - w\| + \sup_{v\in B} \inf_{w\in A} \|v - w\|$$

$\varrho(A, B)$ is called the *Hausdorff distance* of A and B. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of subsets in X. We say that A_n *converge to* A *in the Hausdorff sense*, $A_n \xrightarrow{H} A$, if

$$\lim_{n\to\infty} \varrho(A_n, A) = 0$$

A_n *converge to* A *in the Kakutani sense*, $A_n \xrightarrow{K} A$, if

(5.2)
$$\forall \{x_n\}_{n=1}^{\infty} x_n \in A_n,\; x_n \to x \Rightarrow x \in A$$

and

(5.3)
$$\forall x \in A \;\exists x_n \in A_n,\; n = 1, 2, \ldots\; x_n \to x .$$

Lemma 5.1.

$$A_n \xrightarrow{\text{H}} A \Rightarrow A_n \xrightarrow{\text{K}} \bar{A}$$

Proof. Let $A_n \xrightarrow{\text{H}} A$, $x_n \in A_n$, $n = 1, 2, \ldots$ and $x_n \to x$. Fix an arbitrary $\varepsilon > 0$, then there are an integer n_ε and a point $x'_{n_\varepsilon} \in A$ such that $\|x - x_{n_\varepsilon}\| < \varepsilon/2$ and $\|x_{n_\varepsilon} - x'_{n_\varepsilon}\| < \varepsilon/2$. Consequently $\forall \varepsilon\ \exists x'_{n_\varepsilon} \|x - x'_{n_\varepsilon}\| < \varepsilon$ what means $x \in \bar{A}$.

Conversely if $x \in \bar{A}$ then for any integer k there is an n_k such that

$$\forall n \geqq n_k\ \exists x_n \in A_n : \|x_n - x\| < 1/k$$

From these x_n one can choose a sequence that converges to x. □

Let B, B_n, $n = 1, 2, \ldots$ be subsets of Y and Y_+ be a convex positive cone in Y.

Lemma 5.2.

$$B_n \xrightarrow{\text{H}} B \Rightarrow B_n - Y_+ \xrightarrow{\text{H}} B - Y_+$$

Proof. Let $v \in B_n$, $w \in B$, v_+, $w_+ \in Y_+$, then

$$\inf_{\substack{w \in B \\ w_+ \in Y_+}} \|(v - v_+) - (w - w_+)\| \leqq \inf_{w \in B} \|v - w\|$$

and

$$\inf_{\substack{v \in B_n \\ v_+ \in Y_+}} \|(w - w_+) - (v - v_+)\| \leqq \inf_{v \in B_n} \|v - w\|$$

Consequently

$$\varrho(B_n - Y_+, B - Y_+) \leqq \varrho(B_n, B) \tag{5.4}$$

The assertion of the lemma follows easily from (5.4). □

Now let us have an operator $F : D \to Y$, where $D \subset X$, let $\mathscr{D}_n \subset D$, $\forall n = 1, 2, \ldots$ and $\mathscr{D} \subset D$. Suppose that F is defined on the set $\mathscr{D}_\Delta = \{x \in X \mid \inf_{x' \in \mathscr{D}} \|x - x'\| \leqq \Delta\}$ for some fixed $\Delta > 0$.

Lemma 5.3. If $\mathscr{D}_n \xrightarrow{\text{H}} \mathscr{D}$ and F is uniformly continuous on $\mathscr{D}_\Delta$ then

$$F(\mathscr{D}_n) \xrightarrow{\text{H}} F(\mathscr{D}).$$

Proof. We have

$$\varrho(F(\mathscr{D}_n), F(\mathscr{D})) = \sup_{y \in F(\mathscr{D}_n)} \inf_{z \in F(\mathscr{D})} \|y - z\| + \sup_{z \in F(\mathscr{D})} \inf_{y \in F(\mathscr{D}_n)} \|y - z\| = \tag{5.5}$$

$$= \sup_{v \in \mathscr{D}_r} \inf_{w \in \mathscr{D}} \|F(v) - F(w)\| + \sup_{w \in \mathscr{D}} \inf_{v \in \mathscr{D}_n} \|F(v) - F(w)\|$$

The assertion of the lemma follows from the uniform continuity of F, with regard to (5.5). □

Put

$$\mu_F(A) = F(A) - Y_+$$

then from Lemmas 5.2 and 5.3 it follows

Lemma 5.4. Under the same conditions as in Lemma 5.3 we have

$$\mu_F(\mathscr{D}_n) \xrightarrow{\text{H}} \mu_F(\mathscr{D})$$

Lemma 5.5. Let B, B_n, $n = 1, 2, \ldots$ be subsets of Y such that

$$B - Y_+ \subset B \quad \text{and} \quad B_n - Y_+ \subset B_n \ \forall n = 1, 2, \ldots \tag{5.6}$$

and $B_n \xrightarrow{\text{H}} B$. Then

$$\text{Sup}^{\text{w}} B_n \xrightarrow{\text{K}} \text{Sup}^{\text{w}} B .$$

Proof. We need to verify conditions (5.2) and (5.3). Let $x_n \in \text{Sup}^{\text{w}} B_n$ $\forall n = 1, 2, \ldots,$ $x_n \to x$. For every n there is an $x_{nn} \in B_n$ such that $\|x_{nn} - x_n\| < 1/n$. Consequently $x_{nn} \to x$ and by Lemma 5.1 $x \in \bar{B}$. If $x \notin \text{Sup}^{\text{w}} B$, then there is an $x' \in B$ such that $x' > x$. According to Lemma 5.1 there exists a sequence $\{x'_n\}_{n=1}^{\infty}, x'_n \in B_n$ $\forall n = 1, 2, \ldots$ $\ldots x'_n \to x'$. Hence there is an n_0 such that for all $n \geqq n_0 : x'_n > x_n$, what is a contradiction to $x_n \in \text{Sup}^{\text{w}} B_n$. We have proved (5.2).

Conversely let $x \in \text{Sup}^{\text{w}} B$. Choose an $e \in \text{int } Y_+$ such that

$$\mathscr{B}_1(e) = \{y \in Y \mid \|y - e\| \leqq 1\} \subset Y_+ . \tag{5.7}$$

Put

$$l(x; e) = \{x + te \mid t \in R\} \tag{5.8}$$

and

$$l_n(x; e) = B_n \cap l(x; e) . \tag{5.9}$$

By Lemma 5.1 there is a sequence $\{x'_n\}_{n=1}^{\infty}$, $x'_n \in B_n$ $\forall n$ such that $x'_n \to x$. Using (5.6) we can choose, for any n, $y_n \in (x_n - Y_+) \cap l(x; e)$ such that $y_n \to x$. Put

$$x_n = \sup l_n(x; e) = x + t_n e \ \forall n .$$

From (5.6) it follows that $x_n \in \text{Sup}^{\text{w}} B_n$ and as $\liminf_{n \to \infty} x_n \geqq \lim_{n \to \infty} y_n = x$ we have $t_n \geqq 0$ $\forall n = 1, 2, \ldots$. If x_n does not converge to x there are an $\varepsilon > 0$ and an integer n_ε such that for any $n \geqq n_\varepsilon \|x_n - x\| = \|x + t_n e - x\| = t_n\|e\| > \varepsilon$. Hence $t_n >$ $> \varepsilon/\|e\|$. Consequently

$$\inf_{y \in B} \|x_n - y\| = \inf_{y \in B} \|x + t_n e - y\| = \inf_{y \in B} t_n \|e - (y - x)/t_n\| \geqq$$

$$\geqq \inf_{z \in Y \mid Y_+} t_n \|e - z\| \geqq t_n > \varepsilon/\|e\| \ \forall n \geqq n_\varepsilon \quad (\text{see (5.7)}) .$$

It means $\varrho(B_n, B) \nrightarrow 0$, what is a contradiction to $B_n \xrightarrow{\text{H}} B$. We have proved $x_n \to x$ and thus (5.3). The proof is complete. □

Summarizing Lemmas 5.4 and 5.5 we obtain the following theorem.

Theorem 5.1. Suppose that $\mathscr{D}_n \xrightarrow{\mathrm{H}} \mathscr{D}$ and F is uniformly continuous on $\mathscr{D}_\Delta$. Then

$$\operatorname{Sup}^{\mathrm{w}} \mu_F(\mathscr{D}_n) \xrightarrow{\mathrm{K}} \operatorname{Sup}^{\mathrm{w}} \mu_F(\mathscr{D}) .$$

Lemma 5.6. If $A \subset B \subset Y$ then

$$\forall y \in \operatorname{Sup}^{\mathrm{w}} A \;\; \forall z \in \operatorname{Sup}^{\mathrm{w}} B : y \mathrel{\overline{\gtrless}} z .$$

Proof. Trivial.

Lemma 5.7. Let $B, B_1, B_2, \ldots, B_n, \ldots$ be subsets of Y fulfilling (5.6) and

$$(5.10) \qquad B_n \supset B \;\; \forall n = 1, 2, \ldots .$$

Then $B_n \xrightarrow{\mathrm{H}} B$ implies

$$\operatorname{Sup}^{\mathrm{w}} B = \operatorname{Inf}^{\mathrm{w}} \bigcup_{n=1}^{\infty} \operatorname{Sup}^{\mathrm{w}} B_n .$$

Proof. Let $y \in \operatorname{Sup}^{\mathrm{w}} B$ then by Lemma 5.5 $y \in \overline{\bigcup_{n=1}^{\infty} \operatorname{Sup}^{\mathrm{w}} B_n}$. From Lemma 5.6 it follows that there is no $z \in \bigcup_{n=1}^{\infty} \operatorname{Sup}^{\mathrm{w}} B_n$ with $z < y$ which means $y \in \operatorname{Inf}^{\mathrm{w}} \bigcup_{n=1}^{\infty} \operatorname{Sup}^{\mathrm{w}} B_n$.

Now let $y \notin \operatorname{Sup}^{\mathrm{w}} B$. Then if

(i) $y \in \bar{B}$, there is $y' \in \operatorname{Sup}^{\mathrm{w}} (l(y; e) \cap B) \subset \operatorname{Sup}^{\mathrm{w}} B$ such that $y' < y$, where e and $l(y; e)$ are defined as in the proof of Lemma 5.5 (see 5.8). According to Lemma 5.5 there is some $y'' \in \bigcup_{n=1}^{\infty} \operatorname{Sup}^{\mathrm{w}} B_n$ such that $y'' < y$, thus $y \notin \operatorname{Inf}^{\mathrm{w}} \bigcup_{n=1}^{\infty} \operatorname{Sup}^{\mathrm{w}} B_n$, or (ii) there is $y' \in B$ such that $y' > y$, one can choose an $\varepsilon > 0$ such that $\mathscr{B}_\varepsilon(y) = \{z \in Y \mid \|z - y\| < \varepsilon\} \subset B \subset B_n \;\forall n$, which means that $\mathscr{B}_\varepsilon(y) \cap \operatorname{Sup}^{\mathrm{w}} B_n = \emptyset \;\forall n$. Hence $y \notin \overline{\bigcup_{n=1}^{\infty} \operatorname{Sup}^{\mathrm{w}} B_n}$ and thus $y \notin \operatorname{Inf}^{\mathrm{w}} \bigcup_{n=1}^{\infty} \operatorname{Sup}^{\mathrm{w}} B_n$. The proof is complete. □

In the further development suppose that $F_1 : D \to W$ is affine, $F_2 : B \to Y$, where $F_1(D) \subset B \subset W$, is quasiconcave and $g_n : X \to R$, $n = 1, 2, \ldots$, are real affine functionals. We will extend the duality theory introduced in Section 4 to the class of problems of the following type:

$$(5.11) \qquad \text{find } \operatorname{Sup}^{\mathrm{w}}\{F(x) = F_2[F_1(x)] \mid x \in D \;\&\; g_n(x) \leqq 0 \;\; \forall n = 1, 2, \ldots\} .$$

Instead of the problem (5.11) we will work, as in Section 4, with the problem

$$(5.12) \qquad \text{find } \operatorname{Sup}^{\mathrm{w}} \bigcup_{x \in \mathscr{D}} \mu_F(x) = \operatorname{Sup}^{\mathrm{w}} \mu_F(\mathscr{D}) = S$$

where

$$(5.13) \qquad \mathscr{D} = \{x \in D \mid g_n(x) \leqq 0 \;\; \forall n = 1, 2, \ldots\} .$$

Put

$$(5.14) \qquad \mathscr{D}_n = \{x \in D \mid g_i(x) \leqq 0\,, \quad i = 1, 2, \dots n\}$$

then we have, for all n, the subproblems

$$(5.15) \qquad \text{find } \operatorname{Sup}^w \mu_F(\mathscr{D}_n) = S_n\,.$$

According to Section 4 the T_1-dual problem of (5.15) is

$$(5.16) \qquad \text{find } \operatorname{Inf}^w \bigcup_{(z,w^*)\in\mathscr{L}_n} I_n(z, w^*) = I_n$$

where

$$(5.17) \qquad \mathscr{L}_n = \{(z; w^*) \in R^n \times W^* \mid z_i \leqq 0, i = 1, \dots, n \;\&\; r(z, w^*) =$$
$$= \sup_{x\in D} \left[\sum_{i=1}^n z_i g_i(x) + \langle w^*, F_1(x)\rangle\right] < +\infty \;\&\; \operatorname{Sup}^w\{F_2(w) \mid w \in$$
$$\in F_1(D) : \langle w^*, w\rangle \leqq r(z, w^*)\} \neq \emptyset\}$$

and

$$(5.18) \qquad L_n(z, w) = \operatorname{Sup}^w \bigcup_{\substack{w\in F_1(D)\\ \langle w^*, w\rangle \leqq r(z,w^*)}} \mu_{F_2}(w)\,.$$

Remark 1. By the Corollary to Theorem 4.1, if X, W are of finite dimension D is a polytope and F is lower semicontinuous then

$$(5.19) \qquad S_n = I_n \;\forall n\,.$$

Definition. The problem

$$(5.20) \qquad \text{find } \operatorname{Inf}^w \bigcup_{n=1}^{\infty} \bigcup_{(z,w^*)\in\mathscr{L}_n} L_n(z, w^*) = I$$

where $\mathscr{L}_n$ and L_n are defined in (5.17) and (5.18), is called the T_1-dual for the problem (5.12).

We have immediately

Theorem 5.2. (Weak Duality Principle.)

$$\forall y \in S \;\forall z \in I : y \gtreqqless z$$

Theorem 5.3. (Strong Duality Principle.)
Suppose that X, W are of finite dimension, D is a polytope F_2 is lower semicontinuous and $\mathscr{D}_n \xrightarrow{H} \mathscr{D}$. Then $S = I$.

Proof. By Lemma 5.7 we have

$$(5.21) \qquad S = \operatorname{Sup}^w \mu_F(\mathscr{D}) = \operatorname{Inf}^w \bigcup_{n=1}^{\infty} \operatorname{Sup}^w \mu_F(\mathscr{D}_n)$$

and according to the Corollary of Lemma 4.2

$$\text{(5.22)} \qquad \operatorname{Sup}^{w} \mu_F(\mathscr{D}_n) = \operatorname{Inf}^{w} \bigcup_{H \in P^*_{0,n}} \nu(H)$$

where $P^*_{0,n}$ stands for P^*_0 in problem (Ĩ*) for the problem (5.15). From (5.21), (5.22), with regard to Remark 1 of Section 1, we have

$$S = \operatorname{Inf}^{w} \bigcup_{n=1}^{\infty} \operatorname{Inf}^{w} \bigcup_{H \in P^*_{0,n}} \nu(H) = \operatorname{Inf}^{w} \bigcup_{n=1}^{\infty} \bigcup_{H \in P^*_{0,n}} \nu(H) =$$

$$= \operatorname{Inf}^{w} \bigcup_{n=1}^{\infty} \bigcup_{H \in P^*_{0,n}} \operatorname{Inf}^{w} \nu(H) = \operatorname{Inf}^{w} \bigcup_{n=1}^{\infty} \bigcup_{(z,w^*) \in \mathscr{L}_n} L_n(z, w^*) = I$$

(cf. Lemma 4.3). □

Further suppose that $f_k : D \to R$, $k = 1, \ldots, p$, are convex real functionals. The problem

$$\text{(5.23)} \qquad \text{find} \quad \operatorname{Sup}^{w}\{F(x) \mid x \in D : f_k(x) \leqq 0,\ k = 1, \ldots, p\}$$

is called the *vector quasiconcave programming with convex constraints*. In order to apply the duality theory introduced above we need to transform the convex constraints to the affine ones (maybe of an infinite number).

Definition. Let $f : D \to R$ be a convex functional and $z \in D$. A linear continuous functional $v \in X^*$ is called to be a subgradiant of f at z if

$$f(x) \geqq f(z) - \langle v, x - z \rangle \quad \forall x \in D\,.$$

The set of all subgradients of f at z is denoted by $\partial f(z)$ and called the subdifferential of f at z.

We summarize some facts concerning subgradients and subdifferentials (for details see [5]):

(i) If f is differentiable at z then $\partial f(z) = \{\partial f(z)/\partial x\}$ where $\partial f(z)/\partial x$ is the gradient of f at z.

(ii) If f is continuous at z, then $\partial f(z)$ is nonempty weak*-compact and bounded in X^*.

(iii) If f is continuous on D, then

$$f(x) = \max_{z \in D} \{f(z) + \langle v(z), x - z \rangle\} \quad \forall x \in D$$

where $v(z)$ is an arbitrary vector from $\partial f(z)$ for all $z \in D$.

From (iii) it follows immediately

Lemma 5.8. If f is continuous on D and $D' \subset D$ is dense in D, then

$$f(x) = \sup_{z' \in D'} \{f(z') + \langle v(z'), x - z \rangle\} \quad \forall x \in D$$

and consequently

(5.24) $\{x \in D \mid f(x) \leqq 0\} = \{x \in D \mid f(z') + \langle v(z'), x - z'\rangle \leqq 0 \ \forall z' \in D'\}$

where $v(z')$ is an arbitrary vector from $\partial f(z')$ for all $z' \in D'$.

Lemma 5.9. Suppose that $\dim X < +\infty$, A_n are closed convex subsets of X such that $A_n \supset A_{n+1}$ $\forall n$ and $\bigcap_{n=1}^{\infty} A_n = A$ is bounded, then

$$A_n \xrightarrow{\mathrm{H}} A .$$

Proof. At first we prove that

$$\exists N \ \forall n \geqq N : A_n \ \text{ is bounded.} \tag{5.25}$$

Indeed, if (5.25) does not hold the sets

$$C_n = \{x \in X \mid \|x\| = 1, A_n + tx \subset A_n \ \forall t \geqq 0\}$$

are nonempty and closed for all n. Since $C_{n+1} \subset C_n$ $\forall n$ we have

$$\emptyset \neq \bigcap_{n=1}^{\infty} C_n = C \subset \{x \in X \mid \|x\| = 1 \ \& \ A + tx \subset A \ \forall t \geqq 0\}$$

what is a contradiction to the boundedness of A.

Suppose that $A_n \not\xrightarrow{\mathrm{H}} A$, then there exists an $\varepsilon > 0$ such that

$$B_n = A_n \cap (X \setminus A_\varepsilon) \neq \emptyset \quad \forall n$$

where

$$A_\varepsilon = \{x \in X \mid \inf_{z \in A} \|x - z\| < \varepsilon\} .$$

B_n are compact for all $n \geqq N$ (see (5.25)) and $B_{n+1} \subset B_n$ for all n. Hence

$$\emptyset \neq \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} A_n \cap (X \setminus A_\varepsilon) = A \cap (X \setminus A_\varepsilon) = \emptyset$$

what is absurd. The proof is complete. □

Now let us return to the problem (5.23). From the foregoing lemma it follows immediately

Theorem 5.4. Suppose that $\dim X < +\infty$ and

$$\mathscr{D} = \{x \in D \mid f_k(x) \leqq 0, k = 1, \ldots, p\}$$

is bounded. Let the system $\{z_{k,n}\}_{\substack{k=1,\ldots,p \\ n=1,2,\ldots}} \subset D$ be such that

$$\{x \in D \mid f_k(z_{k,n}) + \langle v_{k,n}, x - z_{k,n}\rangle \leqq 0, k = 1, \ldots, p, n = 1, 2, \ldots\} = \mathscr{D}$$

where $v_{k,n}$ is an arbitrary vector from $\partial f_k(z_{k,n})$. Then

$$\mathscr{D}_n \xrightarrow{\mathrm{H}} \mathscr{D}$$

where

$$\mathscr{D}_n = \{x \in D \mid f_k(z_{k,i}) + \langle v_{k,i}, x - z_{k,i}\rangle \leqq 0, k = 1, \ldots, p, i = 1, \ldots, n\}.$$

We have then, with regard to Lemma 5.8, the following

Corollary. Suppose that $\dim X < +\infty$, $\mathscr{D}$ is bounded and the set $\{z_n\}_{n=1}^{\infty} \subset D$ is dense in D, then

$$\mathscr{D}_n \xrightarrow{\mathrm{H}} \mathscr{D}$$

where

$$\mathscr{D}_n = \{x \in D \mid f_k(z_i) + \langle v_{k,i}, x - z_i\rangle \leqq 0, k = 1, \ldots, p, i = 1, \ldots, n\}$$

and $v_{k,i}$ is an arbitrary vector from $\partial f_k(z_i)$.

On the basis of Lemma 5.8 one can transform the problem (5.23) to problem (5.11) and then apply the duality theory presented above. If $\dim X < +\infty$ and $\mathscr{D}$ is bounded Theorem 5.4 or its Corollary guarantee the strong duality principle.

Example.

(5.26) $$\text{Find } \operatorname{Sup}^{w} \left\{\begin{pmatrix} x_1^3 \\ x_2^3 \end{pmatrix} \mid x_1, x_2 \geqq 0 \;\&\; x_1^2 + x_2^2 \leqq 1\right\}$$

The modified problem of (5.26) is

(5.27) $$\text{find } \operatorname{Sup}^{w} \bigcup_{x \in \mathscr{D}} \mu_F(x) = S$$

where

(5.28) $$\mathscr{D} = \{x = (x_1; x_2) \in R^2 \mid x_1, x_2 \geqq 0 \;\&\; x_1^2 + x_2^2 \leqq 1\}$$

and

$$\mu_F(x) = \{y \in R^2 \mid y \leqq (x_1^3, x_2^3)\}$$

The function $f(x) = x_1^2 + x_2^2 - 1$ is convex and differentiable with

$$\frac{\partial f(z)}{\partial x} = \begin{pmatrix} 2z_1 \\ 2z_2 \end{pmatrix}.$$

By (iii) we have

$$\mathscr{D} = \{x \in R_+^2 \mid 2z_1x_1 + 2z_2x_2 \leqq 1 + z_1^2 + z_2^2 \;\forall z = (z_1; z_2) \geqq 0\} =$$

$$= \bigcap_{z \geqq 0} \left\{x \in R_+^2 \mid \frac{z_1}{\sqrt{(z_1^2 + z_2^2)}} x_1 + \frac{z_2}{\sqrt{(z_1^2 + z_2^2)}} x_2 \leqq \frac{1}{2}\left(\frac{1}{\sqrt{(z_1^2 + z_2^2)}} + \sqrt{(z_1^2 + z_2^2)}\right)\right\} =$$

$$= \bigcap_{\substack{a, b \geqq 0 \\ a^2 + b^2 = 1}} \{x \in R_+^2 \mid ax_1 + bx_2 \leqq 1\}$$

On the curve $\Gamma = \{(a; b) \geqq 0 \mid a^2 + b^2 = 1\}$ we choose an arbitrary countable set

$\Gamma_0 = \{(a_k; b_k) \mid a_k^2 + b_k^2 = 1\}$ that is dense in Γ. According to Theorem 5.4

$$\mathscr{D}_n = \bigcap_{k=1}^{n} \{x \in R_+^2 \mid a_k x_1 + b_k x_2 \leqq 1\} \xrightarrow{\text{H}} \mathscr{D} .$$

We have

$$(5.29)\quad \mathscr{L}_1 = \Big\{(z_1, \ldots, z_n, v_1, v_2) \in R^{n+2} \mid z_i \leqq 0,\ i = 1, \ldots, n \ \&\ \sup_{x \geqq 0} [(\sum_{i=1}^{n} z_i a_i + v_1) x_1 + + (\sum_{i=1}^{n} z_i b_i + v_2) x_2 \dotplus \sum_{i=1}^{n} z_i] = -\sum_{i=1}^{n} z_i \ \&\ \mathrm{Sup}^{\mathrm{w}} \left\{\begin{pmatrix} x_1^3 \\ x_2^3 \end{pmatrix} \mid x_1, x_2 \geqq 0 : v_1 x_1 + + v_2 x_2 \leqq -\sum_{i=1}^{n} z_i\right\} \neq \emptyset \Big\} =$$

$$= \{(z_1, \ldots, z_n, v_1, v_2) \in R^{n+2} \mid z_i \leqq 0,\ i = 1, \ldots, n \ \&\ v = (v_1; v_2) \geqq 0 \ \&$$

$$\sum_{i=1}^{n} z_i < 0 \ \&\ \sum_{i=1}^{n} z_i a_i + v_1 \leqq 0 \ \&\ \sum_{i=1}^{n} z_i b_i + v_2 \leqq 0\}$$

and

$$(5.30)\qquad L_n(z, v) = \mathrm{Sup}^{\mathrm{w}} \bigcup_{\substack{x \geqq 0 \\ v_1 x_1 + v_2 v_2 \leqq -\sum_{i=1}^{n} z_i}} \left\{\begin{pmatrix} x_1^3 \\ x_2^3 \end{pmatrix} - R_+^2\right\}$$

Hence, by Theorem 5.3 the T_1-dual of (5.27) is

$$(5.31)\qquad \text{find } \mathrm{Inf}^{\mathrm{w}} \bigcup_{n=1}^{\infty} \bigcup_{(z,v) \in \mathscr{L}_n} L_n(z, v) = I$$

and we have

$$S = I$$

After a short arrangement we obtain a simpler form of (5.31)

$$(5.32)\qquad \text{find } \mathrm{Inf}^{\mathrm{w}} \bigcup_{n=1}^{\infty} \bigcup_{z \in \mathscr{K}_n} K_n(z) = I$$

where

$$\mathscr{K}_n = \{z = (z_1, \ldots, z_n) \in R^n \mid z \leqq 0 \ \&\ -\sum_{i=1}^{n} z_i = 1\}$$

and

$$K_n(z) = \mathrm{Sup}^{\mathrm{w}} \bigcup_{\substack{x_1, x_2 \geqq 0 \\ (-\sum_{i=1}^{n} z_i a_i) x_1 + (-\sum_{i=1}^{n} z_i b_i) x_2 \leqq 1}} \left\{\begin{pmatrix} x_1^3 \\ x_2^3 \end{pmatrix} - R_+^2\right\}.$$

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