Prem Nath Arora; Subhash Chowdhary
Generalised directed divergence without symmetry

_Kybernetika_, Vol. 20 (1984), No. 2, 147--158

Persistent URL: [http://dml.cz/dmlcz/124496](http://dml.cz/dmlcz/124496)

Terms of use:
© Institute of Information Theory and Automation AS CR, 1984

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.
GENERALISED DIRECTED DIVERGENCE WITHOUT SYMMETRY

P. N. ARORA, SUBHASH CHOWDHARY

The authors have characterized axiomatically the generalized directed divergence (which is a symmetric function of its variables) by considerably weakening the symmetry.

1. INTRODUCTION

Let
\[ A_n = \{ (p_1, p_2, \ldots, p_n); p_k \geq 0, k = 1, 2, \ldots, n, \sum_{k=1}^{n} p_k = 1 \}, \quad n = 2, 3, \ldots, \]
and
\[ A_n^+ = \{ (p_1, p_2, \ldots, p_n); p_k > 0, k = 1, 2, \ldots, n, \sum_{k=1}^{n} p_k = 1 \}, \quad n = 2, 3, \ldots, \]
be the sets of all finite \( n \)-component discrete probability distributions with non-negative elements and positive elements respectively. Let \( P = (p_1, p_2, \ldots, p_n) \), \( Q = (q_1, q_2, \ldots, q_n) \) and \( R = (r_1, r_2, \ldots, r_n) \in A_n \). The generalized directed divergence of three probability distributions \( P, Q \) and \( R \) is defined as

\[
F_d(P, Q, R) = \sum_{k=1}^{n} p_k \log \frac{q_k}{r_k},
\]
where \( F_d : S_n \to R, n = 2, 3, \ldots \), and \( S_n \) be a set of \( 3n \)-tuples of the form \((p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n; r_1, r_2, \ldots, r_n)\) such that \( q_i = 0 \) and \( p_i = 0 \) for all those indices \( i \) for which \( r_i = 0 \) and also \( p_i = 0 \) whenever \( q_i = 0, i = 1, 2, \ldots, n \).

Kannappan and Rathie [3] characterized (1.1) by assuming the following set of postulates.

(Here the base of the logarithm is taken as 2).
Postulate I, (Recursivity). For all probability distributions $P, Q$ and $R \in \mathcal{A}_n$, and $n \geq 3$,

$$(1.2) \quad F_n(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n; r_1, r_2, \ldots, r_n) =$$

$$= F_{n-1}(p_1 + p_2, \ldots, p_n; q_1 + q_2, \ldots, q_n; r_1 + r_2, \ldots, r_n) +$$

$$+ (p_1 + p_2) F_2 \left( \frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}; \frac{q_1}{q_1 + q_2}, \frac{q_2}{q_1 + q_2}; \frac{r_1}{r_1 + r_2}, \frac{r_2}{r_1 + r_2} \right)$$

with $p_1 + p_2 > 0, q_1 + q_2 > 0$ and $r_1 + r_2 > 0$.

Postulate II, $(n = 3)$. $F_3(p_1, p_2, p_3; q_1, q_2, q_3; r_1, r_2, r_3)$ is a symmetric function of its variables $(p_i; q_i; r_i), i = 1, 2, 3$.

Postulate III (Derivibility). The mapping $(x, y, z) \to f(x, y, z), (x, y, z) \in J$ possesses continuous first order partial derivatives with respect to each variable $(x, y, z) \in (0, 1)\times(0, 1)\times(0, 1) \cup \{(0, y, z), 0 \leq y < 1, 0 \leq z < 1\} \cup \{(1, y', z'), 0 < y' \leq 1, 0 < z' \leq 1\}.$

Postulate IV (Normalization).

$$f(\emptyset, \emptyset, \emptyset) = \frac{1}{3} \quad \text{and} \quad f(\emptyset, \emptyset, \emptyset) = 0 .$$

Postulate V (Nullity).

$$f(p, p, p) = 0, \quad p \in (0, 1) .$$

The main object of this paper is to axiomatically characterized (1.1) by considerably weakening the symmetry Postulate II, $(n = 3)$ assumed by Kannappan and Rathie [3] and by many other research workers.

Instead of Postulate II, $(n = 3)$, we assume the following postulate:

Postulate IVa. For all probability distributions $P, Q$ and $R \in \mathcal{A}_n - \mathcal{A}_n^*$, and $n \geq 3$,

$$(1.3) \quad F_n(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n; r_1, r_2, \ldots, r_n) =$$

$$= F_n(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n; r_1, r_2, \ldots, r_n),$$

$$2 \leq j \leq n, \quad \text{if} \quad r_1 > 0 \quad \text{and} \quad r_j = 0 \quad \text{or} \quad q_1 > 0 \quad \text{and} \quad q_j = 0 \quad \text{or} \quad p_1 > 0 \quad \text{and} \quad p_j = 0 \quad \text{holds.}$$

Postulate IVa allows the simultaneous interchange of $p_j$ with $q_j$ with $q_j$ and $r_j$ with $r_j, 2 \leq j \leq n$ is such that either $p_j > 0$ and $p_j = 0$ or $q_j > 0$ and $q_j = 0$ or $r_j > 0$ and $r_j = 0$ holds. It is obvious that Postulate IVa $(n = 3)$ implies Postulate IVa $(n = 3)$. But the converse is not true. For example: Consider $F_n : S_n \to \mathcal{R}$ defined
as
\[ F_n(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n; r_1, r_2, \ldots, r_n) = p_1 q_1 r_1 \quad \text{if} \quad P, Q \quad \text{and} \quad R \in A_n^*; \]
\[ = 1 \quad \text{if} \quad P, Q \quad \text{and} \quad R \in (A_n - A_n^*). \]

Then it is easy to check that \( F_n \) satisfies VI* but not II* (\( n = 3 \)). Thus VI* does not imply that \( F_n, n \geq 2, \) is a symmetric function.

2. CHARACTERIZATION THEOREM

**Theorem.** Let \( F_n : S_n \rightarrow R, n = 2, 3, \ldots, \) satisfy Postulates I* (\( n \geq 3 \)), III, IV, V and VI* (\( n \geq 3 \)). Then \( F_n \) is of the form
\[
F_n(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n; r_1, r_2, \ldots, r_n) = \sum_{k=1}^{n} p_k \log \frac{q_k}{r_k},
\]
where \( p_k \geq 0, q_k \geq 0, r_k \geq 0, k = 1, 2, \ldots, n; \sum_{k=1}^{n} p_k = 1 = \sum_{k=1}^{n} q_k = \sum_{k=1}^{n} r_k. \)

**Proof.** Before proving the main theorem, we shall prove the following lemmas:

**Lemma 1.** Postulates I* (\( n = 3 \)) and VI* (\( n = 3 \)) implies
\[
F_3(0, 1; 0, 1, 0) = 0 = F_3(1, 0; 1, 0, 1). \tag{2.2}
\]
**Proof.** From Postulate VI* (\( n = 3 \)), we have
\[
F_3(0, 1, 0; 1, 0; 1, 0) = F_3(0, 1, 0; 1, 0; 1, 0) = F_3(1, 0, 1; 0, 1, 0) = F_3(1, 0, 1; 0, 1, 0).
\]
which by Postulate I* (\( n = 3 \)) in (2.3), we get (2.2). \( \square \)

**Lemma 2.** Postulates I* (\( n \geq 3 \)) and VI* (\( n \geq 3 \)) implies
\[
F_n(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n; r_1, r_2, \ldots, r_n) = \sum_{k=1}^{n} p_k \log \frac{q_k}{r_k}, \quad n \geq 2.
\]
**Proof.** Let \( p_j \) be the first non-zero element in the probability distribution \( P \) such that \( p_j > 0 \Rightarrow q_j > 0 \Rightarrow r_j > 0, 1 \leq j \leq n, \) and using Postulates VI* (\( n \geq 3 \)), I* (\( n \geq 3 \)) and (2.2), we get
\[
F_n(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n; r_1, r_2, \ldots, r_n) = \sum_{k=1}^{n} p_k \log \frac{q_k}{r_k} = F_{n+1}(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n; r_1, r_2, \ldots, r_n) = \sum_{k=1}^{n} p_k \log \frac{q_k}{r_k} = F_{n+1}(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n; r_1, r_2, \ldots, r_n).
\]
\( \square \)
Lemma 3. Postulates $I_n(n \geq 3)$ and $VI_n(n \geq 3) \Rightarrow F_n$ has $n!$, $n = 2, 3, \ldots$, permutations $\Rightarrow F_n$, $n \geq 2$, is a symmetric function.

Proof. Here we prove the symmetry of $F_n$, $n \geq 2$, by the method of induction on $n$.

When $n = 2$. We have the following cases:

Case 1. When $0 < r_1 < 1$ holds in $F_2$:

Then, $0 < r_2 < 1$ also holds in $F_2$ and it implies that either

(i) $q_1 = 0 \Rightarrow p_1 = 0$, $p_2 = q_2 = 1$ in $F_2$; or (ii) $0 \leq p_1 < 1$, $0 < p_2 \leq 1$,$0 < q_1 < 1$, $0 < q_2 < 1$ in $F_2$.

The proof of (i) is as follows:

(2.5) $F_2(0, 1; 0, 1; r_1, r_2) \overset{(2.5)}{=} F_3(0, 0, 1; 0, 0, 1; 0, r_1, r_2) \overset{(1.3)}{=} F_3(1, 0, 0; 1, 0, 0; r_1, r_2, 0) \overset{(1.3)}{=} F_3(1, 0, 0; 1, 0, 0) + F_3(1, 0; 1, 0; r_2, r_1) \overset{(2.2)}{=} F_3(1, 0; 1, 0; r_2, r_1)$.

Similarly, the proof of (ii) follows.

Case 2. When either $r_1 = 0$ and $r_2 = 1$ or $r_2 = 1$ and $r_1 = 0$ holds in $F_2$:

Then, it implies either $p_1 = 0 = q_1$ and $p_2 = q_2 = 1$ or $p_1 = q_1 = 1$ and $p_2 = q_2 = 0$ in $F_2$.

This case is obviously true from (2.2).

Thus we have proved the symmetry of $F_2$ over $S_2$.

When $n = 3$. We have the following cases:

Case 1. When $0 < p_i < 1$, $0 < q_i < 1$, and $0 < r_i < 1$, $i = 1, 2, 3$ holds in $F_3$:

Then by Postulate $I_n(n = 3)$ and (2.5), we have

(2.6) $F_3(p_1, p_2, p_3; q_1, q_2, q_3; r_1, r_2, r_3) = F_3(p_2, p_1, p_3; q_2, q_1, q_3; r_2, r_1, r_3)$

and

(2.7) $F_3(p_1, p_2, p_3; q_1, q_2, q_3; r_1, r_2, r_3) \overset{(2.6)}{=} F_4(0, p_1, p_2, p_3; 0, q_1, q_2, q_3; 0, r_1, r_2, r_3) \overset{(1.3)}{=} F_4(p_1, p_2, 0; 0, q_1, q_2, 0; r_3, r_1, r_2, 0) \overset{(2.5)}{=} F_4(p_1, p_2, 0; q_1, q_3, q_2, 0; r_1, r_3, r_2, 0) \overset{(1.3)}{=} F_4(0, p_3, p_1, p_2; 0, q_3, q_2, q_1; 0, r_3, r_2, r_1) \overset{(2.6)}{=} F_4(0, p_3, p_1, p_2; q_3, q_2, q_1; r_3, r_2, r_1)$.

Therefore,

(2.8) $F_4(p_1, p_2, p_3; q_1, q_2, q_3; r_1, r_2, r_3) \overset{(2.6)}{=} F_3(p_2, p_1, p_3; q_2, q_1, q_3; r_2, r_1, r_3) \overset{(2.7)}{=} F_3(p_3, p_1, p_2; q_3, q_1, q_2; r_3, r_1, r_2) \overset{(2.6)}{=} F_3(p_3, p_1, p_2; q_3, q_2, q_1; r_3, r_2, r_1)$.

From (2.8), we get the symmetry of $F_3$ over $S_3$. 

150
Case 2. When 

(i) \( p_i = 0, \quad i = 1, 2, 3, \quad 0 < p_j < 1, \quad j + i = 1, 2, 3, \quad 0 < q_j < 1, \quad 0 < r_j < 1, \)

\( j = 1, 2, 3 \) holds in \( F_3 \):

or 

(ii) \( q_i = 0 \Rightarrow p_i = 0, \quad i = 1, 2, 3, \quad 0 < p_j < 1, \quad 0 < q_j < 1, \quad j + i = 1, 2, 3, \)

\( 0 < r_j < 1, \quad j = 1, 2, 3 \) holds in \( F_3 \):

or 

(iii) \( r_i = 0 \Rightarrow q_i = 0 \Rightarrow p_i = 0, \quad i = 1, 2, 3, \quad 0 < p_j < 1, \quad 0 < q_j < 1, \quad 0 < r_j < 1, \)

\( j + i = 1, 2, 3 \) holds in \( F_3 \).

In these subcases, the proof is similar to case 1.

Case 3. When 

(i) \( p_i = 0, \quad p_j = 0, \quad i + j = 1, 2, 3, \quad p_k = 1, \quad k + i + j = 1, 2, 3, \quad 0 < q_k < 1, \)

\( 0 < r_k < 1, \quad k = 1, 2, 3 \) holds in \( F_3 \):

or 

(ii) \( p_i = 0, \quad q_j = 0 \Rightarrow p_j = 0, \quad j + i = 1, 2, 3, \quad p_k = 1, \quad k + i + j = 1, 2, 3, \)

\( 0 < q_k < 1, \quad k + j = 1, 2, 3, \quad 0 < r_k < 1, \quad k = 1, 2, 3 \) holds in \( F_3 \):

or 

(iii) \( p_i = 0, \quad r_j = 0 \Rightarrow q_j = 0 \Rightarrow p_j = 0, \quad i + j = 1, 2, 3, \quad p_k = 1, \quad k + i + j = 1, 2, 3, \)

\( 0 < q_k < 1, \quad k + j = 1, 2, 3, \quad 0 < r_k < 1, \quad k = 1, 2, 3 \) holds in \( F_3 \):

or 

(iv) \( q_i = 0 \Rightarrow p_i = 0, \quad q_j = 0 \Rightarrow p_j = 0, \quad i + j = 1, 2, 3, \quad p_k = 1, \quad k + i + j = 1, 2, 3, \)

\( 0 < q_k < 1, \quad k + j = 1, 2, 3, \quad 0 < r_k < 1, \quad k = 1, 2, 3 \) holds in \( F_3 \):

or 

(v) \( r_i = 0 \Rightarrow q_i = 0 \Rightarrow p_i = 0, \quad r_j = 0 \Rightarrow q_j = 0 \Rightarrow p_j = 0, \quad i + j = 1, 2, 3, \quad p_k = 1, \quad k + i + j = 1, 2, 3, \)

\( 0 < q_k < 1, \quad k + j = 1, 2, 3 \) holds in \( F_3 \):

In case (i), we have

\[ (2.9) \quad F_3(0, 0, 1; q_1, q_2, q_3; r_1, r_2, r_3) = F_3(1, 0, 0; q_3, q_2, q_1; r_3, r_2, r_1) \]

\[ (2.10) \quad F_3(0, 1, 0; q_2, q_3, q_1; r_2, r_3, r_1) = F_3(0, 1, 0; q_1, q_2, q_3; r_1, r_2, r_3) \]

\[ (2.11) \quad F_3(1, 0, 0; q_1, q_2, q_3; r_1, r_2, r_3) = F_3(0, 0, 1; q_3, q_2, q_1; r_3, r_2, r_1) \]

Thus (2.9) shows that \( F_3 \) is a symmetric function. Similarly, the proof of other sub-cases follows from sub case (i).

Case 4. When 

\( r_i = 0 \Rightarrow q_i = 0 \Rightarrow p_i = 0, \quad r_j = 0 \Rightarrow q_j = 0 \Rightarrow p_j = 0, \quad i + j = 1, 2, 3, \quad p_k = q_k = r_k = 1, \quad k + i + j = 1, 2, 3 \) holds in \( F_3 \):

Then, by Postulate VI (n = 3), we have

\[ F_3(0, 0, 1; 0, 0, 1) = F_3(1, 0, 0; 1, 0, 0; 1, 0, 0) = F_3(0, 1, 0; 0, 1, 0; 0, 1, 0) \]

Hence we have proved the symmetry of \( F_3 \) completely.
When \( n = 4 \). We have the following cases:

Case 1. When \( 0 < p_1 < 1, 0 < q_1 < 1 \) and \( 0 < r_1 < 1, i = 1, 2, 3, 4 \) holds in \( F_4 \):

Then, we have

\[
F_A(p_1, p_2, p_3, p_4; q_1, q_2, q_3, q_4; r_1, r_2, r_3, r_4) = \\
F_A(p_2, p_1, p_3, p_4; q_2, q_1, q_3, q_4; r_2, r_1, r_3, r_4)
\]

and

\[
F_A(p_1, p_2, p_3, p_4; q_1, q_2, q_3, q_4; r_1, r_2, r_3, r_4) = \\
F_A(0, p_1, p_2, p_3, p_4; 0, q_1, q_2, q_3, q_4; 0, r_1, r_2, r_3, r_4)
\]

Similarly, we can show

\[
F_A(p_1, p_2, p_3, p_4; q_1, q_2, q_3, q_4; r_1, r_2, r_3, r_4) = \\
= F_A(p_4, p_2, p_3, p_1; q_4, q_2, q_3, q_1; r_4, r_2, r_3, r_1)
\]

Using Postulate \( I_A (n = 4) \) and symmetry of \( F_2 \) and \( F_4 \) in I, II, III, IV, V and VI of (2.13), we have \( 4! = 24 \) permutations of \( F_4 \) are isometric.

Case 2. When

(i) \( p_j = 0, i = 1, 2, 3, 4, 0 < p_j < 1, i \neq j = 1, 2, 3, 4, 0 < q_j < 1, 0 < r_j < 1, j = 1, 2, 3, 4 \) holds in \( F_4 \):

or

(ii) \( q_j = 0 \Rightarrow p_j = 0, i = 1, 2, 3, 4, 0 < p_j < 1, 0 < q_j < 1, i \neq j = 1, 2, 3, 4, 0 < r_j < 1, j = 1, 2, 3, 4 \) holds in \( F_4 \):

or

(iii) \( r_j = 0 \Rightarrow q_j = 0 \Rightarrow p_j = 0, i = 1, 2, 3, 4, 0 < p_j < 1, 0 < q_j < 1, 0 < r_j < 1, j = 1, 2, 3, 4 \) holds in \( F_4 \):

The above sub-cases follows from case 1.
Case 3. When

(i) $p_1 = 0$, $p_2 = 0$, $i + j = 1, 2, 3, 4$, $0 < p_k < 1$, $k \neq i + j = 1, 2, 3, 4$, $0 < q_k < 1$, $0 < r_k < 1$, $k = 2, 3, 4$ holds in $F_4$; or

(ii) $p_1 = 0$, $q_j = 0 \implies p_j = 0$, $i + j = 1, 2, 3, 4$, $0 < p_k < 1$, $k \neq i + j = 1, 2, 3, 4$, $0 < q_k < 1$, $k = 1, 2, 3, 4$, $0 < r_k < 1$, $k = 1, 2, 3, 4$ holds in $F_4$; or

(iii) $p_1 = 0$, $r_j = 0 \implies q_j = 0 \implies p_j = 0$, $i + j = 1, 2, 3, 4$, $0 < p_k < 1$, $k \neq i + j = 1, 2, 3, 4$, $0 < q_k < 1$, $0 < r_k < 1$, $k = 1, 2, 3, 4$ holds in $F_4$; or

(iv) $q_1 = 0 \implies p_1 = 0$, $q_j = 0 \implies p_j = 0$, $i \neq j = 1, 2, 3, 4$, $0 < r_k < 1$, $0 < q_k < 1$, $k \neq j = 1, 2, 3, 4$, $0 < r_k < 1$, $k = 1, 2, 3, 4$ holds in $F_4$; or

(v) $q_1 = 0 \implies p_1 = 0$, $r_j = 0 \implies q_j = 0 \implies p_j = 0$, $i \neq j = 1, 2, 3, 4$, $0 < r_k < 1$, $0 < q_k < 1$, $k \neq j = 1, 2, 3, 4$, $0 < r_k < 1$, $k = 1, 2, 3, 4$ holds in $F_4$; or

(vi) $r_j = 0 \implies q_j = 0 \implies p_j = 0$, $i \neq j = 1, 2, 3, 4$, $0 < r_k < 1$, $0 < q_k < 1$, $k \neq j = 1, 2, 3, 4$, $0 < r_k < 1$, $k = 1, 2, 3, 4$ holds in $F_4$.

Let us assume $p_0 = p_{10}$, $p_2 = p_{20}$ in (i) and using (2.10), (2.11) and (2.12) in $F_4$, we get

$$F_4(p_{10}, p_{20}, p_3, p_4; q_1, q_2, q_3, q_4; r_1, r_2, r_3, r_4) =$$

$$= \frac{(2.10)}{11} F_4(p_j, p_{20}, p_{10}, p_3, p_4; q_3, q_2, q_4; r_j, r_2, r_1, r_3)$$

$$+ \frac{(2.11)}{11} F_4(p_j, p_{20}, p_{10}, p_3, p_4; q_3, q_2, q_4; r_j, r_2, r_1, r_3)$$

$$+ \frac{(2.12)}{11} F_4(p_j, p_{20}, p_{10}, p_3, p_4; q_3, q_2, q_4; r_j, r_2, r_1, r_3)$$

$$+ \frac{(2.13)}{11} F_4(p_j, p_{20}, p_{10}, p_3, p_4; q_3, q_2, q_4; r_j, r_2, r_1, r_3)$$

Now we shall show below that I of (2.14) contributes 4 permutations of $F_4$ which are as follows:

$$F_4(p_{10}, p_{20}, p_3, p_4; q_1, q_2, q_3, q_4; r_1, r_2, r_3, r_4) =$$

$$= \frac{(2.4)}{11} F_4(0, p_{10}, p_{20}, p_3, p_4; q_1, q_2, q_3, q_4; r_1, r_2, r_3, r_4)$$

$$+ \frac{(2.5)}{11} F_4(0, p_{20}, p_{10}, 0, p_3, p_4; q_1, q_2, q_3, q_4; r_1, r_2, r_3, r_4)$$

$$+ \frac{(2.6)}{11} F_4(0, p_{20}, p_{10}, 0, p_3, p_4; q_1, q_2, q_3, q_4; r_1, r_2, r_3, r_4)$$

$$+ \frac{(2.7)}{11} F_4(0, p_{20}, p_{10}, 0, p_3, p_4; q_1, q_2, q_3, q_4; r_1, r_2, r_3, r_4)$$

$$+ \frac{(2.8)}{11} F_4(0, p_{20}, p_{10}, 0, p_3, p_4; q_1, q_2, q_3, q_4; r_1, r_2, r_3, r_4)$$

153
Similarly, we can show that

\[(2.17) \quad F_4(p_1, p_2, p_3, p_4; q_1, q_2, q_3, q_4; r_1, r_2, r_3, r_4) =
\]

\[= F_4(p_3, p_1, p_2, p_4; q_3, q_1, q_2, q_4; r_3, r_1, r_2, r_4).
\]

Now using Postulate I (n = 4) and symmetry of F_2 and F_3 in II, III, IV, V and VI of (2.14) and (2.15), (2.16) and (2.17) in I of (2.14) would yield 4! permutations of F_4 = symmetry of F_4. Similarly, the proof of other subcases follows from sub case (i) of case 3.

Case 4. When

(i) p_i = 0, p_j = 0, p_k = 0, i + j + k = 1, 2, 3, 4, p_i = 1, l + i + j + k = 1, 2, 3, 4 holds in F_4;

or

(ii) p_i = 0, p_j = 0, q_k = 0 \Rightarrow p_k = 0, i + j + k = 1, 2, 3, 4, p_i = 1, l + i + j + k = 1, 2, 3, 4, \ 0 < q_i < 1, l + k = 1, 2, 3, 4, holds in F_4;

or

(iii) p_i = 0, p_j = 0, r_k = 0 \Rightarrow q_k = 0 \Rightarrow p_k = 0, i + j + k = 1, 2, 3, 4, p_i = 1, l + i + j + k = 1, 2, 3, 4, \ 0 < q_i < 1, l + k = 1, 2, 3, 4 holds in F_4;

or

(iv) p_i = 0, q_j = 0 \Rightarrow p_j = 0, q_k = 0 \Rightarrow p_k = 0, i + j + k = 1, 2, 3, 4, p_i = 1, l + i + j + k = 1, 2, 3, 4, \ 0 < q_i < 1, l + j + k = 1, 2, 3, 4, \ l = 1, 2, 3, 4 holds in F_4;

or

(v) p_i = 0, q_j = 0 \Rightarrow q_j = 0, r_k = 0 \Rightarrow q_k = 0 \Rightarrow p_k = 0, i + j + k = 1, 2, 3, 4, p_i = 1, l + i + j + k = 1, 2, 3, 4, \ 0 < q_i < 1, l + j + k = 1, 2, 3, 4, \ 0 < r_i < 1, l + k = 1, 2, 3, 4 holds in F_4;
or

\( \text{(vi) } p_i = 0, \quad r_j = 0 \Rightarrow q_j = 0 \Rightarrow p_j = 0, \quad r_k = 0 \Rightarrow q_k = 0 \Rightarrow p_k = 0, \quad l \neq j + k = 1, 2, 3, 4, \quad p_i = 1, \quad l \neq i + j + k = 1, 2, 3, 4, \quad 0 < q_i < 1, \quad 0 < r_i < 1, \)

\( l \neq j + k = 1, 2, 3, 4 \) holds in \( F_4' \): or

\( \text{(vii) } q_i = 0 \Rightarrow p_i = 0, \quad q_j = 0 \Rightarrow p_j = 0, \quad q_k = 0 \Rightarrow p_k = 0, \quad l \neq j + k = 1, 2, 3, 4, \quad p_i = q_i = 1, \quad l \neq i + j + k = 1, 2, 3, 4, \quad 0 < r_i < 1, \quad l = 1, 2, 3, 4 \) holds in \( F_4' \):

or

\( \text{(viii) } q_i = 0 \Rightarrow p_i = 0, \quad q_j = 0 \Rightarrow p_j = 0, \quad q_k = 0 \Rightarrow p_k = 0, \quad l \neq j + k = 1, 2, 3, 4, \quad p_i = q_i = 1, \quad l \neq i + j + k = 1, 2, 3, 4, \quad 0 < r_i < 1, \quad l \neq j + k = 1, 2, 3, 4 \) holds in \( F_4' \):

or

\( \text{(ix) } q_i = 0 \Rightarrow p_i = 0, \quad r_j = 0 \Rightarrow q_j = 0 \Rightarrow p_j = 0, \quad r_k = 0 \Rightarrow q_k = 0 \Rightarrow p_k = 0, \quad i \neq j + k = 1, 2, 3, 4, \quad p_i = q_i = 1, \quad l \neq i + j + k = 1, 2, 3, 4, \quad 0 < r_i < 1, \quad l \neq j + k = 1, 2, 3, 4 \) holds in \( F_4' \):

Let us assume \( p_0 = 0 = p_{10}, p_2 = 0 = p_{20}, p_3 = 0 = p_{30} \) and \( p_4 = 1 \), in sub-case (i) of case 4 and using (2.10), (2.11) and (2.12), we get

\[
F_4(p_{10}, p_{20}, p_{30}, p_{41}, q_{11}, q_{21}, q_{31}, q_{41}, r_1, r_2, r_3, r_4) =
\]

\[
(2.10)F_4(p_{10}, p_{20}, p_{30}, p_{41}, q_{11}, q_{21}, q_{31}, q_{41}, r_1, r_2, r_3, r_4)
\]

Using Postulate \( L_4 (n = 4) \) and symmetry of \( F_2 \) and \( F_3 \) in III, IV and V of (2.18), and (2.15), (2.16) and (2.17) in I, II and VI of (2.18), we get 4! permutations of \( F_4 \) \( \Rightarrow \) the function \( F_4 \) is a symmetric function. Similarly, the proof of other subcases of case 4 follows from subcase (i) of case 4.

Case 5. When \( r_i = 0 \Rightarrow q_i = 0 \Rightarrow p_i = 0, \quad r_j = 0 \Rightarrow q_j = 0 \Rightarrow p_j = 0, \quad r_k = 0 \Rightarrow q_k = 0 \Rightarrow p_k = 0, \quad i \neq j + k = 1, 2, 3, 4, \quad p_i = q_i = r_i = 1, \quad l \neq i + j + k = 1, 2, 3, 4 \) holds in \( F_4 ' \):

Then symmetry of \( F_4 ' \), obviously, follows by applying Postulate VI \( L_4 (n = 4) \) in \( F_4 ' \).

From case 1 to case 5, discussed above, we conclude that \( F_4 ' \) is a symmetric function for all set of values of \( p 's, q 's \) and \( r 's. \)

155
When \( n = m \)

From the above results, we conclude:

(i) If \( F_2 \) has 2! permutations, then \( F_2 \) is a symmetric function;

(ii) If \( F_3 \) has 3! permutations, then \( F_3 \) is a symmetric function;

(iii) If \( F_4 \) has 4! permutations, then \( F_4 \) is a symmetric function;

Assuming that \( F_{m-1}, m \geq 5 \) is a symmetric function and thus it has \((m-1)!\) permutations, we shall prove that \( F_m \) has \( m! \) permutations which imply \( F_m \) is a symmetric function for \( m \geq 5 \). We proceed as follows:

Case 1. When \( 0 < p_1 < 1, 0 < q_i < 1, i = 1, 2, \ldots, m \) holds in \( F_m \):

Then we have

\[
F_m(p_1, p_2, \ldots, p_m; q_1, q_2, \ldots, q_m; r_1, r_2, \ldots, r_m) =
\]

and by Lemma 2 and Postulate VI \((n \geq 5)\) in the function \( F_m, m \geq 5 \), we get

\[
F_m(p_1, p_2, \ldots, p_m; q_1, q_2, \ldots, q_m; r_1, r_2, \ldots, r_m)
\]

Using Postulate I \((n \geq 5)\) and symmetry of \( F_2 \) in \((2), (3), \ldots, (m-1)\)\( \text{th} \) of \((2.20)\), we get

\[
F_m(p_1, p_2, \ldots, p_m; q_1, q_2, \ldots, q_m; r_1, r_2, \ldots, r_m)
\]

Again using Lemma 2 and Postulate VI \((n \geq 5)\) (as used in obtaining \((2.11)\) and \((2.12)\)) in \((2), (3), \ldots, (m-1)\)\( \text{th} \) of \((2.21)\), we get

\[
F_m(p_1, p_2, \ldots, p_m; q_1, q_2, \ldots, q_m; r_1, r_2, \ldots, r_m)
\]
Using (1) of (2.20) = (2) of (2.20) (i.e. replacement of 1st element of each distribution with third element of each distribution) in (3), (4), ... , (m — 1)th of (2.22), we get

\begin{equation}
F_{m}(p_{3}, p_{4}, p_{5}, \ldots, p_{m}; q_{3}, q_{4}, q_{5}, q_{6}, \ldots)
\end{equation}

Similarly, use of (1) of (2.20) = (3) of (2.20) (i.e. replacement of first element of each distribution with fourth element of each distribution) in (4), (5), ... , (m — 1)th of (2.22), we get

\begin{equation}
F_{m}(p_{4}, p_{5}, \ldots, p_{m}; q_{4}, q_{5}, \ldots, q_{m})
\end{equation}

and so on.

In the end, use (1) of (2.20) = (m — 2)th of (2.20) in (m — 1)th of (2.22), we get

\begin{equation}
F_{m}(p_{m-1}, p_{m}; \ldots, q_{m-1}, q_{m}; r_{m-1}, r_{m})
\end{equation}

Using Postulate \( I_{n} \) \((n \geq 5)\) symmetry of \( F_{2} \) and \( F_{m-1} \) in (2.22), (2.21), (2.23), (2.24), and so on, and (2.25) then each \( F_{m} \) in these would yield \( 2(m-2)! \) permutations of \( F_{m} \) and (2.22), (2.21), (2.23), (2.24), and so on, and (2.25) would yield \( 2(m-1)! \) \((m-2)!(m-2)! \ldots m! \) permutations of \( F_{m} \) respectively. Therefore, the algebraic sum of all these permutations of \( F_{m} \) is \( 2(m-1)(m-2)! \ldots (m-2)! + m! \) which implies that \( F_{m} \) \( m \geq 5 \) is a symmetric function.

Again, we may come across various cases similar to the one, as discussed in the symmetry of \( F_{3} \) and \( F_{4} \). They can be easily verified for the symmetry of \( F_{m} \), \( m \geq 5 \).

Hence we conclude the symmetry of \( F_{m} \), \( n \geq 2 \).

Thus Lemma 3 is proved.

Proof of the main theorem

Now Postulates \( I_{n} \) \((n = 3, 4)\) and \( VI_{n} \) \((n = 3, 4)\) gives \( 3! \) permutations of \( F_{3} \) \Rightarrow symmetry of \( F_{3} \), Kannappan and Rathie [3] has also taken symmetry of \( F_{3} \) as one of the postulate in their proof. Replacing 3-symmetry of \( F_{3} \) by our Postulate \( VI_{n} \), the proof of the theorem follows from their lines of action. Hence the theorem is proved.

Remarks.

1. The authors have proved in this paper that the symmetry of generalized directed divergence (1.1) for \( n \geq 2 \) follows from Postulates \( I_{n} \) \((n \geq 3)\) and \( VI_{n} \) \((n \geq 3)\).
and thus have proved that (1.1) can be characterized without symmetry postulate.

2. It has been analytically proved that $F_n$ has $n!$, $(n \geq 2)$ permutations $\rightarrow F_n$, $(n \geq 2)$ is a symmetric function.

(Received December 30, 1982.)

REFERENCES


Dr. P. N. Arora, Department of Mathematics, Dyal Singh College, Lodhi Road, New Delhi — 110003. India.

Subhash Chowdhary, Department of Mathematics, Hindu College, University of Delhi, Delhi — 110007. India.