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ON STATIONARITY OF A MULTIPLE DOUBLY STOCHASTIC MODEL*

JIŘÍ ANDĚL

A multiple linear process with random coefficients is investigated in the paper. Conditions for existence of such process are derived and its covariance function as well as the matrix of spectral densities are calculated. The results are applied to multiple AR(1) process with random coefficients, where the matrices of coefficients can be described by a stationary process. In this case conditions for existence and stationarity of the AR(1) process are given.

1. INTRODUCTION

Let \( \epsilon_t \) be a white noise such that \( \mathbb{E}\epsilon_t = 0, \mathbb{E}\epsilon_t^2 = \sigma^2 \) and \( \mathbb{E}\epsilon_s\epsilon_t = 0 \) for \( s \neq t \). The classical autoregressive process \( X_t \) is defined by the relation

\[
X_t = \sum_{k=1}^{n} \beta_k X_{t-k} + \epsilon_t, \tag{1.1}
\]

where \( \beta = (\beta_1, \ldots, \beta_n)' \) are the autoregressive parameters. If

\[
z^n - \beta_1 z^{n-1} - \ldots - \beta_n = 0 \quad \text{for} \quad |z| \geq 1, \tag{1.2}
\]

then there exists a unique process \( X_t \) satisfying

\[
X_t = \sum_{k=1}^{\infty} C_k \epsilon_{t-k}, \quad \sum_{k=0}^{\infty} c_k^2 < \infty \tag{1.3}
\]

such that (1.1) holds. This process is called autoregressive and it is stationary. It is known that (1.2) is a necessary and sufficient condition under which there exists a process (1.3) such that (1.1) holds.

A process \( X_t \) can be defined by a more general relation

\[
X_t = \sum_{k=1}^{n} b_k(t) X_{t-k} + \epsilon_t
\]

where
\[ \mathbf{b}(t) = [b_1(t), \ldots, b_n(t)]' \]
are random vectors such that
\[ \mathbb{E}\mathbf{b}(t) = \beta, \quad \text{var} \mathbf{b}(t) = \Lambda. \]
The case when \( \mathbf{b}(t) \) are independent is considerably simpler. Anděl [1] derived a condition under which \( X_t \) is stationary. Nicholls and Quinn [5] proposed a method for estimating parameters and in [6] generalized Anděl's result to multiple autoregressive models. They summarized their work in [7]. Anděl [2] analyzed a model with nonvanishing mean.

If \( \mathbf{b}(t) \) are not independent, the conditions for stationarity are rather complicated. Koubková [4] investigated an AR(1) process with a random coefficient \( b_1(t) \) such that \( b_1(t) \) is a MA(1) process. She proved that \( X_t \) is stationary only if some complicated relations among the moments are satisfied. Tjøstheim [9] derived some conditions under which an AR(1) process with random and dependent coefficients is strictly stationary. His conditions do not guarantee the existence of any moments of \( X_t \).

Pourahmadi [8] presents conditions for stationarity and derives explicit results for the following cases: (i) \( \log b_1(t) \) is a stationary Gaussian process; (ii) \( \log b_1(t) \) is an AR(1) process; (iii) \( \log b_1(t) \) is a MA(q) process.

In our paper we generalize some of Pourahmadi's results to the multiple AR(1) process.

### 2. MULTIPLE LINEAR PROCESS WITH RANDOM COEFFICIENTS

Let \( \mathbf{e}_n \) be a \( p \)-dimensional white noise with \( \mathbb{E}\mathbf{e}_n = 0, \text{var} \mathbf{e}_n = \mathbf{V} \). Let \( \mathbf{B}_n \) be a sequence of \( p \times p \) random matrices, the elements of which have finite second moments. Assume that \( \{\mathbf{B}_n\} \) and \( \{\mathbf{e}_n\} \) are independent. Define
\[
X_t = \sum_{n=-\infty}^{\infty} \mathbf{B}_n \mathbf{e}_{t-n}, 
\]
if the series converges in the quadratic mean. The process \( \{X_t\} \) can be considered as a generalization of the linear process. Denote
\[
\mathbf{W}_n = \mathbb{E}\mathbf{B}_n' \mathbf{B}_n
\]
and
\[
\mathbf{B}_n = \begin{pmatrix} b_{1,n} \\ \vdots \\ b_{p,n} \end{pmatrix},
\]
where \( b_{i,n} \) are row random vectors.

**Lemma 2.1.** The series (2.1) converges in the quadratic mean if and only if
\[
\text{Tr} \mathbf{V} \sum_{n=-\infty}^{\infty} \mathbf{W}_n < \infty. \tag{2.2}
\]
Proof. For a fixed \( t \) and a given \( N \geq 0 \) define

\[
S_N = \sum_{j=0}^{N} B_j e_{t-j}.
\]

For \( m \geq 1 \) we have

\[
E(S_{N+m} - S_N) (S_{N+m} - S_N) = \sum_{j=N+1}^{N+m} \sum_{k=N+1}^{N+m} \text{Tr} \ e_{t-k} e_{t-j} B_j B_k = \sum_{k=N+1}^{N+m} \text{Tr} \ V W_k = \text{Tr} V \sum_{k=N+1}^{N+m} W_k.
\]

Thus \( S_N \) is a Cauchy sequence in the quadratic mean if and only if

\[
\text{Tr} V \sum_{n=0}^{\infty} W_n < \infty.
\]

The convergence of

\[
s_N = \sum_{j=-N}^{-1} B_j e_{t-j}
\]

can be treated analogously. □

Lemma 2.2. Let the condition (2.2) be fulfilled. Then \( X_t \) is a stationary process with vanishing mean. Let \( \gamma^{(s)}_{ij} \) be the \((i,j)\)th element of the covariance function

\[
\gamma^{(s)} = E X_{t+s} X_t^t.
\]

Then

\[
\gamma^{(s)}_{ij} = \text{Tr} V \sum_{k=\infty}^{\infty} E b_{j,k} b_{i,k+s}.
\]

Proof. The relation \( E X_t = 0 \) follows from (2.1) and from Lemma 2.1. Further we have

\[
E X_{t+s} X_t^t = E \sum_{j=\infty}^{\infty} \sum_{k=\infty}^{\infty} B_j e_{t+s-j} e_{t-k} B_k = E \sum_{j=\infty}^{\infty} \sum_{k=\infty}^{\infty} B_j e_{t+s-j} e_{t-k} B_k = E \sum_{k=\infty}^{\infty} B_{k+s} V B_k.
\]

The \((i,j)\)th element of the last expression is

\[
E \sum_{k=\infty}^{\infty} b_{i,k+s} V b_{j,k} = \text{Tr} V \sum_{k=\infty}^{\infty} E b_{j,k} b_{i,k+s}.
\]

It can be checked also directly that (2.2) implies \( \gamma^{(0)}_{ii} < \infty \) for all \( i \). From here it follows that \( \gamma^{(s)}_{ij} \) exist and are finite. □

Lemma 2.3. Let the condition (2.2) be fulfilled and let

\[
\sum_{s=\infty}^{\infty} |\gamma^{(s)}_{ij}| < \infty, \ i, j = 1, \ldots, p.
\] (2.3)
Then there exists the matrix $f(\lambda) = (f_{uv}(\lambda))$ of spectral densities of the process $X_t$ and

$$ f_{uv}(\lambda) = (2\pi)^{-1} \operatorname{Tr} V E \left( \sum_{n=-\infty}^{\infty} b'_{v,n} e^{i n \lambda} \right) \left( \sum_{n=-\infty}^{\infty} b_{u,n} e^{-i n \lambda} \right). $$

**Proof.** It is known (see Brillinger [3], Theorem 2.5.1) that a stationary process $X_t$ satisfying (2.3) possesses the matrix of spectral densities $f(\lambda)$ given by the formula

$$ f(\lambda) = (2\pi)^{-1} \sum_{s=-\infty}^{\infty} \gamma^{(s)} e^{-i s \lambda}. $$

Inserting for $\gamma^{(s)}$ we get after some computations the assertion of Lemma 2.3. □

For applications it is necessary to generalize the model (2.1) in such a way that the matrices $B_n$ are allowed to depend also on $t$.

**Theorem 2.4.** Let $B_{n,t}$ be $p \times p$ random matrices, the elements of which have finite second moments. Let $\{B_{n,t}\}$ and $\{\varepsilon_t\}$ be independent. Define

$$ X_t = \sum_{n=-\infty}^{\infty} B_{n,t} \varepsilon_{t-n}, \quad (2.4) $$

$$ W_{n,t} = E B'_{n,t} B_{n,t}, \quad B_{n,t} = \begin{pmatrix} b'_{1,t} \\ \vdots \\ b'_{p,t} \end{pmatrix}, $$

where $b'_{i,t}$ are row random vectors. Then the series (2.4) converges in the quadratic mean if and only if

$$ \operatorname{Tr} V \sum_{n=-\infty}^{\infty} W_{n,t} < \infty. \quad (2.5) $$

If (2.5) holds, then $E X_t = 0$ and the element $\gamma^{(s)}_{ij}$ of the matrix

$$ \gamma^{(s)} = EX_{t+s}X_t' $$

is given by

$$ \gamma^{(s)}_{ij} = \operatorname{Tr} V \sum_{n=-\infty}^{\infty} E b'_{i,n} b_{j,n+s}. \quad (2.6) $$

**Proof.** The first assertion follows from Lemma 2.1. Formula (2.6) can be proved in the same way as Lemma 2.2. Of course, $\gamma^{(s)}_{ij}$ as well as $\gamma^{(s)}$ generally may depend on $t$.

Notice, however, that the assumptions of Lemma 2.4 do not guarantee the stationarity of the process $X$. It is clear that $X_t$ given by (2.4) is stationary if and only if (2.5) holds for all $t$ and $\gamma^{(s)}_{ij}$ in formula (2.6) does not depend on $t$. This must be verified in special models separately.
3. MULTIPLE AR(1) PROCESS WITH RANDOM COEFFICIENTS

Consider a random process $X_t$ generated by

$$X_t = A_t X_{t-1} + e_t,$$  \hspace{1cm} (3.1)

where $A_t$ are $p \times p$ random matrices and $e_t$ is a $p$-dimensional white noise. Formula (3.1) can be rewritten into the form

$$X_t = \sum_{n=0}^{\infty} B_{n,t} e_{t-n}$$  \hspace{1cm} (3.2)

where

$$B_{0,t} = I, \quad B_{n,t} = \prod_{i=0}^{n-1} A_{t-i} \quad \text{for} \quad n \geq 1.$$

The sequence $A_t$ is called strictly stationary, if for every integer $N \geq 1$ and for arbitrary integers $t_1, ..., t_N$ the joint distribution of $(A_{t_1+h}, ..., A_{t_N+h})$ does not depend on $h$.

**Theorem 3.1.** The relation (3.1) has a solution of the type (3.2) if and only if

$$\text{Tr} \sum_{n=0}^{\infty} \sum_{i=0}^{n-1} E(\prod_{i=0}^{n-1} A_{t-i})' (\prod_{j=0}^{n-1} A_{t-j}) < \infty$$

for all integers $t$.

**Proof.** Theorem 3.1 follows from Theorem 2.4.

**Theorem 3.2.** If $A_t$ is strictly stationary, then (3.1) has a stationary solution of the type (3.2) if and only if

$$\text{Tr} \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} E A_i' A_{i+1} ... A_n A_n ... A_2 A_1 < \infty.$$  \hspace{1cm} (3.4)

**Proof.** We have from (3.3) that

$$\text{Tr} \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} E A_i' A_{i+1} ... A_n A_n ... A_2 A_1 < \infty$$

must hold for all integers $t$. Since $A_t$ is strictly stationary, the left-hand side of (3.5) does not change when we subtract $t - n$ from each index. The proof, that (2.6) does not depend on $t$, is similar.

Generally, it is extremely difficult to verify if the condition (3.4) is fulfilled. Only some special cases allow to write explicit solution.

**Theorem 3.3.** Let $\|A\|$ be a norm of a matrix $A$ satisfying $\|AB\| \leq \|A\| \cdot \|B\|$. If $\|A_t\| \leq d < 1$ for all $t$, then (3.3) and (3.4) hold.

**Proof.** is clear. \hspace{1cm} \(\square\)

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REFERENCES


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