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NONNEGATIVE MULTIVARIATE AR(1) PROCESSES

Jiří Anděl

Conditions for nonnegativity of a p-dimensional AR(1) process \( X_t = UX_{t-1} + e_t \) are investigated in the paper. If all the elements of the matrix \( U \) are nonnegative, a new method for estimating \( U \) is proposed. It is proved that the estimators are strongly consistent. Small-sample properties of the estimators are illustrated in a simulation study.

1. INTRODUCTION

A one-dimensional AR(1) process is given by \( X_t = bX_{t-1} + e_t \) where \( e_t \) is a white noise and \( b \in (-1, 1) \). Assume that \( b \in [0, 1) \) and that \( e_t \) are nonnegative independent identically distributed random variables with a distribution function \( F \). Then, of course, \( X_t \geq 0 \) for all \( t \). Let a realization \( X_1, \ldots, X_n \) be given. Then Bell and Smith [9] proved that

\[
\hat{b} = \min_{2 \leq t \leq n} \frac{X_t}{X_{t-1}}
\]

is a strongly consistent estimator for \( b \) if and only if

\[
F(d) - F(c) < 1
\]

holds for all \( 0 < c < d < \infty \). Anděl [2] derived the distribution of \( \hat{b} \) when \( e_t \) have an exponential distribution. Some moments of \( \hat{b} \) in this case were calculated by Anděl and Zvára [8]. Turkman [11] presents a Bayesian analysis of the model. A generalization to the autoregressive processes of a higher order can be found in [5]. This method was also applied to nonlinear AR process (see [4] and [6]).

In the present paper we deal with multivariate AR(1) processes. First, we derive conditions under which the process is nonnegative. Second, we propose a method for estimating parameters of a nonnegative AR(1) process. It is proved that the estimators are strongly consistent.

2. PRELIMINARIES

Let \( X_t = (X_{1t}, \ldots, X_{pt})' \) be a p-dimensional process given by

\[
X_t = UX_{t-1} + e_t
\]

(2.1)
where $U = (u_{ij})$ is a $p \times p$ matrix and $\mathbf{e}_t = (e_{t1}, \ldots, e_{tp})'$ are random vectors. We make the following assumptions.

A1. All the roots of the matrix $U$ lie inside the unit circle.
A2. The random vectors $\mathbf{e}_t$ are independent identically distributed with a distribution function $F$.
A3. The random vectors $\mathbf{e}_t$ have finite second moments.

Our assumptions ensure that there exists a stationary solution $X_t$ of the equation (2.1) and that it can be written in the form

$$X_t = \mathbf{e}_t + U\mathbf{e}_{t-1} + U^2\mathbf{e}_{t-2} + \ldots$$

where the series converges in the quadratic mean. If we denote $U^b = (u_{ij}^b)$, then (2.2) can be also expressed as

$$X_t = \mathbf{e}_t + \sum_{k=1}^{\infty} \sum_{j=1}^{p} u_{ij}^b \mathbf{e}_{t-k,j} \quad (i = 1, \ldots, p).$$

Let us remark that under A1 - A3 we have

$$\sum_{i} \sum_{j} \left| u_{ij}^b \right| < \infty.$$  

We denote $\mu_i = \mathbb{E} X_{it}, i = 1, \ldots, p.$

3. CONDITIONS FOR NONNEGATIVITY

If all the elements $u_{ij}$ of the matrix $U$ are nonnegative and all the components $e_{it}$ are also nonnegative, then it is clear that $X_{it} \geq 0$ for all $t$ and $i$. On the other hand, these sufficient conditions for nonnegativity of $X_{it}$ are not necessary (cf. Remarks 3.3 and 3.4). It is possible to generalize the results concerning one-dimensional case introduced in Lemma 10.2 in [3] to multidimensional models.

**Theorem 3.1.** Assume that the distribution of $\mathbf{e}_t$ has the property that

$$P\left( \sum_{i} e_{it} e_{it} < \varepsilon \right) > 0$$

holds for every $\varepsilon > 0$ and for every reals $e_{t1}, \ldots, e_{tp}$. If there exist numbers $q \geq 1$ and $c > 0$ such that

$$P\left( \sum_{j} u_{ij}^q \mathbf{e}_{it} < -c \right) > 0$$

for an $i \in \{1, \ldots, p\}$, then with probability 1 there exist infinitely many subscripts $t$ such that $X_{it} < 0$. 

Proof. For $m > q$ introduce the events
\[ Q_{tm1} = \left\{ \omega : e_\ell + \sum_{k=1}^{m} \sum_{j=1}^{p} u_{ij}^{(k)} e_{t-k,j} < -c \right\}, \]
\[ Q_{tm2} = \left\{ \omega : \sum_{k=m+1}^{\infty} \sum_{j=1}^{p} u_{ij}^{(k)} e_{t-k,j} < c / 2 \right\}. \]
From A3 and (2.4) we get that $P(Q_{tm2}) \rightarrow 1$ as $m \rightarrow \infty$. Moreover, $P(Q_{tm1})$ does not depend on $t$. Denote $M_{m+q} = \{1, 2, \ldots, q-1, q+1, \ldots, m\}$ for $m > q$. We have
\[ P(Q_{tm1}) \geq \pi_m, \]
where
\[ \pi_m = P\left( e_\ell < \frac{c}{2m} \sum_{j} u_{ij}^{(k)} e_{t-k,j} < \frac{c}{2m} \right) \]
\[ \text{for } k \in M_{mq}, \sum_{j} u_{ij}^{(k)} e_{t-k,j} < -c \}
\[ = P\left( e_\ell < \frac{c}{2m} \prod_{k \in M_{mq}} P \left( \sum_{j} u_{ij}^{(k)} e_{t-k,j} < \frac{c}{2m} \right) \right), \]
\[ \cdot P\left( \sum_{j} u_{ij}^{(k)} e_{t-k,j} < -c \right) > 0. \]
Let $w_m$ be the smallest integer such that $w_m \pi_m \geq 1$. Introduce the subsets $S_{q+2}, S_{q+3}, \ldots$ of positive integers in the following way. Let $S_{q+2}$ contain the elements of $w_{q+1}$ ($q+2$)-tuples $(1, \ldots, q+2), (q+3, \ldots, 2q+4), \ldots, (1 + (w_{q+1} - 1) (q+2)), \ldots, 2 + q + (w_{q+1} - 1) (q+2))$. Let $S_{q+3}$ contain the elements of $w_{q+2}$ ($q+3$)-tuples starting with
\[ (3 + q + (w_{q+1} - 1) (q+2)), \ldots, 5 + 2q + (w_{q+1} - 1) (q+2)) \]
and so on. The last terms of ($q+2$)-tuples, ($q+3$)-tuples etc. denote $t_1, t_2, \ldots$. If $t_r \in S_m$, then we use the decomposition
\[ X_{t_r,i} = U_{t_r} + Z_{t_r} \]
where
\[ U_{t_r} = e_{t_r,i} + \sum_{k=1}^{m-1} \sum_{j=1}^{p} u_{ij}^{(k)} e_{t-k,j}, \]
\[ Z_{t_r} = \sum_{k=m+1}^{\infty} \sum_{j=1}^{p} u_{ij}^{(k)} e_{t-k,j}. \]
Denote
\[ A_r = Q_{t_{r-1}, t_r-1}, \quad B_r = Q_{t_{r-1}, t_r-2}. \]
The events $A_1, A_2, \ldots$ are independent,
\[ \sum_{r=1}^{\infty} P(A_r) \geq \sum_{m=q+1}^{\infty} w_m \pi_m = \infty, \]
$P(B_r) \rightarrow 1$ as $r \rightarrow \infty$ and the events $A_r, B_r$ are independent. Theorem 8.1 yields that with probability 1 infinitely many events $A_r$ occur and thus also infinitely many events $\{X_t < 0\}$. \qed
Corollary 3.2. Let \( e_1, \ldots, e_p \) be independent nonnegative random variables. Assume that \( P(e_i < \varepsilon) > 0 \) for all \( i = 1, \ldots, p \) and for every \( \varepsilon > 0 \). Further assume that \( P(e_i = 0) < 1 \) for \( i = 1, \ldots, p \). Then the AR(1) process \( X_t \) given by (2.1) has all its components nonnegative if and only if all the elements \( u_{ij} \) of the matrix \( U \) are nonnegative.

Proof. Obviously, if all \( u_{ij} \) are nonnegative, then \( X_t \) has only nonnegative components. Now, assume that there exists a pair \((i,j)\) such that \( u_{ij} < 0 \). Our assumptions ensure that (3.1) is fulfilled. Since \( e_{ij} \) are nonnegative and \( P(e_{ij} = 0) < 1 \), there exists \( c > 0 \) such that

\[
P(u_{ij} e_{ij} < -2c) > 0.
\]

Then we have

\[
P\left( \sum_{n} u_{mn} e_{nm} < -c \right) \geq P\left( u_{ij} e_{ij} < -2c \right) \geq P\left( \sum_{m \neq ij} u_{mn} e_{nm} < c \right).
\]

\[\square\]

Remark 3.3. If all the elements \( e_{ij} \) are nonnegative, then all \( X_t \) can be nonnegative even if some elements \( u_{ij} \) of the matrix \( U \) are negative.

We can demonstrate this fact by the following example. Let \( p = 2 \), \( e_{11} \geq 0 \) and \( e_{12} = e_{21} \). Let \( c \) and \( u \) be numbers such that \( 0 < c < u \) and \( u + c < 1 \). Consider the matrix

\[
U = \begin{pmatrix}
  u & -c \\
 -c & u
\end{pmatrix}.
\]

The roots of \( U \) are \( \lambda_1 = u \pm c \), and thus \( |\lambda_1| < 1, |\lambda_2| < 1 \). Since

\[
U^n = \frac{1}{2}(u + c)^n \begin{pmatrix}
  1 & -1 \\
 -1 & 1
\end{pmatrix} + \frac{1}{2}(u - c)^n \begin{pmatrix}
  1 & 1 \\
 -1 & -1
\end{pmatrix}
\]

and \( e_t = (e_{11}, e_{21})' \), we have

\[
U^n e_{t+1} = (u - c)^n \begin{pmatrix}
  e_{11}, e_{21}
\end{pmatrix}.
\]

Thus \( U^n e_{t+1} \) is a random vector with nonnegative components. Taking into account (2.2) we can see that the same is true for \( X_t \).

Remark 3.4. If all the elements \( u_{ij} \) of the matrix \( U \) are positive, then all the variables \( X_t \) can be positive even if a component of the random vector \( e_t \) is negative.

Choose again \( p = 2 \). Let \( 0 < a < e_{11} < b \) and define \( e_{12} = -\frac{1}{2} e_{11} \). Let \( c \in (0, \frac{1}{2}) \).

Consider the matrix

\[
U = \begin{pmatrix}
  c & c \\
 c & c
\end{pmatrix}.
\]
The roots of $U$ are $\lambda_1 = 0$, $\lambda_2 = 2c$. Both of them lie inside the unit circle. Since

$$U^n = 2^{n-1} c^n \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} , \quad n \geq 1,$$

we get

$$U^n e_{t-n} = 2^{n-2} c^n \begin{pmatrix} e_{t-n,1} \\ e_{t-n,1} \end{pmatrix} , \quad n \geq 1,$$

From (2.2) we have

$$X_{11} = e_{11} + \sum_{n=1}^{\infty} 2^{n-2} c^n e_{t-n,1},$$

$$X_{12} = -\frac{1}{2} e_{11} + \sum_{n=1}^{\infty} 2^{n-2} c^n e_{t-n,1}.$$

It is clear that $X_{11} > 0$. If we take $c = 0.4$, $a = 1$, $b = 2$, then

$$X_{12} > -\frac{1}{2} b + \sum_{n=1}^{\infty} 2^{n-2} c^n a = 0.$$

4. AUXILIARY RESULTS FOR ESTIMATION

Till the end of this paper we assume that not only $A_1 - A_3$, but also the following assumptions $B_1 - B_4$ are satisfied.

$B_1$. All the elements $u_{ij}$ of the matrix $U$ are nonnegative.

$B_2$. Random vectors $e_t$ have only nonnegative components.

$B_3$. $P \left( \epsilon_i < \eta_1, \ldots, \epsilon_{i,p} < \eta \right) > 0$ for all $z > 0$.

$B_4$. There exists a number $\gamma > 0$ such that for every $\eta > 0$ and for each $i \in \{1, \ldots, p\}$

$$P \left( \epsilon_i < \eta, \ldots, \epsilon_{i,p} < \eta, \eta_i > \gamma, \epsilon_{i+1} < \eta, \ldots, \epsilon_{i+p} < \eta \right) > 0.$$

It was already pointed out that $A_1 - A_3$, $B_1$, $B_2$ ensure nonnegativity of all variables $X_{it}$.

**Remark 4.1.** Let $p = 2$. If $B_1$ holds, then $U$ has only real roots. Really, an easy calculation gives

$$|U - \lambda I| = \lambda^2 - (u_{11} + u_{22}) \lambda + u_{11} u_{22} - u_{12} u_{21}$$

and thus the roots are

$$\lambda_{1,2} = -\frac{1}{2} \left\{ u_{11} + u_{22} \pm \left[ (u_{11} - u_{22})^2 + 4u_{12} u_{21} \right]^{\frac{1}{2}} \right\}.$$
Remark 4.2. The assumptions B3 and B4 are independent. This can be shown in an example with \( p = 2 \). If \( \mathbb{P}(\varepsilon_1 = 0, \varepsilon_2 = 0) = 1 \), then B3 is fulfilled but B4 does not hold. If \( \mathbb{P}(\varepsilon_1 = 0, \varepsilon_2 = 5) = \mathbb{P}(\varepsilon_1 = 5, \varepsilon_2 = 0) = \frac{1}{2} \), then B3 is not fulfilled but B4 holds.

Remark 4.3. Consider the case \( p = 2 \). Let \( \xi_i \) be i.i.d. random variables with exponential distribution \( \text{Exp}(\lambda) \) where \( i = 1, 2, 3 \) and \( t = \ldots, -1, 0, 1, \ldots \). If \( \varepsilon_i = \xi_1 + \xi_3, \varepsilon_2 = \xi_2 + \xi_3 \), then the condition B4 is fulfilled, since

\[
\mathbb{P}(\xi_1 + \xi_3 < \eta, \xi_2 + \xi_3 > \gamma) \geq \mathbb{P}(\xi_1 < \frac{\eta}{2}, \xi_3 < \frac{\eta}{2}, \xi_2 > \gamma) = \mathbb{P}(\xi_1 < \frac{\eta}{2}) \mathbb{P}(\xi_3 < \frac{\eta}{2}) \mathbb{P}(\xi_2 > \gamma) > 0
\]

for every \( \eta > 0, \gamma > 0 \). If \( \varepsilon_i = \xi_1 + \xi_2, \varepsilon_2 = \xi_1 \), then \( \mathbb{P}(\xi_1 + \xi_2 < \eta, \xi_1 > \gamma) = 0 \) for every \( 0 < \eta < \gamma \), and thus B4 is not fulfilled.

Theorem 4.4. Define

\[ u_{ij}^n = \min \left( \frac{X_{ii}}{X_{i-1, i}} \right) \]

for \( i, j = 1, \ldots, p \). Then \( u_{ij}^n \to u_{ij} \) a.s. as \( n \to \infty \) for each \( i, j \in \{1, \ldots, p\} \).

Proof. First, consider the case \( i = j = 1 \). Since

\[ X_{ii} = \sum_{\beta=1}^{p} u_{\beta i} X_{i-1, \beta} + \varepsilon_i, \]

we obtain

\[ u_{11}^n = u_{11} + \min_{2 \leq n \leq \infty} \left( \frac{\sum_{\beta=2}^{p} u_{\beta i} X_{i-1, \beta} + \varepsilon_i}{X_{i-1, 1}} \right). \]

Since \( X_{i-1, 1} \geq e_{i-1, 1} \), it is sufficient to prove that

\[ \min_{2 \leq n \leq \infty} \left( \frac{\sum_{\beta=2}^{p} u_{\beta i} X_{i-1, \beta} + \varepsilon_i}{X_{i-1, 1}} \right) / e_{i-1, 1} \to 0 \quad \text{a.s.} \]

Let \( \varepsilon > 0 \) be a given number. Consider the events

\[ Q_\varepsilon = \left\{ \omega : \left( \frac{\sum_{\beta=2}^{p} u_{\beta i} X_{i-1, \beta} + \varepsilon_i}{e_{i-1, 1}} \right) / e_{i-1, 1} < \varepsilon \right\}. \]

Using (2.3) we can write

\[ Q_\varepsilon = \left\{ \omega : \varepsilon_1 + \sum_{\beta=2}^{p} u_{\beta i} \left( e_{i-1, \beta} + \sum_{k=2}^{m} \sum_{r=1}^{p} u_{jr} e_{r-k,r} \right) + \sum_{\beta=3}^{p} u_{\beta i} \sum_{k=2}^{m} \sum_{r=1}^{p} u_{jr} e_{r-k,r} < \varepsilon e_{i-1, 1} \right\}. \]
Denote $A = 2p[1 + (p - 1)(m - 1)]$. It is clear that $Q_t \supset Q_{t1m} \cap Q_{t2m}$ where

$$Q_{t1m} = \{ \omega : e_{t-1,1} > \gamma, e_{t1} < \varepsilon / A, u_{1 \beta} e_{t-1, \beta} < \varepsilon / A \text{ for } \beta = 2, \ldots, p, \}$$

$$Q_{t2m} = \{ \omega : Z_{tm2} < \varepsilon / 2 \}$$

with

$$Z_{tm2} = \sum_{\beta=1}^{p} u_{1 \beta} \sum_{k=m+1}^{\infty} \sum_{r=1}^{m} u_{r \beta}^{(k)} e_{t-k,r}.$$  

From (2.4) we can see that there exists $\Lambda > 0$ such that

$$0 \leq u_{0j}^{(k)} < \Lambda \quad \text{for all } i, j, k.$$  

Therefore $P(Q_{t1m}) \geq \pi_m$ where

$$\pi_m = P(e_{t1} < \varepsilon / A) P\left(e_{t-1,1} > \gamma, e_{t-1, \beta} < \frac{\varepsilon A}{A} \text{ for } \beta = 2, \ldots, p\right) \cdot P\left(e_{t-k,r} < \frac{\varepsilon A}{A^2} \text{ for } r = 1, \ldots, p\right).$$

Our assumptions imply that neither $P(Q_{t1m})$ nor $\pi_m$ depend on $t$. The value of $\gamma$ can be chosen in such a way that $\pi_m > 0$.

It is easy to show that $E Z_{tm2} \to 0$ and $\text{var } Z_{tm2} \to 0$ as $m \to \infty$ for every fixed $t$. Thus $P(Q_{tm2}) \to 1$. Moreover, $P(Q_{tm2})$ also does not depend on $t$.

Let $w_1$ be the smallest integer such that $w_1 \pi_m \geq 1$ ($m = 2, 3 \ldots$). Let the set $S_2$ contain elements of $j_2$ triples

$$(1, 2, 3), \ldots, (3j_2 - 2, 3j_2 - 1, 3j_2),$$

let $S_3$ contain elements of $j_3$ four-tuples

$$(3j_3 + 1, 3j_3 + 2, 3j_3 + 3, 3j_3 + 4), \ldots, (3j_3 + 4j_3 - 3, \ldots, 3j_3 + 4j_3)$$

and so on. The last numbers of the triples, four-tuples etc. denote $t_1, t_2, \ldots$. If $i_n \in S_m$, then we define

$$A_i = Q_{i,n1}, \quad B_i = Q_{i,n2}.$$  

The events $A_1, A_2, \ldots$ are independent,

$$\sum_{i=1}^{\infty} P(A_i) \geq \sum_{n=2}^{\infty} w_n \pi_m = \infty,$$

events $A_i$ and $B_i$ are independent for each $i$, and $P(B_i) \to 1$ as $i \to \infty$. It follows from Theorem 8.1 that with probability 1 infinitely many events $A_i \cap B_i$ occur, and thus also infinitely many events $Q_t$.

The proof for other estimators $u_{0j}^{(k)}$ is quite similar.
Although \( \hat{u}_{ij} \) are strongly consistent estimators for \( u_{ij} \), our experience from similar models (see [5]) leads to the suspicion that the convergence \( \hat{u}_{ij} \rightarrow u_{ij} \) a.s. as \( n \rightarrow \infty \) is too slow and \( \hat{u}_{ij} \) cannot be used in practical situations as reasonable estimators. Simulations really confirmed this fact. In the next section we propose other estimators, which are also strongly consistent, but which are good for moderate values of \( n \).

5. ESTIMATING PARAMETERS

To simplify the notation and the proofs, we describe the estimating procedure in this section only in the case \( p = 2 \). First, we introduce a motivation for our estimators. Let \( \epsilon_{11}, \epsilon_{12} \) be independent random variables such that \( \epsilon_{11} \sim \text{Ex}(\lambda_1), \epsilon_{12} \sim \text{Ex}(\lambda_2) \), where \( \text{Ex}(\lambda) \) denotes the exponential distribution with the density \( f(x) = \lambda e^{-\lambda x} \) for \( x > 0 \). Then the conditional likelihood of \( X_2, \ldots, X_n \), given \( X_1 \), is

\[
\lambda_1^{n-1} \exp \left\{ - \sum_{t=2}^{n} (X_{11} - u_{11} X_{t-1,1} - u_{12} X_{t-1,2}) / \lambda_1 \right\} 
\cdot \lambda_2^{n-1} \exp \left\{ - \sum_{t=2}^{n} (X_{12} - u_{21} X_{t-1,1} - u_{22} X_{t-1,2}) / \lambda_2 \right\}
\]

for

\[
\begin{align*}
X_{11} - u_{11} X_{t-1,1} - u_{12} X_{t-1,2} & \geq 0, \\
X_{12} - u_{21} X_{t-1,1} - u_{22} X_{t-1,2} & \geq 0
\end{align*}
\quad (5.1, 5.2)
\]

\((t = 2, \ldots, n)\). The conditional likelihood reaches its maximum for such \( u_{11} \) and \( u_{12} \) which maximize

\[
u_{11} \sum_{t=2}^{n} X_{t-1,1} + u_{12} \sum_{t=2}^{n} X_{t-1,2}
\]

under the conditions (5.1) with \( u_{11} \geq 0, u_{12} \geq 0 \), and for such \( u_{21} \) and \( u_{22} \) which maximize

\[
u_{21} \sum_{t=2}^{n} X_{t-1,1} + u_{22} \sum_{t=2}^{n} X_{t-1,2}
\]

under the conditions (5.2) with \( u_{21} \geq 0, u_{22} \geq 0 \). Define

\[
X_{11}^0 = n^{-1} \sum_{t=1}^{n} X_{t1}, \quad X_{12}^0 = n^{-1} \sum_{t=1}^{n} X_{t2}.
\]

If \( n \) is large then one can expect that the maximization of (5.3) and (5.4) is nearly the same as the maximization of \( X_{11}^0 u_{11} + X_{12}^0 u_{12} \) and \( X_{21}^0 u_{21} + X_{22}^0 u_{22} \), respectively.
Theorem 5.1. Let \( u_{1}^{*}, u_{2}^{*} \) be a solution of the linear program LP\((n)\)
\[
\max (X_{11}^{u_{1}} v_{11} + X_{12}^{u_{2}} v_{12})
\] (5.5)
under conditions
\[
X_{t} - v_{1} X_{t-1,1} - v_{2} X_{t-1,2} \geq 0 \quad (t = 2, \ldots, n)
\]
with \( v_{1}, v_{2} \geq 0 \), for \( i = 1, 2 \). Then \( u_{i}^{*} \rightarrow u_{i} \) a.s. for all \( i, j = 1, 2 \) as \( n \rightarrow \infty \).

Proof. Let \( i = 1 \). Assume that \( u_{11} > 0 \), \( u_{12} > 0 \). Define
\[
M_{n} = \{(v_{11}, v_{12}) : v_{11} \geq 0, v_{12} \geq 0, X_{11} - v_{11} X_{21,1} - v_{12} X_{21,2} \geq 0 \text{ for } t = 2, \ldots, n\}.
\]

Let \( M \) be the oblong with vertices \((0,0), (u_{11},0), (u_{11},u_{12}), (0,u_{12})\). It is clear that \( M_{2} \supseteq M_{3} \supseteq \ldots \). First we prove that \( M_{n} \rightarrow M \) a.s.

We have
\[
\frac{X_{11}}{X_{21,1}} = u_{11} + \frac{X_{21,2}}{X_{21,1}} u_{12} + \frac{e_{11}}{X_{21,1}}.
\] (5.6)

Theorem 4.4 implies that there exists a sequence \( t \), such that
\[
X_{11} / X_{21,1} \rightarrow u_{11} \text{ a.s.}
\]
In view of (5.6) we can see that
\[
X_{21,2} / X_{21,1} \rightarrow u_{11} \text{ a.s.}
\] (5.7)

Since
\[
\frac{X_{11}}{X_{21,2}} = u_{12} + \frac{X_{21,1}}{X_{21,2}} u_{11} + \frac{e_{11}}{X_{21,2}},
\]
using (5.7) we obtain
\[
X_{11} / X_{21,2} \rightarrow \infty \text{ a.s.}
\]

In this case the straight line \( p \) in Figure 1 approaches the straight line \( q_{1} \). Similarly we can prove that with probability 1 there exists a sequence of straight lines \( p \) converging to \( q_{2} \).
An elementary calculation gives that \( p \) intersects \( q_i \) at the point 
\[
\left( u_{11}, u_{12} + \frac{\epsilon_{11}}{X_{t-1,2}} \right)
\]
and thus no straight line \( p \) intersects \( M \).

Consider the linear program \( LP(n) \) (5.5) for \( i = 1 \). It concerns the problem 
\[
\max (X_{v1}^{n} v_{11} + X_{v2}^{n} v_{12})
\]
on \( M_n \). Since \( M_n \to M \) and \( X_{v1}^{n} \to \mu_1, \ X_{v2}^{n} \to \mu_2 \) a.s. (see [10], Chap. IV.2), the solutions \( (u_{11}^{n}, u_{12}^{n}) \) of \( LP(n) \) converge a.s. to a solution of the linear program 
\[
\max (\mu_1 v_{11} + \mu_2 v_{12})
\]on \( M \). It is clear that the maximum (5.8) on \( M \) is reached at the point \( (u_{11}, u_{12}) \). Thus we have proved that \( u_{11}^{n} \to u_{11}, \ u_{12}^{n} \to u_{12} \) a.s.

If \( u_{11} = 0 \) and/or \( u_{12} = 0 \), the proof is similar. The case \( i = 2 \) is quite analogous. •

6. A SIMULATION STUDY

We simulated the two-dimensional AR(1) process 
\[
X_t = UX_{t-1} + \epsilon_t
\]
with 
\[
U = \begin{pmatrix}
0.7 & 0.3 \\
0.1 & 0.5
\end{pmatrix}
\]
The roots of \( U \) are \( \lambda_1 = 0.8, \lambda_2 = 0.4 \). The white noise \( \epsilon_t = (\epsilon_{1t}, \epsilon_{2t})' \) was constructed in such a way that 
\[
\epsilon_{1t} = \ell_1 \xi_{1t} + \ell_3 \xi_{3t}, \quad \epsilon_{2t} = \ell_2 \xi_{1t} + \ell_3 \xi_{3t}
\]
where \( \ell_1, \ell_2, \ell_3 \) were nonnegative constants and \( \xi_{it} \) were nonnegative i.i.d. variables. Three distributions of \( \xi_{it} \) were examined:

(i) exponential distribution \( Ex(1) \);
(ii) absolutely normal distribution \( AN(0,1) \); i.e. \( \xi_{it} = |U_{it}| \), where \( U_{it} \sim N(0,1) \);
(iii) rectangular distribution \( R(0,1) \) with the density \( f(x) = 1 \) for \( x \in (0,1) \).

The results of simulations are summarized in Tables 1-5. In each case 100 simulations were performed. The tables contain averages of estimates of the elements of the matrix \( U \). The empirical standard deviations are introduced in parentheses.
A simulation of length \( n = 50 \) with \( \ell_1 = \ell_2 = \ell_3 = 1 \) and \( \xi_{it} \sim \text{Ex}(1) \) is depicted in Figure 2.

Fig. 2.

The estimate of the matrix \( U \) for this simulation is

\[
\begin{pmatrix}
0.74 & 0.26 \\
0.11 & 0.53
\end{pmatrix}
\]

A simulation of length \( n = 50 \) with \( \ell_1 = \ell_2 = \ell_3 = 1, \xi_{it} \sim R(0, 1) \) is depicted in Figure 2.

Table 1
\[
\begin{bmatrix}
0.70 & 0.37 \\
0.13 & 0.50
\end{bmatrix}
\]

Table 2
\[
\begin{bmatrix}
0.71 & 0.37 \\
0.17 & 0.46
\end{bmatrix}
\]

Table 3
\[
\begin{bmatrix}
0.68 & 0.44 \\
0.19 & 0.43
\end{bmatrix}
\]

Table 4
\[
\begin{bmatrix}
0.70 & 0.33 \\
0.11 & 0.51
\end{bmatrix}
\]

Table 5
\[
\begin{bmatrix}
0.71 & 0.32 \\
0.11 & 0.51
\end{bmatrix}
\]
The experience from our simulations can be briefly summarized as follows. Tables 1 - 4 show that the estimates are better when the distribution of residuals is nearer to the exponential one. This is not surprising, since our method was motivated by the maximum likelihood estimators for exponential distribution. The best results among Tables 1 - 4 are contained in Table 4. The same quality in the case \( \ell_1 = \ell_2 = \ell_3 = 1, \xi_\ell \sim E x(1) \), is reached only when the length of simulation is enlarged from \( n = 20 \) to \( n = 50 \) (see Table 5).

Let us remark that the least squares estimates of the elements of the matrix \( U \) for the simulation depicted in Figure 2 are

\[
\begin{pmatrix}
0.53 & 0.41 \\
0.05 & 0.48
\end{pmatrix}
\]

(Of course, first of all the average of the both components of the series were subtracted.) The corresponding asymptotic standard deviations are

\[
\begin{pmatrix}
0.14 & 0.17 \\
0.16 & 0.19
\end{pmatrix}
\]

In this case the estimates obtained by the new method are better. Also the empirical standard deviations introduced in Table 5 are smaller than the asymptotic standard deviations of the least squares estimates.

7. ANALYSIS OF REAL DATA

Anděl [1] presents some hydrological data about the small river Volýňka in Czechoslovakia. The mean hourly discharges of the Volýňka river (in m\(^3\)/s) and hourly rainfall in the Volýňka basin were measured for three days. The data are graphically presented in Figure 3.

Denote \( X_{1t} \) the discharges and \( X_{2t} \) the rainfall \((t = 1, \ldots, 72)\). The averages are

\[
\bar{x}_1 = \frac{1}{72} \sum_{t=1}^{72} X_{1t} = 31.78, \quad \bar{x}_2 = \frac{1}{72} \sum_{t=1}^{72} X_{2t} = 0.36
\]

and the empirical variances of the components are

\[
s_1^2 = 207.59, \quad s_2^2 = 0.53.
\]

The least squares estimates of the autoregressive parameters are

\[
\begin{pmatrix}
0.97 & 1.08 \\
0.00 & 0.76
\end{pmatrix}
\]

and their asymptotic standard deviations are

\[
\begin{pmatrix}
0.025 & 0.498 \\
0.004 & 0.075
\end{pmatrix}
\]
The residual variance matrix is

\[
\begin{pmatrix}
9.37 & 0.02 \\
0.02 & 0.21 \\
\end{pmatrix}
\]

Fig. 3.

Applying our new method we get the estimate of the matrix \( U \)

\[
\begin{pmatrix}
0.87 & 1.68 \\
0.00 & 0.00 \\
\end{pmatrix}
\]

The residual variance of the first component is in this case 11.80.

**APPENDIX**

**Theorem 8.1.** Let two sequences of events \( A_1, A_2, \ldots \) and \( B_1, B_2, \ldots \) satisfy the following conditions:

(i) The events \( A_1, A_2, \ldots \) are independent.

(ii) The events \( A_i \) and \( B_i \) are independent for every \( i = 1, 2, \ldots \).

(iii) \( \sum P(A_i) = \infty \).
(iv) $P(B_i) \to 1$ as $i \to \infty$.

Then with probability one infinitely many events $C_i = A_i \cap B_i$ occur.

**Proof.** See [7]. \( \Box \)

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