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Finding a Spanning Tree of a Graph with Maximal Number of Terminal Vertices

BOHDAN ZELINKA

In [1] V. G. Vizing proposes the problem of finding an algorithm for finding a spanning tree of a given finite graph which would have the maximal number of terminal vertices. Here we shall give such an algorithm.

At first we shall define some concepts and prove some theorems. We shall always consider finite undirected graphs without loops and multiple edges. The given problem is closely connected with the problem of finding the most economical communication network. If some spanning tree has the maximal number of terminal vertices, it has the minimal number of the vertices of degree greater than one; this is a relevant requirement for the simplicity of a communication network in the sense of practical applications.

Let G be a graph with the vertex set V . For each set $S \subset V$ we shall define the neighbourhood $N(S)$ of S as the subset of $V - S$ consisting of vertices which are joined with at least one vertex of S . The cardinality of the neighbourhood $N(S)$ of S is called the degree of S and denoted by $q(S)$. It follows from this definition that $q(\emptyset) = q(V) = 0$ and if S is a one-element set $\{u\}$, where $u \in V$, then $q(S)$ is equal to the degree of u in the usual sense. This degree will be denoted by $q(u)$ rather than $q(\{u\})$.

Theorem 1. *Let G be a finite graph with the vertex set V of the cardinality n . Let $r = \max_{S \subset V} q(S)$. Then there exists a set $S_0 \subset V$ such that $|S_0| = n - r$,* $q(S_0) = r$.*

Proof. Let S be some subset of V such that $q(S) = r$. This means that the neighbourhood $N(S)$ of S has the cardinality r ; as $N(S) \subset V - S$, we have $|S| + r \leq n$, thus $|S| \leq n - r$. If $|S| = n - r$, the proof is finished. If $|S| < n - r$, then $|S \cup N(S)| < n$ and there exists at least one vertex of V which is neither in S , nor in $N(S)$.

* $|S_0|$ denotes the cardinality of S_0 .

Let u be such a vertex, let P be the shortest path joining u with a vertex of S , let l be the length of P . As $u \notin N(S)$, we have $l \geq 2$. Let the vertices of P in the direction from a vertex of S to u be $v_0, v_1, \dots, v_l = u$. We have $v_0 \in S, v_1 \in N(S)$. If some v_i were in S for $1 \leq i \leq l$, the subpath of P joining v_i with u would be a path of the length $l - i$ joining u with a vertex of S , which is a contradiction with the minimality of l . If some v_i were in $N(S)$ for $2 \leq i \leq l$, then the subpath of P joining v_i with u together with the edge joining v_i with a vertex of S would form a path of the length $l - i + 1$, which is again a contradiction with the minimality of l . Denote $S' = S \cup \{v_1, \dots, v_{l-1}\}$. The set S' has the cardinality greater than S . Any element of $N(S)$ different from v_1 is also in $N(S')$ and moreover $u \in N(S')$. Therefore $|N(S')| \geq r$, thus, as r is maximal, $|N(S')| = \varrho(S) = r$. If $|S'| = n - r$, the proof is finished, if not, we proceed by the same way as in the preceding case and obtain a set of greater cardinality than S' and of the degree r . After a finite number of such steps we must obtain a set S_0 with $|S_0| = n - r, \varrho(S_0) = r$.

Now let $\mathfrak{C}(G)$ be the set of all subsets of the vertex set V of G which induce connected subgraphs of G .

Theorem 2. Let G be a finite graph with the vertex set V of the cardinality n . Let

$$r = \max \varrho(S), S \in \mathfrak{C}(G).$$

Then there exists a set $S_0 \in \mathfrak{C}(G)$ such that $|S_0| = n - r, \varrho(S_0) = r$.

Proof is the same as the proof of Theorem 1. Here if $S \in \mathfrak{C}(G)$, then also $S' = S \cup \{v_1, \dots, v_{l-1}\} \in \mathfrak{C}(G)$.

Theorem 3. Let G be a finite graph with the vertex set V . Let

$$r = \max \varrho(S), S \in \mathfrak{C}(G).$$

Then if $\varrho(S_0) = r$ for some $S_0 \in \mathfrak{C}(G)$, then S_0 contains no terminal vertices of G .

Proof. Assume that S_0 contains a terminal vertex u of G , i.e., a vertex of the degree 1. As S_0 induces a connected subgraph of G , the vertex v joined with u by an edge in G belongs to S_0 . Therefore u is joined with no vertex of $N(S_0)$, which means $N(S_0) \subset N(S_0 - \{u\})$. Moreover, $N(S_0 - \{u\})$ contains u and therefore $\varrho(S_0 - \{u\}) = r + 1$, which is a contradiction with the maximality of r .

Theorem 4. Let G be a finite graph with the vertex set V of the cardinality at least 3. The maximal number of terminal vertices of a spanning tree of G is equal to the maximal degree of a subset of the vertex set of G inducing a connected subgraph of G .

Proof. Let us denote the maximal number of terminal vertices of a spanning tree of G by r , the maximal degree of a subset of V inducing a connected subgraph of G

by r' . According to Theorem 2 there exists a set $S \subset V$ inducing a connected subgraph of G and such that $\varrho(S) = r'$, $S \cup N(S) = V$. Choose an arbitrary spanning tree T' of the subgraph of G induced by S . At each vertex of $N(S)$ choose an edge joining this vertex with a vertex of S . Add these edges to the spanning tree T' . The resulting graph is evidently a spanning tree of G and all vertices of $N(S)$ are its terminal vertices. As $|N(S)| = \varrho(S) = r'$, we have $r' \leq r$. Now let T be a spanning tree of G with r terminal vertices. Let S_0 be the set of inner vertices of T (vertices whose degree is different from 1). The set S_0 induces a subtree in T , therefore it induces also a connected subgraph in G . Each terminal vertex of T is joined with some vertex of S_0 , therefore it belongs to $N(S_0)$ and $|N(S_0)| \geq r$, thus $\varrho(S_0) \geq r$ and $r' \geq r$. From the inequalities $r' \leq r$, $r' \geq r$ we obtain $r' = r$.

Now we see that to find a spanning tree of G with the maximal number of vertices means to find a set $S \in \mathfrak{C}(G)$ with the maximal degree and such that $S \cup N(S) = V$. When looking for such a set we take the sets with a smaller cardinality sooner than the sets with a greater cardinality. Further, we take into account that for each set $S \in \mathfrak{C}(G)$ we have $\varrho(S) \leq \sum_{u \in S} \varrho(u) - 2|S| + 2$, because the subgraph of G induced by S is connected, and thus it contains at least $|S| - 1$ edges. Obviously we take into account also Theorem 3.

We can describe an algorithm.

Algorithm. Let a finite connected graph G with the vertex set V be given, $|V| = n$. Let $\mathfrak{C}(G)$ be the set of all subsets of V which induce connected subgraphs of G . We shall use two variables k and R which are both integers.

- (A) Put $k := 1, R := 0$.
- (B) By \mathfrak{M}_1 denote the set of all one-element subsets of V . Find $\varrho(S)$ for each $S \in \mathfrak{M}_1$. Go to (C).
- (C) If $\varrho(S) = n - k$ for some $S \in \mathfrak{M}_k$, go to (E), else put $k := k + 1, R := \max_{S \in \mathfrak{M}_k} \varrho(S)$ and go to (D).
- (D) By \mathfrak{M}_k denote the set of all subsets S of V for which

$$\begin{aligned} S &\in \mathfrak{C}(G), \\ |S| &= k, \\ S \cap \{u \in V \mid \varrho(u) = 1\} &= \emptyset, \\ \sum_{u \in S} \varrho(u) &\geq R + 2k - 2. \end{aligned}$$

Find $\varrho(S)$ for each $S \in \mathfrak{M}_k$; if for some sets S, S' from \mathfrak{M}_k the inequality $\sum_{u \in S} \varrho(u) > \sum_{u \in S'} \varrho(u)$ holds, then take S sooner than S' . Go to (C).

- (E) Take a set $S \in \mathfrak{M}_k$ for which $\varrho(S) = n - k$. Choose a spanning tree T of the

360 subgraph of G induced by S . At each vertex of $N(S)$ choose an edge joining it with a vertex of S and add this edge to T . The resulting graph is the required spanning tree.
If we want to obtain all spanning trees of G with the maximal number of terminal vertices, we do (E) in all possible ways.

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