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# MAXIMAL FUZZY TOPOLOGIES

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In this paper we introduce and study maximal fuzzy $P$-spaces where $P$ is fuzzy Lindelöf, fuzzy countably compact, fuzzy compact, fuzzy lightly compact or fuzzy strongly compact. Characterizations are given for maximal fuzzy $P$-spaces where $P$ is fuzzy Lindelöf, fuzzy countably compact, fuzzy compact, or fuzzy strongly compact. Necessary condition is given for maximal fuzzy lightly compact spaces and fuzzy connected spaces.

## 1. INTRODUCTION

This paper can be considered as a continuation of [4] and it presents interesting relations among various notions derived from maximal fuzzy topologies. A fuzzy topological space [5] with property $P$ is said to be maximal $P$ if there is no strictly larger fuzzy topology on $X$ with property $P$. In this paper we shall investigate maximal fuzzy $P$-spaces where $P$ is fuzzy connectedness, fuzzy lightly compactness etc in a manner similar to that of [2].

## 2. FUZZY CONNECTED SPACES

A fuzzy topological space $(X, T)$ is defined to be fuzzy connected [6] if it has no proper fuzzy clopen set. We define such a fuzzy connected space to be maximal fuzzy connected if any fuzzy connected topology stronger than $T$ necessarily coincides with $T$. If $\lambda$ is a fuzzy set in a fuzzy topological space then the closure and the interior of $\lambda$ will be as usual denoted by $\overline{\lambda}$ and $\lambda^0$ respectively. A fuzzy set $\lambda$ is called fuzzy regular open [4] if $\lambda = (\overline{\lambda})^0$. Given any fuzzy topological space $(X, T)$, the fuzzy regular open sets in $T$ form a base for a unique fuzzy topology $T_0$ called the fuzzy semi-regular topology on $X$ associated with $T$. A fuzzy topology $T$ is fuzzy semi-regular [3] $\Leftrightarrow T = T_0$.

Let $(X, T_0)$ be a fuzzy semi-regular space.

Let $E(T_0) = \{T^* | T^* \text{ is a fuzzy topology on } X \text{ and } (T^*)_0 = T_0\}$. For any two elements $T_1, T_2$ in $E(T_0)$ define $T_1 \leq T_2$ if $T_1$ is weaker than $T_2$. It can be shown that $E(T_0)$ has a maximal element. A maximal element of $E(T_0)$ is called a sub-maximal fuzzy topology and $X$ endowed with such a fuzzy topology is referred to as a submaximal fuzzy topological space. In this connection we establish the following results:

**Proposition 1.** A fuzzy topological space $(X,T)$ is fuzzy connected $\Leftrightarrow$ $(X,T_0)$ is fuzzy connected.

Proof. Suppose $T$ is not fuzzy connected. Then there exists a proper fuzzy set $\lambda$ which is both open and closed, so $\lambda$ is regular open and regular closed. Therefore $T_0$ is not fuzzy connected which is a contradiction. The converse follows since $T_0$ is weaker than $T$. □

The following two results can be deduced from the above Proposition 1.

**Proposition 2.** A maximal fuzzy connected space is submaximal.

**Proposition 3.** A fuzzy topology $T$ on $X$ is submaximal $\Leftrightarrow$ every fuzzy set $\lambda$ in $X$ such that $\overline{\lambda} = 1$ belongs to $T$.

## 3. FUZZY LIGHTLY COMPACT SPACE

We define the concept of fuzzy lightly compact space based on the corresponding concept in topology given in [2].

**Definition.** A fuzzy topological space $(X,T)$ is said to be fuzzy lightly compact if for all $\{\lambda_i\}_{i=1}^{\infty} \subset T$ with $\sup\{\lambda_i\} = 1$, there exists an $n_0 \in N$ such that

$$\sup\left\{\overline{\lambda_i}\right\}_{i=1}^{n_0} = 1.$$

In this connection one can prove the following results easily.

**Proposition 4.** A space $(X,T)$ is fuzzy lightly compact $\Leftrightarrow$ $(X,T_0)$ is fuzzy lightly compact.

**Proposition 5.** A maximal fuzzy lightly compact space $(X,T)$ is submaximal.

The converse of Proposition 5 is not true as the following example shows.

**Example.** Let $X = \{x_0, x_1, x_2, \ldots\}$ be a set of points. Let $A$ be any element in $\mathcal{P}(X - x_0)$. Define

$$f_0 : X \longrightarrow [0,1] \quad \text{as} \quad f_0(x) = 0 \quad \text{for all } x \in X,$$

$$f_1 : X \longrightarrow [0,1] \quad \text{as} \quad f_1(x) = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{if } x \neq x_0 \end{cases}$$

$$\text{and} \quad f_A : X \longrightarrow [0,1] \quad \text{as} \quad f_A(x) = \begin{cases} 1 & \text{if } x \in A \cup \{x_0\} \\ 0 & \text{if } x \notin A \cup \{x_0\} \end{cases}.$$

Let $T = \{f_0, f_1, f_A | A \in \mathcal{P}(X - x_0)\}$. Then $(X,T)$ is fuzzy lightly compact and submaximal.

Let us now fix another point $x_1 \in X$ and define

$$g_0 : X \longrightarrow [0,1] \quad \text{as} \quad g_0(x) = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{otherwise} \end{cases}$$

$$g_1 : X \longrightarrow [0,1] \quad \text{as} \quad g_1(x) = \begin{cases} 1 & \text{if } x = x_1 \\ 0 & \text{otherwise.} \end{cases}$$

For all $y \in X \setminus \{x_0, x_1\}$, define

$$g_y : X \longrightarrow [0,1] \quad \text{as} \quad \begin{aligned} g_y(x_0) &= 1 \\ g_y(y) &= 1 \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Let $T'$ be the fuzzy topology generated by the base $\{g_0, g_1, g_y | y \in X \setminus \{x_0, x_1\}\}$. Then $(X, T')$ is fuzzy lightly compact and $T'$ is strictly stronger than $T$ and therefore $(X,T)$ is not maximally fuzzy lightly compact.

## 4. FUZZY COMPACT (COUNTABLY COMPACT, LINDELÖF) SPACES

In this section we give characterizations for maximal fuzzy compact (Countably compact, Lindelöf) spaces.

**Definition.** Let $(X,T)$ be any fuzzy topological space. $(X,T)$ is said to be topologically generated [8] fuzzy compact (Countably compact, Lindelöf) space if there exists a compact (Countably compact, Lindelöf) topology $\mathcal{T}$ on $X$ such that $T = \omega(\mathcal{T}) = \mathcal{F}$, where $\mathcal{F}$ is the set of all continuous functions from $(X, \mathcal{T})$ to $I$.

We make use of the following results from [4] and [8] to prove Proposition 6.

**Theorem A.** [4] Let $(X,T)$ be a fuzzy countably compact (fuzzy compact or fuzzy Lindelöf) and $\delta \notin T$. Then $(X, T(\delta))$ is fuzzy countably compact (fuzzy compact, fuzzy Lindelöf) $\Leftrightarrow$ $1 - \delta$ is fuzzy countably compact (fuzzy compact or fuzzy Lindelöf).

**Theorem B.** [8] If $(X,T)$ is a topologically generated fuzzy compact space, then every fuzzy closed set is fuzzy compact.

**Theorem C.** [8] If $f : (X,T) \longrightarrow (Y,T')$ is fuzzy continuous and $\lambda$ is fuzzy compact fuzzy set in $(X,T)$, then $f(\lambda)$ is fuzzy compact.

**Proposition 6.** The following are equivalent for a topologically generated fuzzy compact (Countably compact, Lindelöf) space $(X,T)$.

1. $(X,T)$ is maximal fuzzy compact (Countably compact, Lindelöf) space $(X,T)$.
2. The set of all fuzzy compact (Countably compact, Lindelöf) sets of $X$ coincides with the set of all fuzzy closed sets of $X$.
3. If $Y$ is topologically generated fuzzy compact (Countably compact, Lindelöf) space and if $f$ is any fuzzy continuous bijection from $Y$ onto $X$, then $f$ is a fuzzy homeomorphism.

Proof. We prove this proposition for fuzzy compact spaces and the proof is similar for the other two cases.

(1) $\Longrightarrow$ (2). Suppose there exist a fuzzy compact set $\lambda$ which is not fuzzy closed. Then $1-\lambda \notin T$ and $(X,\, T(1-\lambda))$ where $T(1-\lambda) = \{[(1-\lambda)\wedge\mu]\vee\nu \mid \mu,\, \nu \in T\}$ is fuzzy compact by Theorem A. This is a contradiction to our assumption (1). Therefore every fuzzy compact set is fuzzy closed. Also from Theorem B it follows that every fuzzy closed set of $X$ is fuzzy compact. Hence (1) $\Longrightarrow$ (2).

(2) $\Longrightarrow$ (3). We need to show that $f^{-1}$ is fuzzy continuous. Let $\lambda$ be a fuzzy closed set in $Y$. Then $(f^{-1})^{-1}(\lambda) = f(\lambda)$ and as $\lambda$ is closed in $Y \Rightarrow \lambda$ is a fuzzy compact set in $Y \Rightarrow f(\lambda)$ is a fuzzy compact (by Theorem C) $\Rightarrow f(\lambda)$ is a fuzzy closed set in $X$ (by assumption (2)). This proves $f^{-1}$ is fuzzy continuous. Hence (2) $\Longrightarrow$ (3).

(3) $\Longrightarrow$ (1). Let $T'$ be any topologically generated fuzzy compact topology on $X$ such that $T \leq T'$. Then the identity map $i : (X,T) \longrightarrow (X,T')$ satisfies the condition (3) and so $T = T'$. That is $(X,T)$ is maximally fuzzy compact. □

**Definition.** [4] Let $(X,T)$ be a fuzzy topological space and $\lambda$ be a fuzzy set in $X$. $\lambda$ is called a fuzzy $G_\delta$-set if

$$\lambda = \bigwedge_{i=1}^{\infty} \lambda_i \quad \text{where each} \quad \lambda_i \in T.$$

A fuzzy topological space $(X,T)$ is called a fuzzy $G_\delta$-space [1] if every fuzzy $G_\delta$-set is fuzzy open.

In this connection, we have the following result.

**Proposition 7.** Let $(X,T)$ be a topologically generated fuzzy Lindelöf space. If $(X,T)$ is maximal fuzzy Lindelöf, then $X$ is fuzzy $G_\delta$-space.

Proof. Let $\lambda$ be any $G_\delta$-set of $X$. Then we can write $\lambda = \bigwedge_{n=1}^{\infty}\lambda_n$ where each $\lambda_n \in T$. Now $1-\lambda = \bigvee_{n=1}^{\infty}(1-\lambda_n)$. From Proposition 6 we find that $1-\lambda$ is closed and so $\lambda \in T$. □

**Remark 1.** In [7] a new definition for the notion of compactness is introduced viz a fuzzy set $\lambda$ of $(X, T)$ is said to be "compact" $\Leftrightarrow$ each filter basis $\mathcal{B}$ such that every finite intersection of members of $\mathcal{B}$ is quasi-coincident with $\lambda$, $(\wedge\overline{\lambda}) \wedge \lambda \neq 0$, $\lambda \in \mathcal{B}$. To distinguish from the above compactness notion let us denote this by "compact*". In a similar manner one can also define "Lindelöf*". Regarding these notions of compact* and Lindelöf* one can prove the following results. For concepts not defined here we refer to [7].

**Result 1.** The following are equivalent for a topologically generated fuzzy compact* (Lindelöf*) space $(X, T)$.

1. $(X, T)$ is maximal fuzzy compact* (Lindelöf*).
2. The set of all fuzzy compact* (Lindelöf*) sets of $X$ coincide with the set of all fuzzy closed sets of $X$.
3. If $Y$ is topologically generated fuzzy compact* (Lindelöf*) space and if $f$ is any fuzzy continuous bijection from $Y$ onto $X$, then $f$ is a fuzzy homeomorphism.

**Result 2.** Let $(X, T)$ be a topologically generated fuzzy Lindelöf* space. $(X, T)$ is maximal fuzzy Lindelöf* and fuzzy Hausdorff $\Leftrightarrow$ $X$ is fuzzy Lindelöf*, fuzzy Hausdorff and fuzzy $G_\delta$ space. Also $X \times X$ is maximal fuzzy Lindelöf* if $X \times X$ is fuzzy Lindelöf*, $X$ is maximal fuzzy Lindelöf* and fuzzy Hausdorff.

**Remark 2.** The study of the product of maximal $P$ spaces where $P$ is fuzzy Lindelöf, fuzzy countably compact etc is rendered uninteresting by the fact that these fuzzy topological properties are not productive in general.

## 5. STRONGLY COMPACT FUZZY TOPOLOGICAL SPACES

**Definition.** [9] Let $\lambda$ be a fuzzy subset of a fuzzy topological space $X$. $\lambda$ is said to be pre-open if $\lambda < (\overline{\lambda})^0$. The set of all pre-open fuzzy sets of $X$ is denoted by $PO(X)$. Let $(Y, S)$ be another fuzzy topological space. Let $T_\Phi$ be a fuzzy topology on $X$ which has $PO(X)$ as a subbase. A mapping $f : X \to Y$ is $\Phi$-continuous if $f : (X, T_\Phi) \longrightarrow (Y, S)$ is continuous. $f$ is said to be $\Phi'$-continuous if $f : (X, T_\Phi) \longrightarrow (Y, S_\Phi)$ is $\Phi$-continuous. A fuzzy topological space $X$ is said to be fuzzy strongly compact if every pre-open cover of $X$ has a finite subcover.

We make use of the following two Theorems from [9] in establishing Proposition 8.

**Theorem D.** Let $(X, T)$ be a fuzzy topological space which is strongly compact. Then each $T_\Phi$-closed fuzzy set in $X$ is strongly compact.

**Theorem E.** Let $X$ and $Y$ be fuzzy topological spaces and let $f : X \to Y$ be $\Phi'$-continuous. If a fuzzy subset $\lambda$ of $X$ is strongly compact relative to $X$, then $f(\lambda)$ is strongly compact relative to $Y$.

**Proposition 8.** For a fuzzy topological space the following are equivalent:

1. $(X, T)$ is maximal fuzzy strongly compact.
2. The class of strongly compact fuzzy sets of $X$ equals the class of $T_\Phi$-closed fuzzy subsets of $X$.
3. If $(Y, S)$ is a fuzzy strongly compact space and if $f$ is any $\Phi'$-continuous bijection from $Y$ onto $X$, then $f$ is a fuzzy homeomorphism.

Proof. (1) $\Longrightarrow$ (2). By Theorem A, $T_\Phi$-closed fuzzy sets are strongly compact. Suppose there exists a fuzzy strongly compact fuzzy set $\lambda$ which is not $T_\Phi$-closed. Then $1-\lambda \notin T_\Phi$ and $(X, T(1-\lambda))$ where $T(1-\lambda) = \{[(1-\lambda) \wedge \mu] \vee \nu \mid \mu, \nu \notin T\}$ is strongly compact which is such that $T < T(1-\lambda)$ which is a contradiction. Hence (1) $\Longrightarrow$ (2).

(2) $\Longrightarrow$ (3). Let $\lambda$ be any $S_\Phi$-closed fuzzy set in $Y$. Then $(f^{-1})^{-1}(\lambda) = f(\lambda)$ and it is sufficient if we show $f(\lambda)$ is $T_\Phi$-closed. Since $\lambda$ is $S_\Phi$-closed, $\lambda$ is fuzzy strongly compact in $Y$ by Theorem D and $f(\lambda)$ is fuzzy strongly compact in $X$ by Theorem E. That is $f(\lambda)$ is $T_\Phi$-closed set in $X$. Hence (2) $\Longrightarrow$ (3).

(3) $\Longrightarrow$ (1). Suppose $T'$ is any strongly compact fuzzy topology on $X$ such that $T' \geq T$. Now the identify map $i : (X, T') \longrightarrow (X, T)$ satisfies the condition (3). Therefore $T = T'$. That is $(X, T)$ is maximal fuzzy strongly compact. □

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