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Brief note on distributivity of triangular fuzzy numbers

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BRIEF NOTE ON DISTRIBUTIVITY OF TRIANGULAR FUZZY NUMBERS[1]

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The general results summarized in [1] and [3] show that fuzzy quantities and more especially fuzzy numbers do not fully preserve some of the classical algebraic properties of addition. The most significant ones are the group property of the opposite elements and one of the distributivity laws. It is shown in [3] that the concept and properties of the opposite element can be easily formulated if we substitute the crisp equality between fuzzy quantities by a weaker type of relation. It is also shown in [3] that this methods does not influence the problem of distributivity, except a very special sort of fuzzy quantities, as shown in [4]. Here we prove that for the triangular fuzzy numbers and for trapezoidal fuzzy intervals the procedure based on the weaker relation leads to the validity of the distributivity.

1. GENERALITIES

Due to [3] and [4] we define *fuzzy quantity* a as a fuzzy subset of the real line R with membership function $\mu_a : R \to [0,1]$ fulfilling

$$\exists\, x_0 \in R,\ \mu_a(x_0) = 1, \tag{1}$$

$$\exists\, x_1,\, x_2 \in R,\ x \notin [x_1, x_2] \Rightarrow \mu_a(x) = 0. \tag{2}$$

The set of all fuzzy quantities is denoted by $\mathbb{R}$. In the whole paper the equality $a = b$ for $a,\, b \in \mathbb{R}$ means the strict equality of membership functions, $\mu_a(x) = \mu_b(x)$ for all $x \in R$.

Applying the extension principle (cf. [1]) we define the *addition* operation over $\mathbb{R}$ by

$$\mu_{a\oplus b}(x) = \sup_{y\in R} \left(\min(\mu_a(y),\, \mu_b(x-y))\right), \tag{3}$$

for any $x \in R,\ a,\, b \in \mathbb{R}$.

If $r \in R$ is a real number then we denote by $\langle r \rangle$ the degenerated fuzzy quantity

$$\begin{aligned} \mu_{\langle r \rangle}(x) &= 1 \quad \text{for } x = r, \\ &= 0 \quad \text{for } x \neq r. \end{aligned} \tag{4}$$

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It is easy to verify (cf. [1],[3]) that operation defined by (3) fulfills some of the group properties, namely

$$a \oplus b = b \oplus a, \quad (a \oplus b) \oplus c = a \oplus (b \oplus c), \quad a + \langle 0 \rangle = a. \tag{5}$$

If $r \in R$ and $a \in \mathbb{R}$ then we define the *crisp product* $r \cdot a$ by

$$\begin{aligned} \mu_{r \cdot a} &= \mu_a(x/r) \quad \text{for } r \neq 0, \\ &= \mu_{\langle 0 \rangle}(x) \quad \text{for } r = 0, \end{aligned} \tag{6}$$

for any $x \in R$. It is not difficult to prove that for $a,\ b \in \mathbb{R},\ r \in R$

$$r \cdot (a \oplus b) = (r \cdot a) \oplus (r \cdot b). \tag{7}$$

In this sense some of the linearity properties of the set $\mathbb{R}$ are fulfilled but the remaining two of them are missing. Namely, if $a \in \mathbb{R}$ and $-a \in \mathbb{R}$ is defined by

$$\mu_{-a}(x) = \mu_a(-x) \quad \text{for all } x \in R, \tag{8}$$

then generally

$$a \oplus (-a) \neq \langle 0 \rangle. \tag{9}$$

Moreover, if $r_1,\ r_2 \in R,\ a \in \mathbb{R}$ then also generally

$$(r_1 + r_2) \cdot a \neq (r_1 \cdot a) \oplus (r_2 \cdot a). \tag{10}$$

These facts are mentioned in [1], [3], [4] and numerous other works.

The discrepancy expressed by (9) can be avoided if we substitute the strict equality between fuzzy quantities by a weaker relation based on the existence and properties of "fuzzy zeros", i.e. of fuzzy quantities symmetric around 0. This procedure is described e.g. in [3].

If $y \in R$ then we say that $a \in \mathbb{R}$ is *y-symmetric* iff

$$\mu_a(y + x) = \mu_a(y - x) \quad \text{for all } x \in R. \tag{11}$$

By $\mathbb{S}_0 \subset \mathbb{R}$ we denote the set of all 0-symmetric fuzzy quantities, and by $\mathbb{S} \subset \mathbb{R}$ the set of all fuzzy quantities which are y-symmetric for some $y \in R$.

We say that $a,\ b \in \mathbb{R}$ are *additively equivalent*, and write $a \sim_{\oplus} b$, iff there exist $s_1,\ s_2 \in \mathbb{S}_0$ such that

$$a \oplus s_1 = b \oplus s_2. \tag{12}$$

Elements from $\mathbb{S}_0$ can be considered for fuzzy zeros, i.e. for elements not influencing the results of the addition (they possess even other useful properties of the zero), and the equivalence defined by (12) means that we ignore the fuzzy zero differences between fuzzy quantities. Then it is easy to verify (cf. [3]) that for any $s \in \mathbb{S}_0,\ a,\ b,\ c \in \mathbb{R}$,

$$a \oplus b \sim_{\oplus} b \oplus a, \quad (a \oplus b) \oplus c \sim_{\oplus} a \oplus (b \oplus c), \quad a \oplus s \sim_{\oplus} a, \quad a \oplus (-a) \sim_{\oplus} s. \tag{13}$$

Of course, $r \cdot (a \oplus b) \sim_{\oplus} (r \cdot a) \oplus (r \cdot b)$ for $r \in R$, a, $b \in \mathbb{R}$, but the complementary distributivity relation

$$(r_1 + r_2) \cdot a \sim_{\oplus} (r_1 \cdot a) \oplus (r_2 \cdot a)$$

is guaranteed only for some special types of a, e.g., for $a \in \mathbb{S}$.

The goal of this paper is to show that for the triangular and trapezoidal fuzzy numbers the method remembered above contributed even to the solution of the distributivity problem.

2. TRIANGULAR FUZZY NUMBERS

In accordance with [1], [2] and [4] a fuzzy quantity $a \in \mathbb{R}$ is called a *triangular fuzzy number* iff its membership function μ_a is determined by a triple of real numbers (a_1, a_0, a_2), $a_1 \le a_0 \le a_2$. For $a_1 < a_0 < a_2$

$$\begin{aligned} \mu_a(x) &= \lambda \quad \text{for } x \in [a_1, a_0],\ x = \lambda\, a_0 + (1-\lambda)\, a_1, \\ \mu_a(x) &= \lambda \quad \text{for } x \in [a_0, a_2],\ x = \lambda\, a_0 + (1-\lambda)\, a_2, \\ \mu_a(x) &= 0 \quad \text{for } x \notin [a_1, a_2]. \end{aligned} \tag{14}$$

This is equivalent to

$$\begin{aligned} \mu_a(x) &= (x - a_1)\,/\,(a_0 - a_1) \quad \text{for } x \in [a_1, a_0], \\ &= (x - a_2)\,/\,(a_0 - a_2) \quad \text{for } x \in [a_0, a_2], \\ &= 0 \qquad\qquad\qquad\qquad \text{for } x \notin [a_1, a_2]. \end{aligned} \tag{15}$$

If $a_1 = a_0$ or $a_2 = a_0$ then we accept the convence $\mu_a(a_0) = 1$, $\mu_a(x) = 0$ for $x < a_0$ or $x > a_0$, respectively. The set of all triangular fuzzy numbers will be denoted by $\mathbb{L} \subset \mathbb{R}$. It can be easily verified that for $r \in R$, a, $b \in \mathbb{L}$, where a, b are determined by (a_1, a_0, a_2), (b_1, b_0, b_2), respectively, also $a \oplus b \in \mathbb{L}$ and $r \cdot a \in \mathbb{L}$. Here $a \oplus b$ is determined by the triple $(a_1 + b_1,\ a_0 + b_0,\ a_2 + b_2)$, $r \cdot a$ is determined by $(r \cdot a_1,\ r \cdot a_0,\ r \cdot a_2)$ for $r > 0$, by $(r \cdot a_2,\ r \cdot a_0,\ r \cdot a_1)$ for $r < 0$, and by $(0, 0, 0)$ for $r = 0$.

As $\mathbb{L} \subset \mathbb{R}$, the triangular fuzzy numbers fulfill (5) and (7).

If $a \in \mathbb{L}$ then obviously also $-a \in \mathbb{L}$, $-a$ is determined by $(-a_2, -a_0, -a_1)$ if a was determined by (a_1, a_0, a_2), it means that

$$a \oplus (-a) \in \mathbb{L} \cap \mathbb{S}_0,$$

and consequently $a \oplus (-a) \sim_{\oplus} \langle 0 \rangle \in \mathbb{L}$.

More attractive is the fact that for the triangular fuzzy numbers the distributivity in the equivalence form is fulfilled. The distributivity $r \cdot (a \oplus b) \sim_{\oplus} (r \cdot a) \oplus (r \cdot b)$ for $r \in R$, a, $b \in \mathbb{L}$, follows from (7) immediately. The complementary one is proved by the following statement.

Theorem 1. If $a \in \mathbb{L}$, $r_1, r_2 \in R$, then

$$(r_1 + r_2) \cdot a \sim_{\oplus} (r_1 \cdot a) \oplus (r_2 \cdot a). \tag{16}$$

Proof. Let $r_2 = 0$ Then

$$(r_1 + r_2) \cdot a = r_1 \cdot a = r_1 \cdot a \oplus \langle 0 \rangle = r_1 \cdot a \oplus r_2 \cdot a.$$

Let $r_1 > 0$, $r_2 > 0$. Then $(r_1 + r_2) \cdot a$ is characterized by $((r_1 + r_2) \cdot a_1, (r_1 + r_2) \cdot a_0, (r_1 + r_2) \cdot a_2) = (r_1 \cdot a_1 + r_2 \cdot a_1, r_1 \cdot a_0 + r_2 \cdot a_0, r_1 \cdot a_2 + r_2 \cdot a_2)$ which triple characterizes $r_1 \cdot a \oplus r_2 \cdot a$, and consequently $(r_1 + r_2) \cdot a = r_1 \cdot a \oplus r_2 \cdot a$. Quite analogously we prove this equality for $r_1 < 0$, $r_2 < 0$. Let $r_1 > 0$, $r_2 < 0$ and $|r_1| > |r_2|$. Then $r_1 + r_2 > 0$, and $(r_1 + r_2) \cdot a$ is characterized by

$$((r_1 + r_2) \cdot a_1, \; (r_1 + r_2) \cdot a_0, \; (r_1 + r_2) \cdot a_2). \tag{17}$$

On the other side, $r_1 \cdot a \oplus r_2 \cdot a$ is characterized by the triple

$$(r_1 \, a_1 + r_2 \, a_2, \; r_1 \, a_0 + r_2 \, a_0, \; r_1 \, a_2 + r_2 \, a_1), \tag{18}$$

as follows from the negativity of r_2. Then (18) is equal to

$$(r_1 \, a_1 + r_2 \, a_1 - r_2 \, a_1 + r_2 \, a_2, \; r_1 \, a_0 + r_2 \, a_0, \; r_1 \, a_2 + r_2 \, a_2 - r_2 \, a_2 + r_2 \, a_1)$$

which is equal to

$$((r_1 + r_2) \cdot a_1 + r_2(a_2 - a_1), \; r_1 \, a_0 + r_2 \, a_0, \; (r_1 + r_2) \cdot a_2 + r_2(a_1 - a_2)).$$

Using (17) we can easily see that this triple characterizes the sum

$$(r_1 + r_2) \cdot a \oplus s' = (r_1 + r_2) \cdot a \oplus r_2 \, s \tag{19}$$

where s' is characterized by

$$(r_2(a_2 - a_1), \, 0, \, r_2(a_1 - a_2)), \tag{20}$$

and s is characterized by

$$(a_1 - a_2, \, 0, \, a_2 - a_1) \tag{21}$$

and, consequently, $s \in \mathbb{S}_0 \cap \mathbb{L}$. Analogously, for $r_1 > 0$, $r_2 < 0$, $|r_1| < |r_2|$

$$(r_1 \cdot a) \oplus (r_2 \cdot a) = (r_1 + r_2) \cdot a \oplus s'' = (r_1 + r_2) \cdot a \oplus r_1 \, s \tag{22}$$

where s'', $s \in \mathbb{S}_0 \cap \mathbb{L} \, s''$ is characterized by the triple

$$(r_1(a_1 - a_2), \, 0, \, r_1(a_2 - a_1)) \tag{23}$$

and s is given by (21). Finally, if $r_1 = -r_2$, $r_1 > 0$, then $(r_1 + r_2) \cdot a = \langle 0 \rangle$ and $r_1 \cdot a \oplus r_2 \cdot a = r_1 \cdot a \oplus (-r_1) \cdot a \in \mathbb{S}_0 \cap \mathbb{L}$. In all cases equivalence (16) is proved. □

If $a \in \mathbb{L} \cap \mathbb{S}$ is y-symmetric for some $y \in R$ then a is characterized by the triple $(y - e,\ y,\ y + e)$ for some $e \in R,\ e > 0$. Expressions (20) and (23) imply that for such $a \in \mathbb{L} \cap \mathbb{S}$ and for $r_1,\ r_2 \in R,\ r_1 > 0,\ r_2 < 0$ the fuzzy quantity $s \in \mathbb{S}_0$ in (19) and (21) is characterized by

$$(2r_2e,\ 0,\ -2r_2e) \quad \text{if } |r_1| > |r_2|,$$

and by

$$(-2r_1e,\ 0,\ 2r_1e) \quad \text{if } |r_1| < |r_2|.$$

3. TRAPEZOIDAL FUZZY NUMBER

Another specific type of fuzzy quantity (cf. [1]) is the *trapezoidal fuzzy number (or interval)* $a \in \mathbb{R}$ with membership function μ_a characterized by a quadruple

$$(a_1,\ a_0,\ a_0',\ a_2), \tag{24}$$

where

$$\begin{aligned} \mu_a(x) &= (x - a_1) / (a_0 - a_1) && \text{for } x \in [a_1, a_0], \\ &= 1 && \text{for } x \in [a_0, a_0'], \\ &= (x - a_2) / (a_0' - a_2) && \text{for } x \in [a_0', a_2]. \\ &= 0 && \text{for } x \notin [a_1, a_2]. \end{aligned} \tag{25}$$

The set of all trapezoidal fuzzy numbers is denoted by $\mathbb{I}$. Evidently $a \oplus b \in \mathbb{I},\ r \cdot a \in \mathbb{I}$, for $a,\ b \in \mathbb{I},\ r \in R$, and they are characterized by quadruples

$$(a_1 + b_1,\ a_0 + b_0,\ a_0' + b_0',\ a_2 + b_2)\,,$$

and

$$\begin{aligned} &(r \cdot a_1,\ r \cdot a_0,\ r \cdot a_0',\ r \cdot a_2) && \text{for } r > 0, \\ &(r \cdot a_2,\ r \cdot a_0',\ r \cdot a_0,\ r \cdot a_1) && \text{for } r < 0, \\ &(0,\ 0,\ 0,\ 0) && \text{for } r = 0, \end{aligned}$$

respectively.

If $a_1 = a_0$ or $a_0' = a_2$ then we accept the convence that $\mu_a(a_0) = 1,\ \mu_a(x) = 0$ for $x < a_0$ or $\mu_a(a_0') = 1,\ \mu_a(x) = 0$ for $x > a_0$, respectively. If $a_0 = a_0'$ then $a \in \mathbb{L}$ and $\mu_a(x) = 1$ iff $x = a_0 = a_0'$.

If $a \in \mathbb{I}$ is characterized by the quadruple (a_1, a_0, a_0', a_2) then there exist $b \in \mathbb{L}$ and $s \in \mathbb{I} \cap \mathbb{S}_0$ characterized by (b_1, b_0, b_2), (s_1, s_0, s_0', s_2) such that $s_1 = s_0,\ s_0' = s_2$,

$$\begin{aligned} &b_1 = a_1 + (a_0' - a_0)/2, \quad b_2 = a_2 - (a_0' - a_0)/2, \\ &b_0 = \frac{a_0 + a_0'}{2}, \\ &s_1 = s_0 = \frac{a_0 - a_0'}{2}, \quad s_2 = s_0' = \frac{a_0' - a_0}{2}, \end{aligned}$$

and

$$a = b \oplus s, \quad \text{i.e.} \quad a \sim_\oplus b,$$

(the sum can be easily realized if we consider b being element of $\mathbb{I}$ characterized by (b_1, b_0, b_0, b_2)).

Analogously to Theorem 1 it is possible to prove the following

Theorem 2. If $a \in \mathbb{I}$, $r_1, r_2 \in R$, then

$$(r_1 + r_2) \cdot a \sim_\oplus r_1\, a \oplus r_2\, a. \tag{26}$$

Proof. The proof is very similar to that one of Theorem 1. If $r_1 \cdot r_2 > 0$ then (26) turns into equality

$$(r_1 + r_2) \cdot a = r_1\, a \oplus r_2\, a. \tag{27}$$

If $r_1 \cdot r_2 < 0$ then

$$r_1\, a \oplus r_2\, a = (r_1 + r_2) \cdot a \oplus r \cdot s$$

where $r = r_1$ if $r_1 + r_2 < 0$, $r = r_2$ if $r_1 + r_2 > 0$, and $s \in \mathbb{S}_0 \cap \mathbb{I}$ is characterized by the quadruple

$$(a_1 - a_2,\, a_0 - a_0',\, a_0' - a_0,\, a_2 - a_1).$$

Other steps of the proof are fully analogous to the previous one. □

4. LINEARITY OF $\mathbb{L}$ AND $\mathbb{I}$

The previous two theorems easily imply the following conclusion

Theorem 3. Sets $\mathbb{L}$ and $\mathbb{I}$ are linear spaces with the addition operation $\oplus$, with the crisp product defined by (6) as the multiplication by real number, and with the additive equivalence relation $\sim_\oplus$ instead of the strict equality. Linear space $\mathbb{L}$ is a subspace of the space $\mathbb{I}$.

Proof. The statement follows from (13), (7), and from Theorems 1 and 2, immediately. □

5. CONCLUSIVE REMARKS

The set of triangular fuzzy numbers $\mathbb{L}$ is a subset of the set of trapezoidal fuzzy intervals $\mathbb{I}$. Being closed regarding the addition and crisp product operations they represent autonomous subspaces whose autonomy is supported by the following fact

If we consider the equivalence relation (12) over $\mathbb{L}$ and $\mathbb{I}$ only, it means if we admit the 0-symmetric fuzzy quantities s_1, s_2 in (12) from $\mathbb{L} \cap \mathbb{S}_0$ and $\mathbb{I} \cap \mathbb{S}_0$, exclusively, then a fuzzy quantity from $\mathbb{L}$ cannot be additively equivalent to any other fuzzy quantity from $\mathbb{R} - \mathbb{L}$, and analogously for $\mathbb{I}$.

This consideration can be inverted in the following sense. If some general fuzzy quantity $b \in \mathbb{R}$ is additively equivalent to some trapezoidal fuzzy interval $a \in \mathbb{I}$, i.e. if $a \sim_{\oplus} b$, then the properties derived for a by means of linear operations can be eventually used for the characterization of b which characterization is adequate proportionally to the adequacy of the applied equivalence. From this point of view the formal analytical properties of the membership functions (like their part-wise linearity, triangularity, etc.) seem to be less essential than the similarity of their fuzziness which similarity means that they differ in "fuzzy zeros" from $\mathbb{S}_0$, only.

The possibility to substitute some fuzzy quantities by additively equivalent triangular fuzzy numbers can be useful in various situations, e.g. in solving fuzzy equivalentions described in [4]. They represent a fuzzy analogy to equations using the additive equivalence instead of equality. Generally, it is possible to solve simple equivalentions like

$$r \cdot x \oplus a \sim_{\oplus} b, \quad r \in R,\ r \neq 0,\ a, b \in \mathbb{R},$$

but more general systems of equivalentions with more variables cannot be generally solved without the validity of both distributivity rules. The results presented above mean that systems of equivalentions can be managed by means of classical algebraic procedures for triangular and trapezoidal fuzzy numbers, and the equivalence between some general fuzzy quantities and their triangular counterparts allows to extend these solutions to wider class of variables.

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