Soo R. Lee; Mohamad Jamshidi
Stability analysis for large scale time delay systems via the matrix Lyapunov function

*Kybernetika*, Vol. 28 (1992), No. 4, 271--283

Persistent URL: [http://dml.cz/dmlcz/124809](http://dml.cz/dmlcz/124809)

Terms of use:
© Institute of Information Theory and Automation AS CR, 1992

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.
STABILITY ANALYSIS FOR LARGE SCALE TIME DELAY SYSTEMS VIA THE MATRIX LYAPUNOV FUNCTION

SOO R. LEE AND M. JAMSHIDI

In this paper, we analyze the stability of large-scale systems with multiple time delays in both isolated parts and interconnections via the scalar approach of the matrix Lyapunov function. This approach of the matrix Lyapunov function estimates the stability of a large-scale interconnected system based on a decomposition-aggregation method. The candidate Lyapunov function here takes advantage of a weighted sum of individual Lyapunov function for each free subsystem and every interconnection related to all in pairs isolated subsystems in case of non-delay.

1. INTRODUCTION

The scalar approach of a matrix Lyapunov function method for the stability estimation of a large-scale interconnected system uses a decomposition-aggregation method [1]. Here the candidate Lyapunov function of the approach takes advantage of a weighted sum of individual Lyapunov functions for each free subsystem and every interconnection among isolated subsystems without time delay. By doing this, we can estimate the influence of interconnections between subsystems on the stability.

In this paper, after considering a nonlinear system stability which involves time delays by some perturbations, the scalar approach of the matrix Lyapunov function method has been taken into account for the stability of large-scale nonlinear systems without time delay [2]. Then, this method is applied to a nonlinear system with time delays following the application to the stability analysis of linear systems with time delays. It is noticed that if a weakly coupled nonlinear system is perturbed with time delays, then those delays do not destroy the stability of the system provided that the original system is stable [3,4]. However, the time delays can be involved in isolated terms as well as interconnections in many cases as pointed out in [3,5] and these delays really have an effect on the stability of the system.
2. STABILITY OF A LARGE-SCALE NONLINEAR SYSTEM

Consider a large-scale nonlinear system

\[ \dot{x}(t) = f[x(t), t] \]  

(1)

where \( x(t) \in \mathbb{R}^n \), \( x(t) = [x_1(t)^T, x_2(t)^T \ldots x_m(t)^T]^T \) and \( f = \mathbb{R}^n \times T \rightarrow \mathbb{R}^n \), \( T \) : time interval \((-\infty, \infty)\). The system is assumed to be decomposed into \( m \) subsystems,

\[ \dot{x}_i(t) = f_i[x_i(t), t] + g_i[x(t), t] \]

(2)

where \( x_i(t) \in \mathbb{R}^{n_i} \), \( f_i : \mathbb{R}^{n_i} \times T \rightarrow \mathbb{R}^{n_i} \),

\[ g_i : \mathbb{R}^n \times T \rightarrow \mathbb{R}^{n_i}, \quad i = 1, 2, \ldots, m, \quad \sum_{i=1}^m n_i = n. \]

The requirements of \( f_i(\cdot) \), \( g_i(\cdot) \) guarantee the existence and uniqueness of the solution of equation (2). The \( x(t) = 0 \) is an equilibrium satisfying

\[ f(0, t) = 0, \quad f_i(0, t) = 0, \quad g_i(0, t) = 0. \]  

(3)

Let us first survey stability conditions for the nonlinear interconnected system (1) which is represented as:

\[ \dot{x}(t) = Ax(t) + g(x, t). \]  

(4)

The equilibrium state of this system is asymptotically stable (A.S.) if the following conditions are satisfied [2]:

(1) \( \mathbf{g}(0, t) = 0 \)

(2) \( \lim_{\|x(0)\| \to 0} \left\| \mathbf{g}(x(0)) \right\| = 0 \)

provided that the stability of linear part is A.S.

The system (4) can be perturbed to have time delay in interconnection as:

\[ \dot{x}(t) = Ax(t) + g[x(t - h), t] \]

(5)

where

\( A \in \mathbb{R}^{n \times n}, \ x(t) \in \mathbb{R}^n, \ 0 \leq h \) constant delay.

The stability of the system above can be estimated by the following theorem.

**Theorem 1.** Consider a nonlinear system (5). If the following conditions for vector function \( g[x(t - h), t] \) hold,

(1) \( \mathbf{g}(0, t) = 0 \)

(2) \( \lim_{\|x(0)\| \to 0} \left\| \mathbf{g}(x(0)) \right\| = 0 \)
(3) solutions of $\dot{x}(t) = Ax(t)$ are A.S.

then the equilibrium state is A.S.

**Proof.** For a candidate Lyapunov function for the system (5)

$$\nu(x, t) = x(t)^T P x(t),$$

where $P$ is a positive definite matrix

$$\dot{\nu}(x, t) = x(t)^T P \dot{x}(t) + \dot{x}(t)^T P x(t) = [A x(t) + g(x(t - h), t)]^T P x(t) + x(t)^T P [A x(t) + g(x(t - h), t)] = x(t)^T [A^T P + P A] x(t) + 2g^T(\cdot) P x(t)$$

where $A^T P + P A$ becomes a negative definite matrix by (3). Further using (2), $2g^T(\cdot) P x = 0$ for small value of $x(t)$ and $\dot{\nu}(x, t) < 0$ with $\nu(x, t) > 0$. This shows the proof of the theorem. □

Let the candidate Lyapunov function for the system (1) be represented as:

$$\nu(x, t) = \left[ d^T \right] [v_{ij}(x_i(t), x_j(t), t)] [d], \quad i, j = 1, 2, \ldots, m$$

(6)

where $d = [d_1, d_2, \ldots, d_m]^T$, $d_i > 0$ are constant scalars and $v_{ij}(\cdot)$ here represents the relationship which the $j$th subsystem affects the $i$th subsystem and $v_{ij}[x_i(t), t] = x_i P_{ij} x_i$, $v_{ij}[x_i(t), x_j(t), t] = x_i^T P_{ij} x_j$ with appropriate dimensions of positive definite matrices $P_{ii}$ and matrices $P_{ij}$. To estimate the lower bound of $\nu(x(t), t)$

$$\nu_i(x_i(t), t) \geq \gamma_i \xi_i^2(x_i)$$

(7a)

$$\nu_{ij}(x_i(t), x_j(t), t) \geq \gamma_{ij} \xi_i(x_i) \xi_j(x_j).$$

(7b)

By letting

$$\gamma_i = \lambda_{mi}(P_{ii})$$

(8a)

$$\gamma_{ij} = -\text{sign}(d_i d_j) \frac{1}{2} \lambda_{ij} \left( P_{ij} P_{ij}^T \right), \quad i \neq j$$

(8b)

and

$$\xi_i(x_i) = \|x_i\|$$

(8c)

where $\lambda_m(\cdot)$ and $\lambda_M(\cdot)$ represent the minimum and maximum eigenvalues of (\cdot) respectively and $\|x_i\|$ is a norm of $x_i$. The lower bound $\nu_i(x(t), t)$ of $\nu(x(t), t)$ is

$$\nu(x(t), t) \geq \nu_i(x)$$

$$\begin{bmatrix}
    d_1 & 0 & \cdots & 0 \\
    0 & d_2 & \cdots & \vdots \\
    \vdots & \ddots & \ddots & \vdots \\
    0 & \cdots & 0 & d_m
\end{bmatrix}
\begin{bmatrix}
    711 & 712 & \cdots & 71m \\
    721 & 722 & \cdots & 72m \\
    \vdots & \ddots & \ddots & \vdots \\
    7m1 & 7m2 & \cdots & 7mm
\end{bmatrix}
\begin{bmatrix}
    d_1 & 0 & \cdots & 0 \\
    0 & d_2 & \cdots & \vdots \\
    \vdots & \ddots & \ddots & \vdots \\
    0 & \cdots & 0 & d_m
\end{bmatrix}
\xi$$

(9)
or as a compact form
\[ \nu_t(x) = \xi^T D^T G D \xi \]  
(10)

where
\[ \xi = [\|x_1\| \|x_2\| \cdots \|x_m\|]^T, \quad D = \text{diag} (d_1 \cdots d_m) \]
and matrix \( G = [\gamma_{ij}] \) for \( i, j = 1, 2, \ldots, m \). If \( G = [\gamma_{ij}] > 0 \), then \( \nu_t(x) \) is positive definite (p. d.). From (6),
\[ \nu(x(t), t) = \sum_{i=1}^{m} \left[ d_i^2 \nu_i(x_i, t) + \sum_{j \neq i} d_i d_j \nu_{ij}(x_i(t), x_j(t), t) \right], \quad \nu_{ij}(-) = \nu_{ji}(-). \]  
(11)

Then the total time derivative of \( \nu(x, t) \) for (2) becomes
\[ \dot{\nu}(x, t) = \sum_{i=1}^{m} \left[ d_i^2 \dot{\nu}_i(x_i, t) + \sum_{j \neq i} d_i d_j \dot{\nu}_{ij}(x_i, x_j, t) \right] \]  
(12)
\[ \dot{\nu}_i(-) = \frac{\partial \nu_i(x_i, t)}{\partial t} + \left( \frac{\partial \nu_i(x_i, t)}{\partial x_i} \right)^T \dot{x}_i(t) \]  
(13)
\[ \dot{\nu}_{ij}(-) = \frac{\partial \nu_{ij}(x_i, x_j, t)}{\partial t} + \sum_{(i,j)} \left[ \left( \frac{\partial \nu_{ij}(x_i, x_j, t)}{\partial x_i} \right)^T [e_i(-) + g_i(-)] \right]. \]  
(14)

Then the upper bound \( \dot{\nu}_M \) of \( \dot{\nu}(x, t) \) with \( \dot{\nu}_i(x_i, t) \) and \( \dot{\nu}_{ij}(x_i, x_j, t) \) can be obtained by collecting terms of degree of \( [q(\cdot)] = 2 \) and terms of degree of \( [r(\cdot)] > 2 \) as
\[ \dot{\nu}_M = q [\psi_1(x_1), \ldots, \psi_m(x_m)] + r [\psi_1(x_1), \ldots, \psi_m(x_m)] \]  
(15)

where \( \psi_i(x_i), i = 1, \ldots, m \) is a positive definite continuous function and
\[ q(0, \ldots, 0) = 0 \quad \text{and} \quad r(0, \ldots, 0) = 0. \]

\( q(\cdot) \) could be written as a quadratic form
\[ q [\psi_1(x_1), \ldots, \psi_m(x_m)] = \psi^T W \psi \]  
(16)
where \( W \) is a constant symmetric matrix
\[ \psi = [\psi_1(x_1), \ldots, \psi_m(x_m)]^T. \]  
(17)

From (16), \( q(\cdot) \) is negative definite near the origin if and only if \( W < 0 \) and this gives \( \dot{\nu}(\cdot) < 0 \) with \( \nu(\cdot) > 0. \)

**Theorem 2.** The equilibrium state \( x(t) = 0 \) of the system (1) is A. S. near the origin if \( G > 0 \) in (10) and \( -W \) in (16) is an \( M \)-matrix [6].

**Proof.** The proof is clear from Theorem 3. \( \square \)
3. MATRIX LYAPUNOV FUNCTION FOR THE STABILITY OF A LINEAR TIME DELAY SYSTEMS

Case 1 – Linear System with Time Delay in Two Interconnections

Based on the discussion of previous section for the stability of a nonlinear system through the scalar approach of the matrix Lyapunov function method, the stability of a linear time delay system is investigated as a special case of a nonlinear system. In this case the degree of the system function is one and \( r(\cdot) \) in (15) does not appear. The time delay terms in the system however, can be analyzed by using linearity with respect to initial state. Consider a simple interconnected system with constant delays in each subsystem,

\[
\begin{align*}
\dot{x}_1(t) &= A_1 x_1(t) + G_{12} x_2(t - h_2) \quad (18a) \\
\dot{x}_2(t) &= G_{21} x_1(t - h_1) + A_2 x_2(t) \quad (18b)
\end{align*}
\]

where \( A_i \in \mathbb{R}^{n_i \times n_i}, \) \( h_i \) is a constant delay for \( i = 1, 2 \) and \( G_{ij} \in \mathbb{R}^{n_i \times n_j} \) for \( i, j = 1, 2 \) \((i \neq j)\).

The candidate Lyapunov function for system (18)

\[
\nu(x, t) = \begin{bmatrix} d_1 & d_2 \end{bmatrix} \begin{bmatrix} \nu_{11}(x_1, t) \\ \nu_{12}(x_1, x_2, t) \\ \nu_{21}(x_2, x_1, t) \\ \nu_{22}(x_2, t) \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = d_1^2 \nu_{11}(x_1) + 2d_1d_2\nu_{12}(x_1, x_2) + d_2^2\nu_{22}(x_2), \quad \nu_{ij} = \nu_{ji} \quad (19)
\]

where

\[
\nu_{ii}(x_i) = x_i^T P_i x_i, \quad \nu_{ij}(x_i, x_j) = x_i^T P_{ij} x_j. \quad (20)
\]

Therefore, \( \dot{\nu}(x_i, t) \) is arranged by substituting \( \dot{x}_i \) given in (18) as follows:

\[
\begin{align*}
\dot{\nu}(x(t), t) &= [d_1 \ d_2] \begin{bmatrix} \nu_{11}(\cdot) \\ \nu_{21}(\cdot) \\ \nu_{22}(\cdot) \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \\
&= [x_1^T \ x_2^T] \begin{bmatrix} d_1^2 (A_1^T P_1 + P_1 A_1) & d_1d_2 (A_1^T P_{12} + P_{12} A_1) \\ d_2^2 (A_1^T P_1 + P_1 A_1) & d_2^2 (A_2^T P_{22} + P_{22} A_2) \end{bmatrix} \begin{bmatrix} x_1 \ x_2 \end{bmatrix} \\
&+ [x_1^T \ x_2^T] \begin{bmatrix} d_1 \ d_2 P_{12} \end{bmatrix} \begin{bmatrix} d_1^2 P_{12} G_{21} \\ d_2^2 P_{12} G_{21} \end{bmatrix} \begin{bmatrix} x_{1a} \ x_{2a} \end{bmatrix} \\
&+ [x_{1a}^T \ x_{2a}^T] \begin{bmatrix} d_1 \ d_2 P_1 \end{bmatrix} \begin{bmatrix} d_1^2 P_1 \ d_2^2 P_1 \end{bmatrix} \begin{bmatrix} x_{1a} \ x_{2a} \end{bmatrix} \\
&- \left( \|x_1\| \|x_2\| \right) \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} \|x_1\| \\ \|x_2\| \end{bmatrix} \\
&- \left( \|x_{1a}\| \|x_{2a}\| \right) \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} \begin{bmatrix} \|x_{1a}\| \\ \|x_{2a}\| \end{bmatrix} \quad (21)
\end{align*}
\]
where \( \alpha_i = \lambda_{\max} \{ d_i^T (A_i + P_i A_i) \} \) for \( i = 1, 2 \)

\[
\alpha_{12} = \|d_1 d_2 (A_1^T P_2 + P_1 A_2)\|_2, \quad \beta_{11} = \lambda_{\max} \{ d_1 d_2 (P_1 G_{21} + G_{11}^T P_2) \},
\]

\[
\beta_{12} = \|d_1^2 (P_1 G_{12} + G_{12}^T P_1)\|_2, \quad \beta_{21} = \|d_2^2 (P_2 G_{21} + G_{21}^T P_2)\|_2,
\]

and

\[
x_{1h_1} = x_1(t - h_1), \quad x_{2h_2} = x_2(t - h_2), \quad \alpha_{12} = \alpha_{21}, \quad \beta_{11} = \beta_{22}.
\]

If we can find \( d_i > 0 \) such that leading principal minors of

\[
W = \begin{bmatrix} -\alpha_{11} & -\alpha_{12} \\ -\alpha_{21} & -\alpha_{22} \end{bmatrix} \quad \text{and} \quad W_b = \begin{bmatrix} -\beta_{11} & -\beta_{12} \\ -\beta_{21} & -\beta_{22} \end{bmatrix}
\]

are positive, that is, \( W \) and \( W_b \) are \( M \)-matrices, then \( \nu(t; \epsilon) < 0 \). This leads the system to be asymptotically stable according to Theorem 2.

**Remark.**

(1) Even if both \( W \) and \( W_b \) are not \( M \)-matrices, the system can be stable if the one which is an \( M \)-matrix is dominant over the other which is not an \( M \)-matrix.

(2) If we can find a scale factor for \( x_i(t - h_i) \) with respect to \( x_i(t) \), then \( W \) and \( W_b \) with the scaling factor can be added up together as in Section 3.5 of Chapter 3 in [3].

**Case 2 - Linear System with Time Delays in Three Interconnections**

The stability of a system with time delays in two interconnections can be extended to the system with time delays in three interconnections. Consider a system whose subsystems are represented by

\[
\begin{align*}
\dot{x}_1 &= A_1 x_1(t) + G_{12} x_2(t - h_2) + G_{13} x_3(t - h_3) \quad (24a) \\
\dot{x}_2 &= G_{21} x_1(t - h_1) + A_2 x_2(t) + G_{23} x_3(t - h_2) \quad (24b) \\
\dot{x}_3 &= G_{31} x_1(t - h_1) + G_{32} x_2(t - h_2) + A_3 x_3(t) \quad (24c)
\end{align*}
\]

where \( A_i, G_{ij}, x_i \) and \( h_i \) are defined similar to Case 1. Thus,

\[
\nu_i(x_i) = x_i^T P_i x_i, \quad \nu_{ij}(x_i, x_j) = x_j^T P_{ij} x_j \quad \text{for } i, j = 1, 2, 3 \text{ and } i \neq j, \quad (25)
\]

\[
\dot{\nu}(x(t)) = [d_1 \ d_2 \ d_3] \begin{bmatrix} \dot{\nu}_{11}(\cdot) \\ \dot{\nu}_{12}(\cdot) \\ \dot{\nu}_{13}(\cdot) \\ \dot{\nu}_{21}(\cdot) \\ \dot{\nu}_{22}(\cdot) \\ \dot{\nu}_{23}(\cdot) \\ \dot{\nu}_{31}(\cdot) \\ \dot{\nu}_{32}(\cdot) \\ \dot{\nu}_{33}(\cdot) \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}, \quad (26)
\]

\[
\nu_i(\cdot) = \nu_{ij}(\cdot) \quad \text{for } i \neq j \text{ and } d_i > 0.
\]
\[
\dot{\mathbf{x}}(t) = \mathbf{x}^T \begin{bmatrix}
\begin{matrix}
d_1 \left( A_1^T P_{11} + P_{11} A_1 \right) & d_1 d_2 \left( A_1^T P_{12} + P_{12} A_1 \right) & d_1 d_2 \left( A_1^T P_{21} + P_{21} A_1 \right) \\
\end{matrix}
\end{bmatrix} \mathbf{x} + 2\mathbf{x}^T \begin{bmatrix}
\begin{matrix}
d_1 \left( A_1^T P_{11} + P_{11} A_1 \right) & d_1 d_2 \left( A_1^T P_{12} + P_{12} A_1 \right) & d_1 d_2 \left( A_1^T P_{21} + P_{21} A_1 \right) \\
\end{matrix}
\end{bmatrix} \mathbf{x}.
\]

where

\[
\mathbf{x} = [x_1, x_2, x_3, x_4, x_5, x_6]^T.
\]

Furthermore,

\[
\dot{\mathbf{x}}_\nu(t) = -\|\mathbf{x}\|^T \begin{bmatrix}
\begin{matrix}
-\alpha_{11} & -\alpha_{12} & -\alpha_{13} \\
-\alpha_{21} & -\alpha_{22} & -\alpha_{23} \\
-\alpha_{31} & -\alpha_{32} & -\alpha_{33}
\end{matrix}
\end{bmatrix} \mathbf{x}_\nu + \|\mathbf{x}\|.
\]

If we can find \( \alpha_{ij}, \beta_{ij}, \gamma_{ij} \) for \( i, j = 1, 2, 3 \) such that the leading principal minors of the matrices

\[
W = \begin{bmatrix}
-\alpha_{11} & -\alpha_{12} & -\alpha_{13} \\
-\alpha_{21} & -\alpha_{22} & -\alpha_{23} \\
-\alpha_{31} & -\alpha_{32} & -\alpha_{33}
\end{bmatrix}
\]

(29a)
and

\[ W_h = \begin{bmatrix}
-(\beta_{11} + \gamma_{11}) & -(\beta_{12} + \delta_{12}) & -(\beta_{13} + \delta_{13}) \\
-(\beta_{21} + \delta_{21}) & -(\beta_{22} + \gamma_{22}) & -(\beta_{23} + \delta_{23}) \\
-(\beta_{31} + \delta_{31}) & -(\beta_{32} + \gamma_{32}) & -(\beta_{33} + \gamma_{33})
\end{bmatrix} \quad (29b)

are positive, then \( \dot{\nu}(x(t)) < 0 \) from (28). This gives the system stability and we can find a general rule for the matrix formation.

**Case 3 – Linear System with Time Delays in Both Isolation and Interconnection**

Consider a linear system with multiple time delays in its ith isolation and interconnection as:

\[
\dot{x}_i(t) = A_i x_i(t) + \sum_{k=1}^{2} F_{ik} x_i(t - h_{ik}) + \sum_{j=1}^{3} B_{ij} x_j(t) + \sum_{j=1}^{3} G_{ij} x_j(t - \tau_{ij}). \quad (30)
\]

For \( i = k = 2 \)

\[
\dot{x}_2(t) = A_2 x_2(t) + \sum_{j=1}^{3} F_{2j} x_j(t - h_{2j}) + \sum_{j=1}^{3} B_{2j} x_j(t) + \sum_{j=1}^{3} G_{2j} x_j(t - \tau_{2j}) \quad (31a)
\]

\[
\dot{x}_2(t) = A_2 x_2(t) + \sum_{j=1}^{3} F_{2j} x_j(t - h_{2j}) + \sum_{j=1}^{3} B_{2j} x_j(t) + \sum_{j=1}^{3} G_{2j} x_j(t - \tau_{2j}) \quad (31b)
\]

where

\[ A_i \in \mathbb{R}^{n \times n}, \quad F_i \in \mathbb{R}^{n \times m}, \]

\[ B_{ij} \in \mathbb{R}^{n \times m} \quad \text{and} \quad G_{ij} \in \mathbb{R}^{n \times m} \quad \text{for} \ i, j = 1, 2 \ (i \neq j) \]

are all real matrices.

Let the candidate Lyapunov function for the system whose subsystems are represented by (31) be:

\[
\nu(x, t) = [d_1 \ d_2] \begin{bmatrix}
\nu_{11}() \\
\nu_{21}()
\end{bmatrix} \begin{bmatrix}
d_1 \\
d_2
\end{bmatrix} = d_1^2 \nu_{11}() + 2d_1 d_2 \nu_{12}() + d_2^2 \nu_{22}(), \quad \nu_{12}() = \nu_{21}(). \quad (32)
\]

Then

\[
\dot{\nu}(x, t) = [d_1 \ d_2] \begin{bmatrix}
\nu_{11}() \\
\nu_{21}()
\end{bmatrix} \begin{bmatrix}
d_1 \\
d_2
\end{bmatrix} = d_1^2 \dot{\nu}_{11}() + 2d_1 d_2 \dot{\nu}_{12}() + d_2^2 \dot{\nu}_{22}() \quad (33)
\]

with

\[ \nu_{11}(x_i, t) = x_i^T P_{11} x_i, \quad \nu_{12}(x_i, x_j, t) = x_i^T P_{12} x_j. \]

From \( \nu_{11}(x_i, t) \) and \( \nu_{12}(x_i, x_j, t) \), we can obtain \( \dot{\nu}(x, t) \) along the (31).

\[
\dot{\nu}(t) = \begin{bmatrix}
-d_1^2 Q_1 + d_1 d_2 (B_{11}^T P_{21} + P_{12} B_{21}) \\
d_1 d_2 (P_{11} B_{12} + B_{21}^T P_{22} + A_1 P_{12} + P_{12} A_2)
\end{bmatrix} x
\]

\[
+ \begin{bmatrix}
d_2^2 Q_2 + d_1 d_2 (B_{21}^T P_{22} + A_2 P_{12} + P_{12} A_1) \\
d_2^2 Q_2 + d_1 d_2 (B_{21}^T P_{22} + A_2 P_{12} + P_{12} A_1)
\end{bmatrix} x
\]

\[
= \begin{bmatrix}
d_1^2 Q_1 + d_1 d_2 (B_{11}^T P_{21} + P_{12} B_{21}) \\
d_1 d_2 (P_{11} B_{12} + B_{21}^T P_{22} + A_1 P_{12} + P_{12} A_2)
\end{bmatrix} x
\]

\[
+ \begin{bmatrix}
d_2^2 Q_2 + d_1 d_2 (B_{21}^T P_{22} + A_2 P_{12} + P_{12} A_1) \\
d_2^2 Q_2 + d_1 d_2 (B_{21}^T P_{22} + A_2 P_{12} + P_{12} A_1)
\end{bmatrix} x
\]
\begin{align*}
&\dot{x} + x^T \begin{bmatrix} d_1^2 P_{11} & d_1 d_2 P_{12} \\ d_2 d_1 P_{21} & d_2^2 P_{22} \end{bmatrix} x_A + x_2 \begin{bmatrix} d_1^2 P_{11} & d_1 d_2 G_{12} P_{12} \\ d_2 d_1 G_{21} P_{21} & d_2^2 G_{22} P_{22} \end{bmatrix} x_r \\
&\quad + x^T \begin{bmatrix} d_1 d_3 P_{13} G_{21} - d_2 d_3 G_{12} P_{12} \\ d_2^2 P_{23} G_{21} - d_2 d_3 G_{12} P_{22} \end{bmatrix} x_r + x_2 \begin{bmatrix} d_1 d_3 G_{13} P_{13} - d_2 d_3 G_{22} P_{12} \\ d_2^2 G_{13} P_{13} - d_2 d_3 G_{22} P_{22} \end{bmatrix} x
\end{align*}

where

\[ x \equiv [x_1^T, x_2^T]^T, \quad x_A \equiv [x_{1A}^T, x_{2A}^T]^T, \quad x_r \equiv [x_{1r}^T, x_{2r}^T]^T, \]

and \( x_{1A} \equiv x_i(t - h_{ij}), \quad x_{1r} \equiv x_i(t - \tau_{ij}) \)

\[ A_i^T P_i + P_i A_i = -Q_i, \quad i = 1, 2. \]

If \( B_{12} = B_{21} \), then

\begin{align*}
\dot{\nu}(x) &\leq -\left[ \|x_1^T\| \|x_2^T\| \right] \begin{bmatrix} -\alpha_{11} & -\alpha_{21} \\ -\alpha_{12} & -\alpha_{22} \end{bmatrix} \begin{bmatrix} \|x_1\| \\ \|x_2\| \end{bmatrix} \\
&\quad - \left[ \|x_1^T\| \|x_2^T\| \right] \begin{bmatrix} -\beta_{11} & -\beta_{21} \\ -\beta_{12} & -\beta_{22} \end{bmatrix} \begin{bmatrix} \|x_{1A}\| \\ \|x_{2A}\| \end{bmatrix} - \left[ \|x_1^T\| \|x_2^T\| \right] \begin{bmatrix} -\gamma_{11} & -\gamma_{12} \\ -\gamma_{21} & -\gamma_{22} \end{bmatrix} \begin{bmatrix} \|x_{1r}\| \\ \|x_{2r}\| \end{bmatrix}
\end{align*}

where

\begin{align*}
\alpha_{11} &= \lambda_{\max} \left[ -d_1^2 Q_1 + d_1 d_2 \left( B_{11}^T P_{11} + P_{12} B_{12} \right) \right] \\
\alpha_{12} &= \alpha_{21} = \lambda_{\max} \left[ d_1 d_3 \left( P_{13} + B_{13}^T P_{13} \right) + A_1 P_{12} + P_{12} A_2 \right] \\
\alpha_{22} &= \lambda_{\max} \left[ -d_2^2 Q_2 + d_2 d_3 \left( B_{22}^T P_{22} + P_{21} B_{21} \right) \right] \\
\beta_{11} &= \lambda_{\max} \left[ d_1^2 \left( P_{11} F_{ii} + F_{ii}^T P_{11} \right) \right] \\
\beta_{12} &= \lambda_{\max} \left[ d_1^2 \left( P_{11} + F_{ii}^T P_{11} \right) \right] \\
\beta_{21} &= \lambda_{\max} \left[ d_2^2 \left( P_{21} + G_{21}^T P_{21} \right) \right] \\
\gamma_{11} &= \lambda_{\max} \left[ d_1 d_3 \left( P_{13} G_{12} + G_{12}^T P_{13} \right) \right] \\
\gamma_{12} &= \lambda_{\max} \left[ d_2 d_3 \left( P_{21} G_{12} + G_{12}^T P_{21} \right) \right] \\
\gamma_{21} &= \lambda_{\max} \left[ d_1 d_3 \left( P_{13} + G_{13}^T P_{13} \right) \right] \\
\gamma_{22} &= \lambda_{\max} \left[ d_2 d_3 \left( P_{21} + G_{21}^T P_{21} \right) \right]
\end{align*}

and \( x_{1A}, x_{1r} \) are \( x_i(t - h_{ij}) \) and \( x_i(t - \tau_{ij}) \) for \( i \neq j \), respectively. If further \( x_{1A} = x_{1r}, x_{2A} = x_{2r}, \) and \( d_i = 1, \quad i = 1, 2, \) then

\begin{align*}
\dot{\nu}_M(x, t) &\leq -\left[ \|x_1^T\| \|x_2^T\| \right] \begin{bmatrix} -\alpha_{11} & -\alpha_{12} \\ -\alpha_{21} & -\alpha_{22} \end{bmatrix} \begin{bmatrix} \|x_1\| \\ \|x_2\| \end{bmatrix} \\
&\quad - \left[ \|x_1^T\| \|x_2^T\| \right] \begin{bmatrix} -\beta_{11} & -\beta_{12} \\ -\beta_{21} & -\beta_{22} \end{bmatrix} \begin{bmatrix} \|x_{1A}\| \\ \|x_{2A}\| \end{bmatrix}
\end{align*}

If the leading principal minors of

\[ W = \begin{bmatrix} -\alpha_{11} & -\alpha_{12} \\ -\alpha_{21} & -\alpha_{22} \end{bmatrix} \]

(37a)
are positive definite, then the system (31) is stable.

Remark.
(1) If we can scale \( \|x_1(t - h_{11})\| \) and \( \|x_3(t - h_{33})\| \) in (36) to \( \|x_1\| \) and \( \|x_3\| \), respectively, then the stability can be examined by the similar way to the method of Section 3.5 in Chapter 3 in [3].
(2) If \( F_i = 0 \) for \( i = 1, 2 \) and \( B_{13} = B_{23} = 0 \) in (31), then the result is the same as the Case 1, the linear system with time delays in two interconnections. That is,

\[ MX_0 - K\] 

\[ MX_0 - K\]

equals to equation (23).

4. MATRIX LYAPUNOV FUNCTION APPLICATIONS TO NONLINEAR TIME DELAY SYSTEMS

Case 1 – Nonlinear System with Time Delays in Interconnections

Consider a large-scale system

\[ \dot{x} = f[x(t), x_1(t - \tilde{h}_1), \ldots, x_m(t - \tilde{h}_m), t], \] 

(38)

where \( \tilde{h}_i = [h_{11}, h_{12}, \ldots, h_{1m}]^T \). And its ith subsystem is represented by

\[ \dot{x}_i(t) = f[x_i(t), x(t), x_i(t - \tilde{h}_1), \ldots, x_m(t - \tilde{h}_m), t]. \] 

(39)

The candidate Lyapunov function for the system (38) can be expressed as

\[ V(x, t) = \sum_{i} d_i^2 \dot{v}_i(x_i, t) + \sum_{i} d_i d_j \dot{v}_{ij}(x_i, x_j, t), \quad d_i > 0. \] 

(40)

From this

\[ \dot{V}(x, t) = \sum_{i} \left[ d_i^2 \left( \frac{\partial v_i}{\partial t} + \left( \frac{\partial v_i}{\partial x_i} \right)^T \dot{x}_i \right) + \sum_{j} d_i d_j \left( \frac{\partial v_{ij}}{\partial t} + \left( \frac{\partial v_{ij}}{\partial x_i} \right)^T \dot{x}_i + \left( \frac{\partial v_{ij}}{\partial x_j} \right)^T \dot{x}_j \right) \right] \] 

(41)
Separating the terms of degree 2 from the terms of degree greater than 2,

\[
\mathbf{S}_i(x, t) = \sum_{i=1}^{m} q_i \left( \frac{\partial \nu_i}{\partial t} + \left( \frac{\partial \nu_i}{\partial x_i} \right)^T \mathbf{f}_i + \left( \frac{\partial \nu_i}{\partial x_j} \right)^T \mathbf{g}_i \right)
\]

where \( q_i, q_{ij} \) are negative definite continuous functions of degree 2 and \( r_i, r_{ij} \) are negative definite continuous functions whose degree is greater than 2. Then the stability near the origin can be evaluated from \( q_1(\cdot) \) and \( q_{ij}(\cdot) \) by constructing quadratic forms:

\[
q_1 \left( \mathbf{f}_1(x), \mathbf{f}_2(x), \ldots, \mathbf{f}_m(x) \right) = \xi^T W_1 \xi
\]

\[
q_{ij} \left( \mathbf{f}_1(x_i), \mathbf{f}_2(x_i), \ldots, \mathbf{f}_m(x_i) \right) = \xi^T W_{1i} \xi
\]

If \( W_1 < 0 \) and \( W_{1i} < 0 \), then \( \dot{\mathbf{S}}(\mathbf{x}, t) < 0 \) near the origin. This says that the system is A. S. near the origin. Similarly we can analyze the stability of a large-scale system with time delays in isolated states and interconnections.

Case 2 - Nonlinear System with Time Delay in Both Isolation and Interconnection

We are considering the following system:

\[
\dot{x}(t) = f(x(t), t, x_i(t - h_i), \ldots, x_m(t - h_m), t)
\]

where \( h_i = [h_1, h_2, \ldots, h_m]^T \). Its initial function is given as

\[
x(\tau) = \phi(\tau), \quad t_0 - \bar{h} \leq \tau \leq t_0
\]

with the maximum delay \( \bar{h} \) and its initial condition

\[
f(0, 0, \ldots, 0) = 0
\]

and with ith subsystem,

\[
\dot{x}_i(t) = f_i(x_i, t, x_i(t - h_i), t) + g_i(x(t), x_i(t - h_1), \ldots, x_m(t - h_m))
\]
The candidate Lyapunov function (40) for the system (44a) is used again:

\[
\dot{V}(x, t) = \sum_{i=1}^{m} \dot{\nu}(x_i, t) + \sum_{j=1}^{m} d_{ij} \dot{\nu}(x_i, x_j, t) \quad (41)
\]

The term \(\dot{\nu}(x, t)\) using \(\dot{\nu}(x_i, t)\) and \(\dot{\nu}(x_i, x_j, t)\) with \(\dot{x}_i(t)\) and \(\dot{\dot{x}}_i(t)\) from (44d) is arranged to lead to the followings. We are assuming that the upper bound \(\dot{\nu}_M(x, t)\) is obtained from (41) as

\[
\dot{\nu}_M(x, t) = q_{21} [\xi_1(x_1), \ldots, \xi_m(x_m)] + q_{22} [\xi_1(x_1(t - h_1)), \ldots, \xi_m(x_m(t - h_m))] + r_{21} [\xi_1(x_1), \ldots, \xi_m(x_m)] + r_{22} [\xi_1(x_1(t - h_1)), \ldots, \xi_m(x_m(t - h_m))] \quad (45)
\]

where \(q_{21}()\), \(q_{22}()\) are negative definite continuous functions of degree 2 and \(r_{21}\) and \(r_{22}\) are negative definite continuous functions of degree over 2. Similarly to the Case 1,

\[
q_{21} [\xi_1(x_1), \ldots, \xi_m(x_m)] = \xi^T W_2 \xi \quad (46a)
\]

\[
q_{22} [\xi_1(x_1(t - h_1)), \ldots, \xi_m(x_m(t - h_m))] = \xi^T W_{2h} \xi. \quad (46b)
\]

Based on Theorem 2 and the Case 2 in this section, the following theorem can be established for the system (44).

**Theorem 3.** The equilibrium state \(x_e = 0\) of the time delay system (44) is asymptotically stable if \(-W_2 \) and \(-W_{2h}\) in (46) are \(M\)-matrices, provided that \(G > 0\) in (10).

**Proof.** From the candidate Lyapunov function (40) for the system (44), it is not difficult to get the upper bound \(\dot{\nu}_M(x, t)\) in (45). Since \(-W_2\) and \(-W_{2h}\) in (46) are \(M\)-matrices and the degree of \(q_{21}()\) and \(q_{22}()\) are lower than those of \(r_{21}()\) and \(r_{22}()\), respectively, the system is guaranteed to be asymptotically stable near the origin. \(\Box\)

**Remark.** The difference of equation (45) from (42) is that the equation (45) should have time delay terms coming from the isolated part in (44d).

5. CONCLUSIONS

The stability of a large-scale system with time delays is evaluated by the way of scalar approach of a matrix Lyapunov function method and an \(M\)-matrix. By using a decomposition-aggregation method, the contribution of the interconnections between subsystems to the stability is well visualized especially in linear systems with time delays.

Sufficient conditions for asymptotic stability of the equilibrium state of the system are obtained by examination of \(M\)-matrices consisting of delay and nondelay terms together and the \(M\)-matrices here has been considered for the quadratic terms rather than the terms over degree two to watch the stability of the system near the origin.
In nonlinear time delay systems, the method can be applied to the stability analysis of large-scale nonlinear systems whose subsystems with time delays are possibly unstable. If the system could not be well formed like (43) or (46), then we could deal with the problem using a computer program [3].

(Received January 28, 1991.)

REFERENCES


Professor Soo R. Lee, Department of Electrical and Electronics Eng., The Military College, Korea Third Military Academy 771-849, Korea.

Professor M. Jamshidi, Department of Electrical and Computer Eng., The University of New Mexico, Albuquerque, New Mexico 87131, U.S.A.