Tomáš Havránek
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The Computation of Characteristic Vectors of Logical–Probabilistic Expressions

TOMÁS HAVRÁNEK

This paper is related to paper [2], knowledge of which is essential for a full understanding of the following text. Particular case of the computation of the characteristic vector of logical-probabilistic expressions are considered here.

The notion of logical-probabilistic expression was introduced in paper [2] for describing nets with logical and probabilistic elements. A logical-probabilistic expression $\Phi$ is a triple $[F, \Omega_F, \mathcal{P}_F]$, where $F$ is a logical-probabilistic form (LP-form), $\Omega_F$ is a space of random events, and $\mathcal{P}_F$ is a system of probabilities on $\Omega_F$.

An LP-form is a generalized logical form of the propositional calculus in which a new kind of unary connectives is introduced; let every such connective (probabilistic connective) be denoted by one of the symbols $\varphi_1, \varphi_2, \ldots$ (probabilistic connectives in a given LP-form must be denoted by different symbols). Let the connectives $\varphi_1, \ldots, \varphi_n$ occur in a given LP-form $F$. A space of random events (denoted $\Omega_F$) is equal to $\{\omega_0, \omega_1\}$. The value of the associated function $\varphi_i$ is 1 or 0, depending on the random events $\omega_0$ and $\omega_1$.

Further $\Omega_F$ is equal to $X_{i=1}^n \Omega_i$. The system of probabilities $\mathcal{P}_F$ is

$$\{P(\gamma; \omega)\}_{\omega \in \{0, 1\}^n}, \gamma = (\gamma_1, \ldots, \gamma_n).$$

Let $x = (x_1, \ldots, x_n)$ be variables occurring in $F$. Then for their given evaluation $\sigma \in \{0, 1\}^n$, $\gamma_i$ is equal to the evaluation of a subform $F'$ such that $\varphi_i(F')$ is a subform of $F (\varphi_i(F') < F)$, for $i = 1, \ldots, n$.

For a more precise description of the notion of LP-expression (and corresponding LP-net) see Definitions 8 and 5 in [2]. It is useful if $\mathcal{P}_F$ fulfills condition 4) from Definition 5 in [2].

As formulated and proved in Theorem 2 of [2], the probabilistic properties of an LP-expression $\Phi$ can be described by the characteristic vector $p_\Phi$. $p_\Phi$ is equal to the vector of probabilities $(P_x(\Omega))_{x \in \{0, 1\}^n}$, where $x$ are possible evaluations of the variables.
Remark. The previous description of \( p_\Phi \) is valid only if every variable occurring in \( F \) occurs in the interior of some probabilistic connective \( \varphi_i \). Otherwise if \( x_1, \ldots, x_m \) are variables occurring in \( F \), but not occurring in the interior of any \( \varphi_i \), then \( \Omega_i \subseteq \Sigma' \times \Omega_F \), where \( \Sigma' \) is the space of values of these variables.

The general method of computation of \( p_\Phi \) was described in the previous paper [2]. But this general method is too complicated to be used in many real cases. Theorem 3 from [2] made it possible for us to restrict the subject to the computation of the characteristic vectors of LP-expressions which are in probabilistic disjunctive normal form (PDNF). Roughly speaking in LP-expression \([F, \Omega_F, \mathcal{P}_\Phi]\) is in PDNF, if for every \( \varphi_i \) occurring in \( F \) \((\varphi_i(F) < F) \) is its interior \((F')\) of the form

\[
F' = (F_1 \& \ldots \& F_{n_i}) \vee \ldots \vee (F_{n_{i-1}} \& \ldots \& F_{n_i})
\]

where \( F_j \) are either variables or of the form \( \varphi_j(F_j) \).

The following considerations will be devoted to the computation of \( p_\Phi \) for several particular cases of LP-expressions in PDNF. These particular cases are given by restrictions on \( \mathcal{P}_\Phi \) (e.g. stochastic independence).

Remark. It is very useful to bear in mind, that an LP-expression is a description of a net with logical and probabilistic elements (LP-net). For more particulars see Definitions 5 and 8 and Theorem 1 from [2].

Theorem 1. Consider an LP-expression \( \Phi = [F, \Omega_F, \mathcal{P}_\Phi] \), where

\[
F = \varphi(F_1 \& \ldots \& F_{n_i}) \vee \ldots \vee (F_{n_{i-1}} \& \ldots \& F_{n_i})
\]

and

\[
\mathcal{P}_\Phi = \{P(\gamma; \omega)\}_{\gamma, \omega}.
\]

Let \( l \) be the number of conjunctive members in \( F \).

Then: 1) For every conjunctive member

\[
K_i = (F_{n_{i-1}+1} \& \ldots \& F_{n_i})
\]

(more precisely, for the corresponding subexpression) we have \( p_{K_i} = p_{K_i,l} \), where \( p_{K_i,l} \) is the last column of the matrix \( P_{K_i} (F_i = (F_{n_{i-1}+1}, \ldots, F_{n_i})) \).

2) For the subform \( F' = \text{int}(\varphi, F) \) we have \( p_{F'} = I^T - p_{K,0} \), where \( p_{K,0} \) is the first column of the matrix \( P_K (K = (K_1, \ldots, K_l)) \) and \( I = (1, \ldots, 1) \).

Moreover, let

\[
P(\gamma; \omega) = P_1(\gamma_1; \omega_1) \cdot P_2(\gamma_2; \omega_2)
\]

(for every \( \gamma \in [0, 1] \)), where

\[
\gamma_1 = (\gamma_1, \ldots, \gamma_{n-1}), \quad \omega_1 = (\omega_1, \ldots, \omega_{n-1})
\]
Then:

3) \( P_F = (I^T - P_{K,0}) P_1 + P_{K,0} P_0 \),

where \( p_0, p_1 \) are probabilistic parameters of \( \varphi_F \).

Remarks. 1) We suppose that the number of probabilistic connectives in \( F \) is \( n \).

For conventions concerning the enumeration of these connectives, see [2], convention 1).

2) An LP-subexpression \([F', \Omega_{F'}, \mathcal{P}_{F'}] \) of a given LP-expression \([F, \Omega_F, \mathcal{P}_F] \) is determined by a subform \( F' \prec F \).

3) \( \text{Int}(\varphi_F, F) \) is the \( F' \) for which \( \varphi_F(F') \prec F \).

4) It is necessary to explain the meaning of symbols \( P_{K,0}, P_{K,1} \). We have

\[ P_K = (P_{K,0}, P_{K,1}) \in \{0, 1\}^n, \]

where

\[ P_{K,0} = \mathcal{P}(\text{func}_F F_1(\sigma, \omega) = \zeta_1, \ldots, \text{func}_F K_0(\sigma, \omega) = \zeta_0). \]

The right-hand side of the preceding equation is equal to

\[ \mathcal{P}(\sigma; \Omega_{K,1}, \ldots, \Omega_{K,n}), \]

where (for \( i = 1, \ldots, l \) \( \Omega_{K,i} \) is a set which plays the same role for \( K_i \) as the set \( \Omega_i \) (if \( \zeta_i = 1 \)) or \( \Omega_0 \) (if \( \zeta_i = 0 \)) for \( F \) (see above; for more details see [2], the remark before Theorem 6 and the method of computation of \( p_0 \) in Part II). Analogously for \( P_{K,1} \).

5) Probabilistic parameters of a probabilistic connective \( \varphi_j \) (in a given \( \Phi \)) are defined by the following equalities:

\[ p_0^j = \sum_{(\omega, \omega') = 1} P(\gamma_2, \ldots, \gamma_{j-1}, 0, \gamma_{j+2}, \ldots, \gamma_k, \omega), \]

\[ p_1^j = \sum_{(\omega, \omega') = 1} P(\gamma_2, \ldots, \gamma_{j-1}, 1, \gamma_{j+1}, \ldots, \gamma_k, \omega). \]

In the preceding theorem, then,

\[ p_0^j = P_j(0; 1) \quad \text{and} \quad p_1^j = P_j(1; 1). \]

We will denote \( p_0^j, p_1^j \) by \( p_{\varphi_j} \).

Proof of Theorem 1: We can use Theorem 5 from [2].

1) According to this theorem we obtain \( P_{K} = P_{F,0} P_{K}; \) we know that \( p_0 = 0, 0, \ldots, 0, 1)^T \) and the assertion is evident.

2) Again, according to Theorem 5, \( P_{\text{Int}(\varphi,F)} = P_{K} P_F \). It is well known that \( P_F = (0, 1, \ldots, 1)^T \) and thus

\[ P_{\text{Int}(\varphi,F)} = \sum_{(\omega, \omega') = 1} P_K P_F = \sum_{\zeta \in \{0, 1\}^n} p_{K,\zeta F}. \]
Recall that the matrix $P_K$ is stochastic; hence, the right hand-side of (1) equals $1 - p_{K,a,0}$ and we obtain the second assertion.

3. We have $P_{int(a,b)} = (P_{K,a} - P_{K,0})$ and so we obtain the third assertion (by Theorem 5).

**Examples.** 1) Consider an LP-expression $\Phi = [F, \Omega_F, \mathcal{P}_F]$, where $F \simeq \varphi_1(x_1) \& \varphi_2(x_2)$; we assume stochastic independence of $\Phi$ and we denote $q = 1 - p$. Let $p_{\varphi_1} = p_{\varphi_2} = (1 - p, p)$. Then:

$$P = \begin{pmatrix} p^2 & pq, pq, q^2 \\ pq, p^2, q^2, pq \\ q^2, pq, pq, p^2 \end{pmatrix}$$

and $p_k = (0, 0, 1)^T$. Thus

$$p_F = p_{p_k} = (q^2, pq, pq, p^2)^T.$$

2) Analogously, for $F \simeq \varphi_1(x_1) \lor \varphi_2(x_2)$. There is $p_F = (0, 1, 1, 1)^T$ and thus

$$p_F = p_{p_F} = \begin{pmatrix} q^2 + 2pq \\ p^2 + pq + q^2 \\ p^2 + pq + q^2 \\ p^2 + 2pq \end{pmatrix} = \begin{pmatrix} 1 - p^2 \\ 1 - pq \\ 1 - pq \\ 1 - q^2 \end{pmatrix}.$$ 

3) Let us now consider an LP-expression $\Phi = [F, \Omega_F, \mathcal{P}_F]$, where $F \simeq \varphi_3(x_1) \lor \varphi_3(x_2)$. Then we have $K_1 \simeq \varphi_3(x_1)$ and $K_2 \simeq \varphi_3(x_2)$. We assume stochastic independence and $p_{\varphi_2} = p_{\varphi_3} = (q, p)$ again. Now we obtain ($P_k$ is equal to the matrix (2))

$$P_F = \begin{pmatrix} 1 - p^2 \\ 1 - pq \\ 1 - q^2 \end{pmatrix} = \begin{pmatrix} p + (q - p) p^2 \\ p + (q - p) pq \\ p + (q - p) q^2 \end{pmatrix}$$

Let us have an LP-expression $\Phi = [F, \Omega_F, \mathcal{P}_F]$, where $F$ is

$$\varphi_3((F_2 \& \ldots \& F_n) \lor \ldots \lor (F_{n-1} \& \ldots \& F_n)).$$

We will assume further that $x = (x_1, \ldots, x_n)$ are variables occurring in $F$ and $x_i = (x_{i1}, \ldots, x_{in})$ are variables occurring in $F_i (i = 1, \ldots, n_i)$. We can now formulate the following theorem.

**Theorem 2.** Consider an LP-expression $\Phi$ such as in Theorem 1. Let, moreover, all variables be different (i.e. each variable cannot occur more then once on $F$)
and let

\[ P(\gamma; \omega) = \prod_{i=1}^{n} P(\gamma_i; \omega_i), \]

where, if \( \varphi_1, \ldots, \varphi_k \) are probabilistic connectives occurring in \( F \), \( \gamma_i \) is \((\gamma_{i_1}, \ldots, \gamma_{i_k})\) and \( \omega_i \) is \((\omega_{i_1}, \ldots, \omega_{i_k})\).

Values of the variables \( x_i \) will be denoted by \( \sigma_i \) (if we have the evaluation \( \sigma \) on \( x \)).

Then

\[ P_{\text{ant}(\omega_i; F); \sigma} = 1 - \prod_{j=1}^{l} (1 - \prod_{i=j+1}^{n} P_{i; \omega}), \]

where \( P_{i; \omega} = P(\sigma_i; \Omega_i) \), and

\[ P_{F; \sigma} = P_0 + (P_0 - P_0) \prod_{j=1}^{l} (1 - \prod_{i=j+1}^{n} P_{i; \omega}). \]

Remark: It is helpful to note that

\[ P(\sigma_i; \Omega_i) = P(\text{func}_F F(\sigma_i; \omega) = 1) \]

as we know from [2]. According to our assumptions

\[ \text{func}_F F(\sigma_i; \omega) \]

is equal to \( \text{func}_F F(\sigma; \omega) \).

Example. If we return to point 3) of the preceding example, we can see that Theorem 2 immediately gives

\[ \begin{pmatrix}
    p + (q - p)(1 - q)(1 - q) \\
    p + (q - p)(1 - q)(1 - p) \\
    p + (q - p)(1 - q)(1 - p) \\
    p + (q - p)(1 - p)(1 - p)
\end{pmatrix} = \begin{pmatrix}
    p + (q - p)p^2 \\
    p + (q - p)pq \\
    p + (q - p)pq \\
    p + (q - p)q^2
\end{pmatrix}. \]

Remark. The situation described in Theorem 2 is well known to everyone who deals with unreliable logical nets.

Proof of theorem 2. As a consequence of stochastical independence we have

\[ P_{K; \sigma} = P_{F; \sigma} \prod_{i=1}^{s} P_{i; \sigma}. \]

Furthemore

\[ P_{K; \Omega} = 1 - P_{K; \sigma} = 1 - \prod_{i=1}^{s} P_{i; \sigma}, \]

and with respect to stochastic independence

\[ P_{K; 0} = \prod_{j=1}^{l} (1 - \prod_{i=j+1}^{n} P_{i; \sigma}). \]
(pK_{\sigma, \beta} \) is a member of the first column of P_K) and thus
\[
\prod_{j=1}^{n_j} (1 - \prod_{i=j+1}^{n_j} p_{i; \sigma_i}).
\]
The second assertion is a consequence of point 3) from the previous theorem.

Remarks. 1) The assumption that the variables must be different is not essential. But for practical computation it is useful to have different variables and to denote them in such a way that \( x_1 = (x_{11}, \ldots, x_{1n}) \), \( x_2 = (x_{21}, \ldots, x_{2n}) \). Then \( \sigma = (\sigma_1, \ldots, \sigma_n) \) and the computation of matrices \( P_P, P_K \) is simpler. We must consider this fact from the point of view of the inductive computation of \( p_0 \) for an LP-expression \( \Phi \) which is in PDNF.

On the other hand it is possible to transform every LP-form \( F \) to the LP-form \( F' \) in which all variables are different. Then we obtain the result for the original LP-expression by omitting some members in \( p_0 \).

2) For Theorem 2 it is sufficient that
\[
P(\sigma; \omega_1^*, \ldots, \omega_n^*) = \prod_{i=1}^{n_j} P(\sigma_i; \omega_i^*),
\]
where \( P(\sigma; \omega_1^*, \ldots, \omega_n^*) \) is the joint probability of evaluations of \( F_1, \ldots, F_n \) if the variables are evaluated by \( \sigma \) (\( \omega_i^* \) is a variable with two possible values \( \Omega_i^0, \Omega_i^1 \); see point 4) of the remark after Theorem 1).

![Diagram](image)

Fig. 1.

Examples. 1) Let us consider an LP-net \([N, \Omega_N, \mathcal{P}_N]\) (fig. 1). Assume stochastic independence and let the probabilistic parameters of all probabilistic elements (connectives) be (0.4, 0.6). The corresponding LP-expression is \([F(N), \Omega_N, \mathcal{P}_N]\),
where

\[ F(N) = \psi_5(((\psi_1(\neg x_1) \land x_2) \lor (\psi_2(x_1) \land \psi_3(x_3))) \lor (\psi_4(x_2 \land x_3))). \]

Then

\[ F_1 = \psi_1(\neg x_1), \quad F_2 = x_2, \quad F_3 = \psi_2(x_1), \quad F_4 = \psi_3(x_3), \quad F_5 = \psi_4(x_2 \land x_3) \]

and

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<tr>
<th>( \sigma )</th>
<th>( p_{F_1} )</th>
<th>( p_{F_2} )</th>
<th>( p_{F_3} )</th>
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where \( \zeta \) are, for comparison, the values of expression \( F(N') \) without probabilistic elements (connectives).

2) Consider an LP-net \([N, \Omega_N, \psi_N]\) (fig. 2). Assume stochastic independence. The
corresponding LP-expression is \([F(N), \Omega_N, \mathcal{P}_N]\), where \(F(N)\) is

\[
\phi_a((\varphi_3(\neg x_1 \lor x_3 \lor \varphi_1(\neg x_2)) \land x_2) \lor (\varphi_7(x_1 \land \varphi_2(\neg x_2) \land \varphi_3(x_1)) \land \varphi_3(x_2) \land x_3).
\]

For all probabilistic connectives let the probabilistic parameters be \((0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1)\). Then

\[
F_1 \simeq \varphi_6(\neg x_1 \lor x_3 \lor \varphi_1(\neg x_2)), \quad F_2 \simeq x_2, \quad F_3 \simeq \varphi_7(x_1 \land \varphi_2(\neg x_2) \land \varphi_3(x_1))
\]

\[
F_4 \simeq \varphi_8(x_2), \quad F_5 \simeq \varphi_9(x_2 \land x_3)
\]

(for the second step of induction) and \(p_F\) is

\[
(0.25, 0.25, 0.77, 0.88, 0.2, 0.31, 0.33, 0.88, 0.77)^T;
\]

for comparison, the characteristic vector of this net without probabilistic elements is \((0, 1, 0, 0, 0, 0, 0, 1)^T\).

3) As an example, we can compute the characteristic vector of LP-expression \(\Phi'\) from the proof of Theorem 4 of [2]. There we had \(\Phi' = [F', \Omega_{F'}, \mathcal{P}_{F'}]\), where

\[
F' \leq \bigvee_{i=1}^{2m} \varphi_i(x_1^i) \land x_2^i \land \ldots \land x_m^i
\]

(where

\[
x_j^i = \begin{cases} x_j & \text{if } e_j = 1, \\ \neg x_j & \text{if } e_j = 0. \end{cases}
\]

The LP-expression \(\Phi'\) was assumed to be stochastically independent, i.e.

\[
P(\gamma; \omega) = \prod_{i=1}^{2m} P(\gamma_i; \omega_i),
\]

and \(P(0; 1) = 0, P(t; 1) = p_i (i = 1, \ldots, 2m)\). According to Theorem 2 we have

\[
p_{F'\sigma} = 1 - \prod_{i=1}^{2m} (1 - \prod_{j=\sigma_i + 1}^{\sigma_i + 1} p_{j; \sigma_j}),
\]

where \(\sigma_j = \sigma_{i}^j\) if \(j = (i - 1) m + l (l = 1, \ldots, m)\). For

\[
j = (i - 1) m + l (l \geq 2) \quad p_{j; \sigma_j} = \begin{cases} 0, & \text{if } \sigma_i = e_i^j; \\ 1, & \text{if } \sigma_i = e_i^j. \end{cases}
\]

and for

\[
j = (i - 1) m + 1 \quad p_{j; \sigma_j} = \begin{cases} 0, & \text{if } \sigma_i = e_i^j; \\ p_0, & \text{if } \sigma_i = e_i^j. \end{cases}
\]

Then

\[
1 - \prod_{j=(i-1)m+1}^{im} p_{j; \sigma_j} = \begin{cases} 1, & \text{if } \sigma = e', \\ 1 - p_0, & \text{if } \sigma = e'. \end{cases}
\]
and thus
\[ p_{F \sigma} = p_i \ (\sigma = \epsilon). \]

As we mentioned in [2], the class of stochastically independent LP-expressions is not closed with respect to the transformation to PDNF. During this transformation, stochastically independent groups of functionally equivalent probabilistic connectives are formed. The same problem can arise if we consider logical nets with stochastically independent probabilistic elements, but, in addition, with elements of forkjunction. If we have such a net, e.g. having the structure of fig. 3, we can transform this net to that net which will be an LP-net according to our concept (see [2], Def. 2). Let us consider the net from fig. 3 and assume that \( N_1, N_2, N_3 \) are nets without elements of forkjunction. Let \( N'_1 \) be a new net such that \( N'_1 = [N_1, \Omega_{N_1}, \Phi_{N_1}] \) and \( N'_2 = [N_1, \Omega_{N_1}, \Phi_{N_1}] \) are functionally equivalent, i.e. \( P(\text{func}_{N_1} (\sigma, \omega) = \text{func}_{N_1} (\sigma, \omega)) = 1 \) (for every \( \sigma \)) (see [2], Def. 10). Then we transform the net from fig. 3 to the net having the structure of fig 4. The new net is a net in our sense of word.

Generally we proceed in the same way as in the process of transforming an LP-expression to PDNF. If we need the new subnet, we take new functionally equivalent probabilistic elements and substitute them in the given structure. Transforming the net, we proceed by induction on degrees of probabilistic elements. For more particulars see [2], Lemma 2, Theorems 3 and 7 and their proofs.

Now, we must consider the computation of characteristic vectors for these cases, i.e., when we have an LP-expression for which

\[
P(\gamma'; \omega) = \begin{cases} P(\gamma; \omega), & \text{if for } \gamma_i = \gamma_{i1} = \ldots = \gamma_{i\alpha_i} \\ \text{is } \omega_i = \omega_{i1} = \ldots = \omega_{i\alpha_i}, & (i = 1, \ldots, n), \\ 0 & \text{in other cases} \end{cases}
\]

(\( \gamma' = (\gamma_1, \ldots, \gamma_n, \gamma_{11}, \ldots, \gamma_{m}), \gamma = (\gamma_1, \ldots, \gamma_n), \) analogously for \( \omega', \omega \)).
Then
\[ \varphi_i = (\varphi_{i1}, \varphi_{i2}, \ldots, \varphi_{in}) \]
is such a group of functionally equivalent probabilistic connectives.

We can proceed according to Theorem 1; for computation of probabilities \( p_{F_i;\sigma} \) and \( p_{G_i;\sigma} \) we must consider the occurrence of functionally equivalent subexpressions.

First, we can consider some particular cases:

1) Assume \( F_i(x_1) \equiv f_1 F_2(x_2) \). Then (for \( F_i = (F_1, F_2) \))

\[ p_{F_{i,\sigma}} = p_{F_{i,\sigma}} = p_{F_{i,\sigma}} \]

\( (p_{F_{i,\sigma}}, p_{F_{i,\sigma}} \) are members of \( p_{F_1} \) and \( p_{F_2} )

2) Assume \( F_i(x_1) \equiv f_{F_2}(x_2) \), \( F_i = (F_1, \varphi(F_2)) \), and let probabilistic parametres of \( \varphi_j \) be \( p_0, p_1 \). Then, using the general method of computation (see [2], Part II), we obtain

\[ p_{F_{i,\sigma}} = \sum_{\gamma \in \{0, 1\}} p(\gamma; \Omega_{F_1}, \Omega_{F_2}, 1) = \]

\[ = P(1; \Omega_{F_1}, \Omega_{F_2}, 1) = P(1; \Omega_{F_1}, 1), \]

because

\[ P(0; \Omega_{F_1}, \Omega_{F_2}, 1) = 0. \]

Then \( p_{F_{i,\sigma}} \) is the probability for subexpression corresponding to \( F_1, F_2 \) and the probabilistic connective \( \varphi_j \). Using stochastic independence the right-hand side of (3) is then equal to \( p_{F_{i,\sigma}} p_{F_2} \).

Now we can formulate an auxiliary theorem which solves a more general case.

**Theorem 3.** Consider two subexpressions with subforms

\[ F_1 \equiv G(F_1, x_1) \quad \text{and} \quad F_2 \equiv G_2(F_2, x_2), \]

where \( F_1 \equiv F_2 \) and for every \( \gamma \) let

\[ p(\gamma; \omega) = p_1(\gamma_1; \omega_1) p_2(\gamma_2; \omega_2) p_3(\gamma_3; \omega_3), \]

where \((\gamma_1, \omega_1)\) corresponds to probabilistic connectives from \( F_1, (\gamma_2, \omega_2) \) to probabilistic connectives from \( G_1 \), and \((\gamma_3, \omega_3)\) to probabilistic connectives from \( G_2 \).

Then for every \( \sigma \in \{0, 1\}^m \) \((F_i = (F_1, F_2))\), we have following:

\[ p_{F_{i,\sigma}} = p_{F_{i,\sigma}} p_{G_{i,\sigma}} \]

Proof. If we apply the disjointness of random events, we have

\[ p_{F_{i,\sigma}} = \sum_{\lambda \in \{0, 1\}^2} p(\sigma_0, \lambda, \sigma_1, k, \sigma_2; \Omega_1, \Omega_2, \Omega_3, \Omega_4) \]
where $P'(o_0, l, o_1, k, o_2; o_3, o_4)$ is the joint probability for the subexpressions corresponding to $F_1$, $F_2$, $G_1(y, x_1)$ and $G_2(y, x_2)$.

Furthermore,

$$P_{F_1, o} = \sum_{i \in \{0,1\}} P'(o_0, l, o_1, l, o_2; o_3, o_4)$$

and by applying stochastic independence we complete the proof.

In the same way we can proceed in some more complicated cases, e.g., if $F_1$ is $G_1(F'(x_1), F'(x_2), x_3)$ and $F_2$ is $G_2(F_2'(x_1), F_2'(x_2), x_3)$, where $F_1 \equiv F_2$ and $F_1' \equiv F_2'$.

Moreover, we can proceed in the same way in the computation of probabilities $P_{K,x,o}$.

**Conclusion.** In all cases we compute the characteristic vector of an LP-expression (LP-net) with the help of its PDNF (more precisely, with the help of a functionally equivalent LP-expression in PDNF). We can apply points 1) and 2) from Theorem 1. In real cases there can be some simplifications:

1) If a probabilistic connective is stochastically independent on the probabilistic connectives from its int $(\varphi, F)$, we can apply point 3) from Theorem 1, or its modification for joint probabilities, if the connective is not the last one.

2) If the PDNF is stochastically independent, we can apply Theorem 2 inductively.

3) If a given PDNF contains stochastically independent groups of functionally equivalent probabilistic connectives $\varphi^1, \varphi^2, ..., \varphi^r$ such that for every $k \leq n$ and every $i, j \leq n$, if $\varphi_i \in \varphi^k$ and $\varphi_j \in \varphi^k$, then $d(\varphi_i, F) = d(\varphi_j, F)$ ($d(\varphi, F)$ means the degree of probabilistic connective in $F$; see [2]), we can use all three points of Theorem 2, because the independence of $\varphi_i$ on int $(\varphi_i, F)$ is preserved, and the probabilities $P_{F_1, o, 1}$ and $P_{K,x,o}$ can be computed analogously as in Theorem 3. It is possible to formulate a general theorem for this case, but it would be too complicated and incomprehensible. If, for $F_1, ..., F_k$ for which we compute $P_{F_1, o, 1}$ or $P_{F_1, o}$, it holds that no pair $F_i, F_j (i \neq j)$ contains a pair of functionally equivalent subexpressions $F_i < F_i$ and $F_j < F_j$, we can apply Theorem 2.

**Remarks.** 1) The condition of stochastic independence of $\varphi_i$ on int $(\varphi_i, F)$ is preserved if we transform a net with stochastically independent probabilistic elements and elements of fork junction to our LP-net ($F$ describes the structure of our new net). In the original net notion of the degree of probabilistic connective is meaningless.

2) If we compute $P_o$ recursively for a given LP-expression, we must use Theorem 1, paying attention to the joint probabilities. In a given inductive step (at the beginning) we have the following situation:

We have subforms $F_1, ..., F_n$ as in Theorem 1 and moreover, subforms $G_1, ..., G_k$.

$$F' \simeq (F_1 \& ... \& F_n) \lor ... \lor (F_{n-1} \& ... \& F_n)$$
is then the interior of a probabilistic connective $\varphi_t$. We must consider the joint probabilities

$$P(\sigma, \gamma'; \omega_1^*, \ldots, \omega_k^*, \omega_1^G, \ldots, \omega_k^G, \omega),$$

where

$$\gamma' = (\gamma_1, \ldots, \gamma_n), \quad \omega' = (\omega_1, \ldots, \omega_n).$$

Now we have to proceed computation from point 1) of Theorem 1 for $K_t \equiv F_1 \& \ldots \& F_n$ (variables $\omega_1^*, \ldots, \omega_n^*$) for any given value of other variables. We use matrices

$$P_F(\gamma'; \omega_1^*, \ldots, \omega_k^G, \omega')$$

and we obtain the joint probabilities

$$P(\sigma, \gamma'; \omega_1^*, \ldots, \omega_k^*, \omega_1^G, \ldots, \omega_k^G, \omega')$$

and now we repeat this computation for $K_2$ etc. In the same way we apply point 2) to the joint probabilities $P(\sigma, \gamma'; \omega_1^*, \ldots, \omega_k^*, \omega_1^G, \ldots, \omega_k^G, \omega')$ for any given value of $\gamma', \omega_1^G, \ldots, \omega_k^G, \omega'$.

3) The computation is simpler for an LP-expression in PDNF, obtained from a stochastically independent LP-expression. Functionally equivalent subexpressions can occur only in $K \equiv K_1 \lor \ldots \lor K_n$.

We can see that

$$P(\sigma, \gamma'; \omega_1^*, \ldots, \omega_k^*, \omega') = P_1(\sigma; \omega_1^*, \ldots, \omega_k^*) P_2(\sigma; \omega_1^G, \ldots, \omega_k^G) \prod_{i=1}^n P(\gamma_i; \omega_i)$$

and for every $F_i \equiv F_j$ and for $\omega_i^* = \omega_j^*$

$$P_1(\sigma; \omega_1^*, \ldots, \omega_n^*) = 0.$$ 

We will now turn our attention to the vectors of LP-expressions. First, we define a characteristic matrix of a probabilistic operator (for the concept of probabilistic operator see [2]) as a matrix $P = (p_{a,b})$, where $p_{a,b} = P(a)$ (a is a symbol from input alphabet, $b$ a symbol from the output alphabet). Analogously we define a characteristic matrix of a vector of LP-expressions. Let us consider a vector $\Phi = [F, \Omega_F, \Phi_F]$, where $F = (F_1, \ldots, F_k)$ and $F$ contains variables $x_1, \ldots, x_n$. Then we call the matrix

$$P = (p_{ij})_{i=1}^{2^n}, \quad \text{where} \quad p_{ij} = P(\sigma; \Omega_{ij}, \ldots, \Omega_{i})$$

($\sigma, \zeta$ are binary forms of the numbers $i - 1, j - 1$), the characteristic matrix of $\Phi$ (see Theorem 5 of [2]).

We now formulate a theorem about the computation of this matrix for a particular case.
Theorem 4. Let us take a vector \( \Phi = [F, \Omega_F, \mathcal{P}_F] \), where \( F = (F_1, \ldots, F_k) \) and for every \( \gamma \)

\[
P(\gamma; \omega) = \prod_{i=1}^{k} P(\gamma_i; \omega_i),
\]

where

\[
\gamma_i = (\gamma_{i1}, \ldots, \gamma_{it}) \quad \text{and} \quad \omega_i = (\omega_{i1}, \ldots, \omega_{it})
\]

for \( F_i \) containing \( \varphi_{i1}, \ldots, \varphi_{it} \) (We call such a vector weakly stochastically independent). We put \( \sigma_i = \sigma_{i1}, \ldots, \sigma_{it} \) for \( F_i \) containing \( x_{i1}, \ldots, x_{it} \). Let \( p_1 = (p_{1;\omega}) \) be the characteristic vector of \( \Phi_1 = [F_1, \Omega_{F_1}, P_{F_1}] \). Then

\[
p_{ij} = \prod_{i=1}^{k} (p_{1;\omega})^{\gamma_i} (1 - p_{1;\omega})^{1 - \gamma_i}.
\]

Proof. We know that \( p_{ij} = P(\sigma_i; \Omega_{ij}, \ldots, \Omega_{it}) \); stochastic independence implies

\[
p_{ij} = P(\sigma_i; \Omega_{ij}), \ldots, P(\sigma_i; \Omega_{it}).
\]

We have

\[
P(\sigma_i; \Omega_{ij}, \ldots, \Omega_{it}) = p_{1;\omega}, P(\sigma_i; \Omega_{ij}) = 1 - p_{1;\omega}
\]

and thus

\[
p_{ij} = \prod_{l=1}^{k} p_{1;\omega} \prod_{l=1}^{k} (1 - p_{1;\omega}). \quad \square
\]

In other cases we have to compute these probabilities inductively and simultaneously for all LP-expressions in the vector analogously as for a single LP-expression.

A stochastically independent LP-net with more than one output can be transformed into a vector of LP-nets. The method is analogous to constructing the canonical LP-expression in [2]. We transform our net into a logical net (remembering the position of probabilistic elements). And now we transform this net into a vector of logical nets in the way described, e.g., in [3]. Then we can put probabilistic elements in these nets. Since we needed new subnets structurally equivalent to the original ones in the preceding step, we have to use some new functionally equivalent elements.

Then we have a vector of LP-nets, where two different nets \( N_i, N_j \) can contain functionally equivalent subnets, i.e. \( F(N_i), F(N_j) \) obtain subexpressions \( F_1 = F_2, F_1 < F(N_i), F_2 < F(N_j) \). We know how to compute the characteristic vectors of LP-expressions which can contain subexpressions \( F_1 \equiv F_2 \). The characteristic matrix of the vector of LP-expressions is then computed similarly to the subsequent example.

Example. Consider a vector \( \Phi = [F, \Omega_F, \mathcal{P}_F] \), where

\[
F \simeq \begin{pmatrix} (F_1(x)) \\ (F_2(x)) \end{pmatrix}
\]
and
\[ F_1 \simeq G_1(F_1(x), x), \quad F_2 \simeq G_2(F_2(x), x). \]

Let
\[ F_1 = F_2 \]
and let
\[ P(y; \omega) = P_{G_2}(y; \omega_2) P_{G_1}(y; \omega_1). \]

If we denote \( \xi = (\xi_1, \xi_2) \), \( \sigma = (\sigma_1, \ldots, \sigma_m) \) the binary forms of the numbers \( j - 1 \), \( i - 1 \), then
\[
P_{ij} = (p_{G_1; \sigma, \omega})^{\xi_1} (1 - p_{G_1; \sigma, \omega})^{1 - \xi_1} (p_{G_2; \omega, \omega})^{\xi_2} (1 - p_{G_2; \omega, \omega})^{1 - \xi_2},
\]
\[
(1 - p_{F, \sigma}) + (p_{G_1; \sigma, \omega})^{\xi_1} (1 - p_{G_1; \sigma, \omega})^{1 - \xi_1} (p_{G_2; \omega, \omega})^{\xi_2} (1 - p_{G_2; \omega, \omega})^{1 - \xi_2} p_{F, \sigma},
\]
where \( p_{G_1}, p_{G_2}, p_F \) are the characteristic vectors of \( G_1(y, x), G_2(y, x), \) and \( F_1(x) \).

Proof. We have
\[
P_{ij} = P(\sigma; \Omega_1, \Omega_2) =
\]
\[
\sum_{l, k \in \{0,1\}^*} P'(\sigma, l, \sigma, \Omega_1^l, \Omega_2^k, \Omega_1^l, \Omega_2^k).
\]

If we consider that \( P'(\sigma, l, \sigma, \Omega_1^l, \Omega_2^k, \sigma) = 0 \) for \( l = k \), then the right-hand side of (5) equals
\[
\sum_{l, k \in \{0,1\}^*} P'(\sigma, l, \sigma, \Omega_1^l, \Omega_2^k, \sigma),
\]
and if we apply the independence condition we complete the proof. \( \square \)

Lastly, we will pay attention to probabilistic automata. Let us have a vector of LP-expressions
\[ \Phi = [P, \Omega, \Phi], \]
where
\[ \Phi \simeq (G_1, G_2), \quad G_1 \simeq (F_1(x, z), \ldots, F_i(x, z)) \]
and
\[ G_2 \simeq (G_1(x, z), \ldots, G_i(x, z)). \]

We can compute
\[ P(\text{func}_z(x, z) = \delta, \xi; x = \sigma, z = \xi). \]

If we have an output alphabet \([\Omega_{\sigma, \delta\xi}]\) \( ([\Omega_{\sigma, \delta}] = \delta) \) and a state alphabet \([\Omega_{\delta, \xi}]\) \( ([\Omega_{\delta, \xi}] = \xi) \), we have a probabilistic automaton. If \( P(\gamma; \omega) = P_{G_1}(\gamma; \omega_1) P_{G_2}(\gamma; \omega_2) \) we have a probabilistic Mealy automaton. We have these automata with the output alphabet \([\xi]\), the input alphabet \([\sigma]\) and the state alphabet \([\delta]\), where \( \sigma \) are values of variables \( x(t - 1) \), \( \delta \) are values of \( z(t) \) and \( \xi \) are values of \( z(t - 1) \). If we include a new kind of primitive elements — delay elements and the rules for their connection
(as in [1] or [3]), we have a net realizing a probabilistic automaton. We can transform
this net by the elimination of elements of forkjunction into a net which can be described
by the vector $\Phi = [F, Q_\varphi, P_\chi]$, $F = F(N)$, where $F$ is as in (6), and where
both $\text{func}_{\varphi}(x(t), z(t - 1)) = y(t)$ and $(x(t), z(t - 1)) = z(t)$ are analogous to the
canonical equations of the deterministic automata (see [3]). We can calculate the
characteristic matrix $P = (p_{a \cdot z \cdot e \cdot a})$ in the same way as in the calculation of characteristic matrix of the vector of LP-expressions.

The results which are described in this paper can be applied to the problem of
realization of probabilistic automata. These problems will be considered in another
paper [4].

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