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A New Generalization of Entropy and its Characterization

BHU DEV SHARMA, ISHWAR SINGH

We generalize branching property by considering the difference of entropies before and after grouping as nonhomogeneous functions of group probability. Starting from a suitable set of axioms which are modifications of those considered by Fadeev and later by Vajda for studying Shannon's entropy and degree α-entropy, new entropy is obtained, which is quite general and contains several parameters. Functional equations resulting from such an approach have also been formed.

1. INTRODUCTION

Shannon [6], in his fundamental paper introduced a quantitative measure of information, called entropy, which, for a discrete probability distribution

$$P = (p_1, p_2, \ldots, p_n), \quad p_i \geqq 0, \quad \sum_{i=1}^{n} p_i = 1,$$

is given by

$$H_n(P) = -\sum_{i=1}^{n} p_i \log p_i \tag{1.1}$$

Shannon characterized this measure taking a reasonable set of postulates. The quantity has been subsequently characterized in various ways by several authors (see Aczel [1]). By suitably changing the postulates and sometimes otherwise, generalizations of this measure have been studied.

Among several ways of characterization two elegant approaches are to be found in the work of Fadeev [4] and Chaundy and McLeod [2]. Basic postulate in Fadeev's approach is the branching property, viz:

$$H_n(p_1, p_2, \ldots, p_n) = H_{n-1}(p_1 + p_2, p_3, \ldots, p_n) + + (p_1 + p_2) H_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}\right) \tag{1.2}$$

$n = 3, 4, \ldots$ for the distribution

$$P = (p_1, p_2, \ldots, p_n), \; p_i \geqq 0, \; \sum_{i=1}^{n} p_i = 1 .$$

Both of the above mentioned approaches have been extensively employed and generalized (refer Sharma and Taneja [7]). A generalization of (1.2) taken by Havrda and Charvát [4] is

$$(1.3) \qquad H_n(p_1, p_2, \ldots, p_n) = H_{n-1}(p_1 + p_2, p_3, \ldots, p_n) +$$

$$+ (p_1 + p_2)^\alpha H_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}\right), \quad \alpha > 0, \quad \alpha \neq 1 .$$

In Section 2, of this paper we consider a broad generalization of branching property and in Section 3 we obtain the form of entropy resulting from it. In Section 4, we obtain a functional equation arising from the generalized branching property taken in Section 2.

2. GENERALIZED BRANCHING PROPERTY

A close observation of (1.2) and (1.3) reveals that it is equivalent to tacitly assuming that, difference of the entropies before and after grouping the events, i.e.

$$(2.1) \qquad H_{n+1}(p_1, \ldots, p_{n-1}, p_n q, p_n(1 - q)) - H_n(p_1, \ldots, p_n),$$

is a homogeneous linear function of p_n (group probability) for Shannon's entropy and of positive degree $\alpha(\neq 1)$ for Havrda-Charvát's entropy.

Taneja [9] has considered a unified form of the linear and degree α branching property by considering

$$(2.2) \qquad H_{n+1}(p_1, p_2, \ldots, p_n q, p_n(1 - q)) =$$

$$= H_n(p_1, p_2, \ldots, p_n) + g(p_n) H_2(q, 1 - q)$$

and has shown that under continuity, symmetry and boundary conditions: $H_2(\frac{1}{2}, \frac{1}{2}) = 1$, $H_2(1, 0) = 0$, $g(p)$ can take only two forms viz. $g(p) = p$ and $g(p) = p^\alpha$, $\alpha \neq 1$, $\alpha > 0$ leading to the two cases, viz. Shannon's measure and Havrda-Charvát's measure

$$(2.3) \qquad H_n^\alpha(P) = \frac{\sum_{i=1}^{n} p_i^\alpha - 1}{2^{1-\alpha} - 1}, \quad \alpha > 0, \quad \alpha \neq 1 .$$

This measure has been studied later by Vajda [10] and Daroczy [3].

As a natural generalization, it occurs to examine the case when the difference of entropies before and after grouping is not necessarily a homogeneous function of the group probability. In this situation we may considei that (2.1) is a function of the type

$$(2.4)\qquad \mu_0 p_n\, h_0(q) + \mu_1 p_m^{\alpha_1}\, h_1(q) + \ldots + \mu_l p_n^{\alpha_l}\, h_l(q)\,,$$
$$\alpha_i > 0\,,\quad \alpha_i \neq 1\,,\quad l \geqq 1\,,$$

where $h_0(q), h_1(q), \ldots, h_l(q)$ are some suitable functions of q and the μ's are arbitrary constants. In other words we can take the branching property in the form

$$(2.5)\qquad H_{n+1}(p_1, p_2, \ldots, p_n q, p_n(1-q)) = H_n(p_1, p_2, \ldots, p_n) + $$
$$+ \mu_0 p_n\, h_0(q) + \mu_1 p_n^{\alpha_1}\, h_2(q) + \ldots + \mu_l p_n^{\alpha_l}\, h_l(q)\,.$$

In fact we start with an apparently more general form of branching property given by

$$(2.6)\qquad H_{n+1}(p_1, p_2, \ldots, p_n q, p_n(1-q)) =$$
$$= H_n(p_1, p_2, \ldots, p_n) + \sum_{t=0}^{l} \mu_t\, g_t(p_n)\, h_t(q)$$

and consider that $h_t(q)$ is the same as $H_2^{(\mu_t, g_t)}(q, 1-q)$, an information type measure of the probability distribution $(q, 1-q)$ depending on a parameter t. Further, to indicate the dependence of the measure $H_n(p_1, \ldots, p_n)$ on the constants μ's and the functions g's, in what follows, we write $H_n^{(\mu_0,\ldots,\mu_l; g_0,\ldots,g_l)}(p_1, \ldots, p_n)$ in place of just $H_n(p_1, \ldots, p_n)$. Thus we can announce axioms for the measure $H_n^{(\mu_0,\ldots,\mu_l; g_0,\ldots,g_l)}\,.$ $.\,(p_1, \ldots, p_n)$ as follows:

(i) The measure $H_n^{(\mu_0,\ldots,\mu_l; g_0,\ldots,g_l)}(p_1, \ldots, p_n)$ is a symmetric function of $p_1, \ldots, p_n$.

(ii) The measure $H_n^{(\mu_0,\ldots,\mu_l; g_0,\ldots,g_l)}(p_1, \ldots, p_n)$ is a continuous function of its arguments.

(iii) The measure $H_n^{(\mu_0,\ldots,\mu_l; g_0,\ldots,g_l)}(p_1, \ldots, p_n)$ satisfies the branching property

$$H_{n+1}^{(\mu_0,\ldots,\mu_l; g_0,\ldots,g_l)}(p_1, \ldots, p_{i-1}, v_{i1}, v_{i2}, p_{i+1}, \ldots, p_n) =$$
$$(2.7)\qquad = H_n^{(\mu_0,\ldots,\mu_l; g_0,\ldots,g_l)}(p_1, p_2, \ldots, p_n) + \sum_{t=0}^{l} \mu_t\, g_t(p_i)\,.\, H_2^{(\mu_t, g_t)}\left(\frac{v_{i1}}{p_i}, \frac{v_{i2}}{p_i}\right)$$

where $v_{i1} + v_{i2} = p_i > 0\,,\quad i = 1, 2, \ldots, n\,.$

(2.8) (iv)
$$H_2^{(\mu_0,\ldots,\mu_l; g_0,\ldots,g_l)}(\tfrac{1}{2}, \tfrac{1}{2}) = 1\,,$$
$$H_2^{(\mu_0,\ldots,\mu_l; g_0,\ldots,g_l)}(1, 0) = 0\,.$$

Before coming to the characterization of entropy under the above axioms, we need to examine the nature of measures $H_2^{(\mu_t, g_t)}(q, 1-q)$ and the functions $g_0, \ldots, g_l$. As a first step let us take $\mu_0 = \mu \neq 0$ and $\mu_1 = \mu_2 = \mu_3 = \ldots = 0$, $g_1(x) = g(x)$. The resulting measure $H_n^{(\mu, g)}(p_1, \ldots, p_n)$ satisfies then the relation

$$H_{n+1}^{(\mu,g)}(p_1, \ldots, p_{n-1}, p_n q, p_n(1-q)) = \\ = H_n^{(\mu,g)}(p_1, \ldots, p_n) + \mu\, g(p_n)\, H_2^{(\mu,g)}(q, 1-q)\,. \tag{2.10}$$

This measure is also to be symmetric and continuous in its arguments. If we set $\mu\, H_2^{(\mu,g)}(q, 1-q) = h'(q)$ in (2.10) then (2.10) reduces to

$$H_{n+1}^{(\mu,g)}(p_1, \ldots, p_n) = H_n^{(\mu,g)}(p_1, \ldots, p_n) + g(p_n)\, h'(q) \tag{2.11}$$

which (refers Taneja [9]) requires that

$$g(pq) = g(p)\, g(q) \tag{2.12}$$

and the function h' is to be one of the following forms:

$$h'(q) = h_0'(q) = -q \log_2 q - (1-q) \log_2 (1-q) \tag{2.13}$$

when $g(p) = p$,
and

$$h'(q) = \frac{g(q) + g(1-q) - 1}{2g(\frac{1}{2}) - 1} \tag{2.14}$$

when $g(p) = p^\alpha$, $\alpha \neq 1$.

The result (2.13) which we obtain for $g(p) = p$ is in fact limiting form of the one in (2.14). To see this let us write (2.14) as

$$h'(q) = \frac{q^\alpha + (1-q)^\alpha - 1}{2^{1-\alpha} - 1} \tag{2.15}$$

when $g(p) = p^\alpha$, $\alpha \neq 1$, then $\lim\limits_{\alpha \to 1} h'(q) = h_0(q)$.

For the purpose of achieving compact representation, $h_0(q)$ will also be written as

$$\frac{g_0(q) + g_0(1-q)}{2g_0(\frac{1}{2}) - 1} = \lim_{\alpha \to 1} \frac{q^\alpha + (1-q)^\alpha - 1}{2^{1-\alpha} - 1} = \\ = -q \log_2 q - (1-q) \log_2 (1-q) \tag{2.16}$$

so that for the purposes of writing results finally we have symbolic representation $-q \log q$ for $g_0(q)$ and $g_0(1) = 0$.

Theorem. The axioms (1) to (4) determine the measure, which for a probability distribution $P = (p_1, \ldots, p_n)$, $p_i \geqq 0, \sum_{i=1}^{n} p_i = 1$, is given by

$$H_n^{(\mu_0,\ldots,\mu_l;g_0,\ldots,g_l)}(p_1, \ldots, p_n) = \frac{\sum_{t=0}^{l} \mu_t[g_t(1) - \sum_{i=1}^{n} g_t(p_i)]}{\sum_{t=0}^{l} \mu_t[2g_t(\frac{1}{2}) - g_t(1)]}, \tag{3.1}$$

where

$$g_t(pq) = g_t(p)\, g_t(q) \tag{3.2}$$

for $t = 0, 1, 2, \ldots, l$.

Before proving the theorem, we prove, in the lemmas below, some intermediary results based on the above axioms.

Lemma 1. If $v_k \geqq 0$, $k = 1, 2, \ldots, m; \sum_{k=1}^{m} v_k = p_i > 0$ then

$$\begin{aligned} &H_{n+m-1}^{(\mu_0,\ldots,\mu_l;g_0,\ldots,g_l)}(p_1, \ldots, p_{i-1}, v_1, \ldots, v_m, p_{i+1}, \ldots, p_n) = \\ &= H_n^{(\mu_0,\ldots,\mu_l;g_0,\ldots,g_l)}(p_1, \ldots, p_n) + \sum_{t=0}^{l} \mu_t\, g_t(p_i)\, H_m^{(\mu_t,g_t)}\left(\frac{v_1}{p_i}, \ldots, \frac{v_m}{p_i}\right). \end{aligned} \tag{3.3}$$

Proof. The result will be proved by induction. The statement clearly holds for $m = 2$, in view of axiom (iii). Let us suppose that the result is true for integers less than or equal to m. We have

$$\begin{aligned} &H_{n+m}^{(\mu_0,\ldots,\mu_l;g_0,\ldots,g_l)}(p_1, \ldots, p_{i-1}, v_1, \ldots, v_{m+1}, p_{i+1}, \ldots, p_n) = \\ &= H_{n+1}^{(\mu_0,\ldots,\mu_l;g_0,\ldots,g_l)}(p_1, \ldots, p_{i-1}, v_1, L, p_{i+1}, \ldots, p_n) + \\ &+ \sum_{t=0}^{l} \mu_t\, g_t(L)\, H_m^{(\mu_t,g_t)}\left(\frac{v_2}{L}, \frac{v_3}{L}, \ldots, \frac{v_{m+1}}{L}\right), \end{aligned}$$

where $L = v_2 + v_3 + \ldots + v_{m+1} > 0$. Hence

$$\begin{aligned} &H_{n+m}^{(\mu_0,\ldots,\mu_l;g_0,\ldots,g_l)}(p_1, \ldots, p_{i-1}, v_1, \ldots, v_{m+1}, p_{i+1}, \ldots, p_n) = \\ &= H_n^{(\mu_0,\ldots,\mu_l;g_0,\ldots,g_l)}(p_1, \ldots, p_n) + \sum_{t=0}^{l} \mu_t\, g_t(p_i)\, H_2^{(\mu_t,g_t)}\left(\frac{v_1}{p_i}, \frac{L}{p_i}\right) + \\ &+ \sum_{t=0}^{l} \mu_t\, g_t(L)\, H_m^{(\mu_t,g_t)}\left(\frac{v_2}{L}, \frac{v_3}{L}, \ldots, \frac{v_{m+1}}{L}\right), \end{aligned}$$

where $p_i = v_1 + L < 0$.

In view of (3.2), the terms in the two summations on the right are

$$\sum_{t=0}^{l} \mu_t\, g_t(p_i) \left[H_2^{(\mu_t, g_t)} \left(\frac{v_1}{p_i}, \frac{L}{p_i} \right) + g_t \left(\frac{L}{p_i} \right) H_m^{(\mu_t, g_t)} \left(\frac{v_2}{L}, \ldots, \frac{v_{m+1}}{L} \right) \right]$$

or

$$\sum_{t=0}^{l} \mu_t\, g_t(p_i)\, H_{m+1}^{(\mu_t, g_t)} \left(\frac{v_1}{p_i}, \frac{v_2}{p_i}, \ldots, \frac{v_{m+1}}{p_i} \right)$$

as from (2.11) it is easy to prove (see Taneja [8]) that

(3.4)

$$H_{m+1}^{(\mu_t, g_t)} \left(\frac{v_1}{p_i}, \ldots, \frac{v_{m+1}}{p_i} \right) = H_2^{(\mu_t, g_t)} \left(\frac{v_1}{p_i}, \frac{L}{p_i} \right) + g_t \left(\frac{L}{p_i} \right) H_m^{(\mu_t, g_t)} \left(\frac{v_2}{L}, \ldots, \frac{v_{m+1}}{L} \right).$$

Thus the result is proved by induction. Now by repeated application of Lemma 1, we have

Lemma 2. If $v_{ij} \geqq 0$, $j = 1, 2, \ldots, m_i$, $\sum_{j=1}^{m_i} v_{ij} = p_i$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} p_i = 1$, then

(3.5) $$H_{m_1+m_2+\ldots+m_n}^{(\mu_0, \ldots, \mu_l; g_0, \ldots, g_l)}(v_{11}, v_{12}, \ldots, v_{1m_1}, \ldots, v_{n1}, \ldots, v_{nm_n}) =$$

$$= H_n^{(\mu_0, \ldots, \mu_l; g_0, \ldots, g_l)}(p_1, \ldots, p_n) + \sum_{t=0}^{n} \mu_t \sum_{i=1}^{n} g_t(p_i)\, H_{m_i}^{(\mu_t, g_t)} \left(\frac{v_{i1}}{p_i}, \ldots, \frac{v_{im_i}}{p_i} \right).$$

Next let

(3.6) $$F(n) = H_n^{(\mu_0, \ldots, \mu_l; g_0, \ldots, g_l)} \left(\frac{1}{n}, \ldots, \frac{1}{n} \right).$$

The value of $F(n)$ is given in Lemma 4 below. However we shall need the following result due to Taneja [9].

Lemma 3. If

$$F^{(\mu_t, g_t)}(n) = H_{n+1}^{(\mu_t, g_t)} \left(\frac{1}{n}, \ldots, \frac{1}{n} \right),$$

then

(3.7) $$F^{(\mu_t, g_t)}(n) = \lambda \left[g_t(1) - n g_t \left(\frac{1}{n} \right) \right]$$

where

$$\lambda = [\mu_t[g_t(1) - \tfrac{1}{2}\, g_t(\tfrac{1}{2})]] .$$

Lemma 4. The function $F(n)$ defined in (3.6) is given by

$$F(n) = K\sum_{t=0}^{l}\mu_t\left[g_t(1) - ng_t\left(\frac{1}{n}\right)\right], \tag{3.8}$$

where

$$K = [\sum_{t=0}^{l}\mu_t\{g_t(1) - 2\,g_t(\tfrac{1}{2})\}]^{-1}. \tag{3.8}$$

Proof. In the result of lemma 2, replace m_i by m and set $v_{ij} = 1/mn$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, where m and n are some positive integers. This, using symmetry, gives

$$F(mn) = F(n) + n\sum_{t=0}^{l}\mu_t\,g_t\left(\frac{1}{n}\right)F^{(\mu_t,g_t)}(m), \tag{3.10}$$

$$F(mn) = F(m) + m\sum_{t=0}^{l}\mu_t\,g_t\left(\frac{1}{m}\right)F^{(\mu_t,g_t)}(n). \tag{3.11}$$

Putting $m = 1$ in (3.11) and using the fact that $F(1) = 0$, (from (2.9)), we get

$$F(n) = \sum_{t=0}^{l}\mu_t\,g_t(1)\,F^{(\mu_t,g_t)}(n). \tag{3.12}$$

Expressions (3.12) and (3.7) prove the result (3.8). The value of K is determined by (2.8).

This completes the proof of lemma 4.

Proof of the Theorem. We next prove the theorem for rational numbers: the result, in general, then follows for real numbers for continuity axiom (ii). So let $r_i/m = p_i$ where $r_i \geq 0$ and $m > 0$ are integers, $\sum_{i=1}^{n} r_i = m$, $i = 1, 2, \ldots, n$. By Lemma 2 we have

$$F(m) = H_m^{(\mu_0,\ldots,\mu_l;g_0,\ldots,g_l)}\Big(\underbrace{\frac{1}{m},\ldots,\frac{1}{m}}_{r_1},\ldots,\underbrace{\frac{1}{m},\ldots,\frac{1}{m}}_{r_n}\Big) =$$

$$= H_n^{(\mu_0,\ldots,\mu_l;g_0,\ldots,g_l)}(p_1,\ldots,p_n) + \sum_{t=0}^{l}\mu_t\sum_{i=1}^{n}g_t(p_i)\,H_{r_i}^{(\mu_t,g_t)}\left(\frac{1}{r_i},\ldots,\frac{1}{r_i}\right)$$

or

$$H_n^{(\mu_0,\ldots,\mu_l;g_0,\ldots,g_l)}(p_1,\ldots,p_n) = F(m) - \sum_{t=0}^{l}\mu_t\sum_{i=1}^{n}g_t(p_i)\,F^{(\mu_t,g_t)}(r_i), \tag{3.13}$$

where

$$F(r_i) = H_{r_i}^{(\mu_t,g_t)}\left(\frac{1}{r_i},\ldots,\frac{1}{r_i}\right). \tag{3.14}$$

Expressions (3.13) and (3.8) give

$$H_n^{(\mu_0,\ldots,\mu_l;g_0,\ldots,g_l)}(p_1,\ldots,p_n) = K\sum_{t=0}^{l}\mu_t\left[g_t(1) - m\, g_t\left(\frac{1}{m}\right)\right] -$$

$$- K\sum_{t=0}^{l}\mu_t\sum_{i=1}^{n}g_t(p_i)\left[g_t(1) - r_i\, g_t\left(\frac{1}{r_i}\right)\right] =$$

$$= K\sum_{t=0}^{l}\mu_t\left[g_t(1) - m\, g_t\left(\frac{1}{m}\right) - \sum_{i=1}^{n}g_t(p_i) + \sum_{i=1}^{n}r_i\, g_t\left(\frac{p_i}{r_i}\right)\right] =$$

$$= K\sum_{t=0}^{l}\mu_t\left[\sum_{i=1}^{n}g_t(p_i) - g_t(1)\right],$$

where K is as determined earlier.

This completes the proof of the theorem.

Particular Cases

(i) If we take $\mu_0 \neq 0$ and all other μ's $= 0$, then (3.1) reduces to

$$(3.15)\qquad H_n^{(\mu_0,g_0)}(p_1,\ldots,p_n) = \frac{\sum_{i=1}^{n} g_0(p_i) - g_0(1)}{2\, g_0(\frac{1}{2}) - g_0(1)} = -\sum_{i=1}^{n} p_i \log p_i\,,$$

what is Shannon's entropy.

As explained earlier, here we have used symbolic representation explained in (2.16) according to which in the final result we have

$$g_0(p_i) = -p_i \log p_i\,,$$
$$g_0(1) = 0\,.$$

(ii) Next take $\mu_1 = \mu \neq 0$, and other μ's $= 0$. If we take $g_1(p_i) = p_i^\alpha$, $\alpha > 0$, $\alpha \neq 1$, then (3.1) reduces to Havrda - Charvát entropy given in (2.15).

(iii) Also if we take $\mu_2 = 2$, $\mu_3 = -2$ and all other μ's $= 0$, $g_2(p_i) = p_i^\alpha$, $g_3(p_i) = p_i^\beta$, then (3.1) reduces to entropy of type (α, β) (refer [6])

$$H_n(p_1,\ldots,p_n;\alpha,\beta) = (2^{1-\alpha} - 2^{1-\beta})^{-1}\sum_{i=1}^{n}(p_i^\alpha - p_i^\beta)\,,$$
$$\alpha \neq \beta\,,\quad \alpha,\beta > 0\,.$$

4. FUNCTIONAL EQUATION

Given a complete probability distribution $P = (p_1,\ldots,p_n)$; $p_i \geqq 0$, $\sum_{i=1}^{n} p_i = 1$ of a discrete finite random variable $X = (x_1, x_2, \ldots, x_n)$, let the generalized entropy be taken so as to satisfy the axioms given in Section 2. Applying branching property for $n = 3$, and writing generalized entropy with several parameters μ's and functions g's obtained earlier as simply $H_n(p_1, \ldots, p_n)$, we get

$$(4.1) \quad H_2(p_1 + p_2, p_3) + \sum_{t=0}^{l} \mu_t\, g_t(p_1 + p_2)\, H_2^{(\mu_t, g_t)}\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}\right) =$$

$$= H_2(p_1 + p_3, p_2) + \sum_{t=0}^{l} \mu_t\, g_t(p_1 + p_2)\, H_2^{(\mu_t, g_t)}\left(\frac{p_1}{p_1 + p_3}, \frac{p_3}{p_1 + p_3}\right).$$

Let us define

$$(4.2) \qquad f(x) = H_2(1 - x, x),$$

$$(4.3) \qquad f^{(\mu_t g_t)}(x) = H_2^{(\mu_t, g_t)}(1 - x, x).$$

So that setting $p_3 = x$, $p_2 = y$ and $p_1 + p_2 + p_3 = 1$ the relations (4.1), (4.2) and (4.3) lead to the functional equation

(4.4)

$$f(x) + \sum_{t=0}^{l} \mu_t\, g_t(1 - x) f^{(\mu_t, g_t)}\left(\frac{v}{1 - x}\right) = f(y) + \sum_{t=0}^{l} \mu_t\, g_t(1 - y) f^{(\mu_t, g_t)}\left(\frac{x}{1 - y}\right).$$

From the symmetry of measures in (3.1) and (4.3), (4.4) we also have

$$(4.5) \qquad f(x) = f(1 - x),$$

$$(4.6) \qquad f^{(\mu_t, g_t)}(x) = f^{(\mu_t, g_t)}(1 - x).$$

Further, expressions (4.6), (4.5) and (2.9) give the boundary conditions

$$(4.7) \qquad f(1) = f(0) = 0,$$

$$(4.8) \qquad f^{(\mu_t, g_t)}(1) = f^{(\mu_t, g_t)}(0).$$

The measure $H_n^{(\mu_0, \ldots, \mu_l; g_0, \ldots, g_l)}(p_1, \ldots, p_n)$ for P may now be defined in terms of the solutions of (4.4), (4.5), (4.6) when (4.7) and (4.8) hold, as

$$(4.9) \qquad H_n^{(\mu_0, \ldots, \mu_l; g_0, \ldots, g_l)}(p_1, \ldots, p_n) = \sum_{t=0}^{l} \mu_t \sum_{i=2}^{n} g_t(s_i) f\left(\frac{p_i}{s_i}\right),$$

where $s_i = p_1 + p_2 + \ldots + p_i$ and g satisfies the equation

$$(4.10) \qquad g_t(pq) = g_t(p)\, g_t(q)$$

(c.f. Daroczy [3] (1970)). Detailed studies in this direction will be reported in a forthcoming paper.

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