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ON PSEUDOPARABOLIC OPTIMAL CONTROL PROBLEMS

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An optimal control problem for a pseudoparabolic equation is considered. Control parameters appear in coefficients of operators of a state equation. The existence theorem, the conditions for the uniqueness and the sensitivity analysis are presented.

1. OPTIMIZATION IN COEFFICIENTS FOR PSEUDOPARABOLIC EQUATIONS

We start with some functions spaces. Let $T > 0$, X be a Banach space with a norm $\|\cdot\|_X$. We denote by $C(0,T;X)$ the space of all continuous and by $C^1(0,T;X)$ the space of all continuously differentiable functions $f : [0,T] \to X$. $L_2(0,T;X)$ denotes the space of all measurable functions $f : (0,T) \to X$, such that $f(\cdot) \in L_2(0,T;X)$. Further, we denote by $W_2^1(0,T;X)$ the space of all $f \in L_2(0,T;X)$ with a distributive derivative $f' \in L_2(0,T;X)$. If X is a Hilbert space with the inner product $(\cdot,\cdot)_X$, then $W_2^1(0,T;X)$ is the Hilbert space with the inner product $(f,g)_{1,2} = \int_0^T [(f(t),g(t))_X + (f'(t),g'(t))_X]\,\mathrm{d}t$.

Let V be the Hilbert space with the inner product $(\cdot,\cdot)$ and the norm $\|.\|$, v^* its dual space with the duality pairing $\langle\cdot,\cdot\rangle$ and with the norm $\|\cdot\|_*$, $L(V,V^*)$ the Banach space of all linear bounded operators from V into V^*. Let U be a reflexive Banach space of control with a norm $\|\cdot\|_U$ and $U_{ad} \subset U$ be a convex closed and bounded set od admissible controls. We assume the families of operators $A_i(t,u) : V \to V^*$, $t \in [0,T]$, $u \in U$, $i = 0,1$; fulfilling the assumptions

$$A_0(\cdot,u) \in C(0,T;L(V,V^*)) \tag{1}$$

$$A_1(\cdot,u) \in C^1(0,T;L(V,V^*)) \tag{2}$$

$$\langle A_i(t,u)y,z\rangle = \langle A_i(t,u)z,y\rangle, \quad i = 0,1 \tag{3}$$

$$\langle A_1(t,u)y,y\rangle \ge c_1\|y\|^2, \; c_1 > 0 \tag{4}$$

$$\langle [2A_0(t,u) - A_1'(t,u)]y,y\rangle \ge c_2\|y\|^2, \; c_2 > 0 \tag{5}$$

for all $t \in [0,T]$, $u \in U$; y, $z \in V$

$$u_n \rightharpoonup u \text{ in } U \text{ weakly } \Rightarrow A_i(\cdot,u_n) \to A_i(\cdot,u) \tag{6}$$

in $C(0,T;L(V,V^*))$, $i = 0,1$

Let $f \in C(0,T;V^*)$, $f_0 \in V^*$. We shall deal with a following optimal control problem:

$$A_1(t,u)\, y'_t(t,u) + A_0(t,u)\, y(t,u) = f(t) \tag{7}$$

$$A_1(0,u)\, y(0,u) = f_0 \tag{8}$$

$$J(\overline{u}) = \min_{u \in U_{ad}} J(u), \tag{9}$$

with

$$J(u) = \|Dy(T,u) - z_d\|_X^2 + j(u), \quad u \in U_{ad}, \tag{10}$$

where X is a Hilbert space, $z_d \in X$, $D \in L(V,X)$ and $j : U \to R$ is a weakly lower semicontinuous functional.

The state initial value problem (7), (8) can be due to the assumption (4) expressed as the initial value problem for the first order ordinary differential equation in the Hilbert space V

$$y' + B(t,u) = g(t,u), \quad y(0) = g_0$$

with

$$B(t,u) = A_1^{-1}(t,u)\, A_0(t,u), g(t) = A_1^{-1}(t,u)\, f(t), \quad g_0 = A_1^{-1}(0,u)\, f(0).$$

Using the theory of the ordinary differential equations in Hilbert spaces (see [3]) we obtain the existence and uniqueness of a solution $y \in C^1(0,T;V)$. The function $y := y(\cdot,u) \in C^1(0,T;V)$ is simultaneously a unique solution of the state initial value problem (7), (8). Hence, the cost functional $u \to J(u)$ is correctly defined.

The main result of this part is the existence theorem for the control problem (7)–(10).

Theorem 1. There exists at least one solution $\overline{u} \in U_{ad}$ of the Optimal control problem (7)–(10).

Proof. Let $y(\cdot,u) \in C^1(0,T;V)$ be a solution to the state problem (7), (8). Using the assumptions (2), (3) we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\langle A_1(t,u)\, y(t,u),\, y(t,u)\rangle + \langle [2A_0(t,u) - A_1'(t,u)]y(t,u),\, y(t,u)\rangle = \\ &= \; 2\langle f(t),\, y(t,u)\rangle \end{aligned} \tag{11}$$

We introduce the function $\varphi \in C^1(0,T;R)$ by

$$\varphi(t) = \langle A_1(t,u)\, y(t,u),\, y(t,u)\rangle, \quad t \in [0,T],\ u \in U_{ad} \tag{12}$$

Further we set

$$c_3 = \sup_{(t,u)\in[0,T]\times U_{ad}} \|A_1(t,u)\|_{L(V,V^*)} \tag{13}$$

Using the assumptions (4), (5) we arrive from (11) at the inequality

$$\varphi'(t) + c_2 c_3^{-1}\varphi(t) \le 2\|f(t)\|_* c_1^{-1/2}\varphi(t)^{1/2}$$

and

$$\varphi'(t) + \frac{1}{2} c_2 c_3^{-1} \varphi(t) \le 2c_1^{-1} c_2^{-1} c_3 \|f(t)\|_*^2 \tag{14}$$
for all $t \in [0,T]$ and $u \in U_{ad}$.

The estimate (14) and the initial condition (8) imply

$$\langle A_1(t,u)\, y(t,u),\, y(t,u)\rangle = \varphi(t) \le$$
$$\le c_1^{-1} \|f_0\|_*^2 e^{-\alpha t} + 2c_1^{-1} c_2^{-1} c_3 \int_0^t \|f(s)\|_*^2 e^{-\alpha(t-s)} \mathrm{d}s, \tag{15}$$

where

$$\alpha = \frac{1}{2} c_2 c_3^{-1} > 0.$$

The assumption (4) then implies the estimate

$$\|y(t,u)\| \le c_1^{-1} \|f_0\|_* + 2T^{1/2} c_1^{-1} c_2^{-1/2} c_3^{1/2} \|f\|_{C(0,T;V^*)} \tag{16}$$
for all $t \in [0,T]$, $u \in U_{ad}$

If we denote

$$c_4 = \sup_{(t,u)\in[0,T]\times U_{ad}} \|A_0(t,u)\|_{L(V,V^*)},$$

then it follows directly from the equation (7) that

$$\|y'(t,u)\| \le c_1^{-2} c_4 \|f_0\|_* + \left(c_1^{-1} + 2T^{1/2} c_1^{-2} c_2^{-1/2} c_3^{1/2} c_4\right) \|f\|_{C(0,T;V^*)} \tag{17}$$
for all $t \in [0,T]$, $u \in U_{ad}$

The estimates (16), (17) imply that the set of functions $y(\cdot,u) : [0,T] \to V$ is bounded both in $C^1(0,T;V)$ and in $W_2^1(0,T;V)$, which is a Hilbert space. Using the standard compactness method in U_{ad} and in $W_2^1(0,T;V)$ we obtain due to (6) and the weak lower semicontinuity of the cost functional J the existence of an optimal control $\overline{u} \in U_{ad}$, what concludes the proof. □

Remark 1. The existence of an optimal control can be verified also for other types of cost functionals (see [1]) and even for the pseudoparabolic variational inequality. ([2]).

2. SENSITIVITY ANALYSIS WITH RESPECT TO TIME

In order to perform the sensitivity analysis for the control problem (7)–(10) we add some differentiability assumptions. We assume that the operators

$$A_i(t,\cdot) : U_{ad} \to L(V,V^*), \quad i = 0,1;$$

are twice differentiable in the sense of Fréchet and their derivatives are estimated by

$$\left\| \frac{\mathrm{d}}{\mathrm{d}u} A_i(t,u) \right\|_{L(U,L(V,V^*))} \le \beta_i \tag{18}$$

$$\left\| \frac{\mathrm{d}^2}{\mathrm{d}u^2} A_i(t,u) \right\|_{L(U\times U,L(V,V^*))} \le \gamma_i \tag{19}$$

for all $t \in [0,T]$, $u \in U_{ad}$; $i = 0,1$.

In order to simplify our considerations we introduce the operators

$$\mathcal{A}(u) \in L(C^1(0,T;V),\ C(0,T,V^*) \times V)$$

by

$$\mathcal{A}(u)\, y = [A_1(t,u)\, y' + A_0(t,u)\, y,\ A_1(0,u)\, y(0)].$$

We define the norm in $C(0,T;V^*) \times V$ by

$$\|[f(\cdot),g]\| = \|f\|_{C(0,T;V^*)} + \|g\|$$

The operator

$$\mathcal{A} : U_{ad} \to L(C^1(0,T;V), C(0,T;V) \times V^*)$$

is then twice differentiable in the sense of Fréchet and

$$\|\mathcal{A}'(u)\| \le 2\beta_1 + \beta_0 \tag{20}$$

$$\|\mathcal{A}''(u)\| \le 2\gamma_1 + \gamma_0 \tag{21}$$

for every $u \in U_{ad}$.

Theorem 2. The mapping $y(\cdot) : U_{ad} \to C^1(0,T;V)$ defined by (7), (8) is differentiable in the sense of Fréchet and its derivative fulfils the equation

$$\mathcal{A}(u)[y'_u(u)\, v] = -[\mathcal{A}'(u)v]\, y(u) \tag{22}$$

for all $u \in U_{ad}$, $v \in U$.

Proof. Let $z \in C^1(0,T;V)$ be a unique solution of the equation

$$\mathcal{A}(u)\, z = -[\mathcal{A}'(u)v]\, y(u) \tag{23}$$

We shall verify that $z = y'_u(u)v$.

Let us denote

$$r(v) = y(u+v) - y(u) - z, \quad v \in U \tag{24}$$

The function $r(v) \in C^1(0,T;V)$ is a solution of the equation

$$\mathcal{A}(u)\, r(v) = \Phi(v), \tag{25}$$

where

$$\begin{aligned} \Phi(v) &= -[\mathcal{A}(u+v) - \mathcal{A}(u) - \mathcal{A}'(u)\, v]\, y(u+v) \\ &\quad -[\mathcal{A}'(u)\, v]\, [y(u+v) - y(u)] \end{aligned} \tag{26}$$

Applying the a priori estimates (16), (17) with respect to $r(v)$ as the solution of (25) we obtain the estimate

$$\|r(v)\|_{C^1(0,T;V)} \leq M_1 \|\Phi(v)\|_{C(0,T;V^*)\times V^*}, \tag{27}$$

where

$$M_1 = c_1^{-1}\left(1 + c_1^{-1}c_4\right)\left(1 + 2T^{1/2}c_2^{-1/2}c_3^{1/2}\right)$$

the estimates (19), (20) and the Lagrange theorem imply the estimate

$$\begin{aligned}\|\Phi(v)\|_{C(0,T;V^*)\times V^*} &\leq (2\beta_1 + \beta_0)\,\|v\|_U^2 \|\,y(u+v)\|_{C^1(0,T;V)} + \\ &+ (2\gamma_1 + \gamma_0)\,\|v\|_U \|\,y(u+v) - y(u)\|_{C^1(0,T;V)}\end{aligned} \tag{28}$$

The difference $y(u+v) - y(u)$ fulfils the equation

$$\mathcal{A}(u)[y(u+v) - y(u)] = [\mathcal{A}(u) - \mathcal{A}(u+v)]y(u+v)$$

In the same way as above we obtain the estimate

$$\|y(u+v) - y(u)\|_{C^1(0,T;V)} \leq M_2\|v\|_U, \tag{29}$$

where

$$M_2 = M_1^2(2\beta_1 + \beta_0)\,(\|f\|_{C(0,T;V^*)} + \|f_0\|_{V^*})$$

Finally, we have from (26), (27), (28) the estimate

$$\|r(v)\|_{C^1(0,T;V)} \leq M_3\|v\|_U^2, \tag{30}$$

where

$$M_3 = M_1^2(1 + 2\gamma_1 + \gamma_0)(2\beta_1 + \beta_0)\,(\|f\|_{C(0,T;V^*)} + \|f_0\|_{V^*})$$

The estimate (29) implies the relation

$$\begin{aligned}&\lim_{\|v\|\to 0}\left[\|r(v)\|_{C^1(0,T;V)}\|v\|_U^{-1}\right] = \\ &\lim_{\|v\|\to 0}\left[\|y(u+v) - y(u) - z\|_{C^1(0,T;V)}\|v\|_U^{-1}\right] = 0\end{aligned}$$

and hence

$$z = y'_u(u)v$$

what completes the proof. □

Let us assume further that the functional $j : U \to R$ is differentiable with a strongly monotone derivative, i.e.,

$$\langle j'(u) - j'(v),\, u - v\rangle_U \geq N\|u - v\|_U^2,\ N > 0 \quad \text{for all } u,\, v \in U \tag{31}$$

We shall verify that for sufficiently great N there exists a unique optimal control $\overline{u}$.

The functional $J : U_{ad} \to R$ is due to Theorem 2 differentiable in the sense of Fréchet and its derivative has the form

$$\langle J'(u), v\rangle_U = (Dy(T,u) - z_d,\ D[y'_u(u)v](T))_X + \langle j'(u), v\rangle_U \quad \text{for all } u \in U_{ad},\ v \in U$$

An optimal control $\overline{u}$ fulfils then the variational inequality

$$\langle J'(\overline{u}),\ v - \overline{u}\rangle_U \geq 0 \quad \text{for all } v \in U_{ad},$$

or

$$(Dy(T,\overline{u}) - z_d,\ D[y'(u)(v-u)(T)])_X + \langle j'(\overline{u}),\ v - \overline{u}\rangle_U \geq 0 \quad \text{for all } v \in U_{ad}. \tag{32}$$

Let $\overline{u}_1, \overline{u}_2$ be two optimal controls. Then the inequality (32) implies the inequality

$$\langle J'(\overline{u}_1) - J'(\overline{u}_2), \overline{u}_1 - \overline{u}_2\rangle_U \leq 0,$$

and hence

$$\begin{aligned} &\langle j'(\overline{u}_1) - j'(\overline{u}_2),\ \overline{u}_1 - \overline{u}_2\rangle_U \leq \\ \leq\ & (Dy(T,\overline{u}_1) - z_d,\ D[y'(\overline{u}_1)(\overline{u}_2 - \overline{u}_1)(T)])_X + \\ +\ & (Dy(T,\overline{u}_2) - z_d,\ D[y'(\overline{u}_2)(\overline{u}_1 - \overline{u}_2)(T)])_X = \\ -\ & (Dy(T,\overline{u}_1) - z_d,\ D[y(T,\overline{u}_2) - y(T,\overline{u}_1) - y'(\overline{u}_1)(\overline{u}_2 - \overline{u}_1)(T)])_X - \\ -\ & (Dy(T,\overline{u}_2) - z_d,\ D[y(T,\overline{u}_1) - y(T,\overline{u}_2) - y'(\overline{u}_2)(\overline{u}_1 - \overline{u}_1)(T)])_X - \\ -\ & \|D[y(T,\overline{u}_1) - y(T,\overline{u}_2)]\|_X^2 \end{aligned}$$

The estimates (16), (29) and the strong monotonicity (31) then imply the inequality

$$(N - M_4)\,\|\overline{u}_1 - \overline{u}_2\|_U^2 \leq 0,$$

where

$$M_4 = 2M_3\|D\|_{L(V,X)}\left[c_1^{-1}\|f_0\|_* + 2T^{1/2}c_1^{-1}c_2^{-1/2}c_3^{1/2}\|f\|_{C(0,T,V^*)} + \|z_d\|_X\right]$$

The inequality (32) implies

Theorem 3. If $N > M_4$, then there exists a unique solution $\overline{u}$ to the optimal control problem (7)–(10).

We proceed with the sensitivity analysis with respect to T.

Let $N > M_4$ and $0 < t_1 < t_2 \leq T$. We denote by u_1 and u_2 solution to the control problems

$$J_i(u_i) = \min_{v \in U_{ad}} J_i(v), \tag{33}$$

where

$$J_i(v) = \|Dy(t_i, v) - z_d\|_X^2 + j(v), \quad i = 1, 2$$

Optimal controls u_1, u_2 fulfil the variational inequalities

$$(Dy(t_i, u_i) - z_d,\ D[z'(u_i)(v - u_i)(t_i)])_X + \langle j'(u_i),\ v - u_i\rangle_U \geq 0$$
$$\text{for all } v \in U_{ad},\ i = 1, 2$$

We obtain in the same way as in (32) the estimate

$$\begin{aligned}
& (N - M_4)\|u_1 - u_2\|_U^2 \leq \\
\leq\ & \|z_d\|_X \|D\|_{L(V,X)} (\|y(t_2, u_1) - y(t_1, u_1)\| + \|y(t_2, u_2) - y(t_1, u_2)\|) + \\
+\ & \frac{1}{2} (\|Dy(t_2, u_1)\|_X^2 - \|Dy(t_1, u_1)\|_X^2 + \|Dy(t_1, u_2)\|_X^2 - \|Dy(t_2, u_2)\|_X^2)
\end{aligned}$$

and with respect to the estimate (15)

$$\|u_1 - u_2\|_U^2 \leq M_5 [\|y(t_2, u_1) - y(t_1, u_1)\| + \|y(t_2, u_2) - y(t_1, u_2)\|], \tag{34}$$

where

$$M_5 = (N - M_4)^{-1} \|D\|_{L(V,X)} \left[(1 - \|D\|_{L(V,X)}\|z_d\|_X + \frac{1}{2} M_3^{-1} M_4\right]$$

Using the estimate

$$\|y(t_2, u) - y(t_1, u)\| \leq \sup_{t \in [0,T]} \|y_t'(t, u)\|(t_2 - t_1), \quad u \in U_{ad}$$

we obtain, considering (17), the estimate

$$\|u_2 - u_1\|_U^2 \leq 2M_5 M_1 \left[\|f_0\|_* + \|f\|_{C(0,T;V^*)}\right] (t_2 - t_1)$$

Hence we have verified the following results on the sensitivity analysis.

Theorem 4. Let $N > M_4$ and u_τ be the unique optimal control with respect to the cost functional

$$J_\tau(v) = \|Dy(\tau, v) - z_d\|_X^2 + j(v), \quad v \in U_{ad},\ 0 < \tau \leq T.$$

Then the mapping $\tau \to u_\tau$ is Hölder continuous and it holds

$$\|u_s - u_t\|_U \leq M[s - t]^{1/2}, \quad s, t \in (0, T];$$

where

$$M = \left[2M_1 M_5(\|f_0\|_* + \|f\|_{C(0,T;V^*)}\right]^{1/2}.$$

Remark 2. Using the previous method it is possible to investigate the behaviour of the mapping $\tau \longrightarrow u_\tau$ for $\tau \longrightarrow \infty$.

Let the assumptions (1)-(6) hold for every $T > 0$ with constants c_1, c_2 not depending on T and moreover

$$\lim_{t\to\infty} \|A_0(t,u) - A_0(\infty,u)\|_{L(V,V^*)} = \lim_{t\to\infty} \left\| \frac{\mathrm{d}}{\mathrm{d}t} A_1(t,u) \right\|_{L(V,V^*)} = \lim_{t\to\infty} \|f(t) - f(\infty)\|_* = 0.$$

Using the a priori estimate (16) we can verify the relation

$$\lim_{t\to\infty} \|y(t,u) - y(\infty,u)\| = 0,$$

where $y(\infty,u)$ fulfils the elliptic equation

$$A_0(\infty,u)\, y(\infty,u) = f_\infty$$

If we define the corresponding control problem

$$J_\infty(u_\infty) = \min_{v\in U_{ad}} J_\infty(v),\ J_\infty(v) = \|Dy(\infty,v) - z_d\|_X^2 + j(v);$$

then it can be verified in the same way as above the relation analogous to (34)

$$\|u_\tau - u_\infty\|_U^2 \le M_5 \left[\|y(\infty,u_\tau) - y(\tau,u_\tau)\| + \|y(\infty,u_\infty) - y(\tau,u_\infty)\|\right], \quad \tau > 0;$$

and with respect to (34) we have

$$\lim_{\tau\to\infty} \|u_\tau - u_\infty\|_U = 0$$

It means that the optimal control u_τ tends as a function of τ to the solution of the corresponding optimal control problem with the elliptic equation as the state problem.

Remark 3. The whole theory can be applied to the optimal design of a viscoelastic plate with respect to its variable thickness. The operators $A_r(t,u) : V \longrightarrow V^*$ have the form ([1], [2])

$$\langle A_r(t,u)\, y, z\rangle = \int\int_\Omega u^3(x)\, A_{ijkl}^{(r)}(t)\, y_{,ij}\, z_{,kl}\, \mathrm{d}x_1 \mathrm{d}x_2,\ r = 0,1;$$
$$V \subset W_2^2(\Omega),\ y_{,ij} = \frac{\partial^2 y}{\partial x_i \partial x_j}$$

Remark 4. J. Sokolowski ([4], [5]) investigated the differentiability of the mapping $\tau \longrightarrow u_\tau$ for the case of a parabolic state problem with control parameters in the right-hand side.

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