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Valéria Menyhértová

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On Adaptive Replacement Policies

VALÉRIA MENYHÉRTOVÁ

Renewals of a machine component by means of preventive and after failure replacements are studied. It is assumed that the service time distribution is specified up to an unknown parameter. Properties of the maximum likelihood estimates of this parameter are presented.

0. INTRODUCTION

One of the problems of production management is the preventive replacement of machine components so that the costs needed to secure a smooth progress of production are minimal.

Recently a sequential construction of an optimal policy in the case that $F(x)$ – the distribution function of failure times of machine components – is unknown, was produced ([1]). Further results on the asymptotic behaviour of the cost in this situation were published by P. Mandl in [2].

In this paper the problem of preventive replacement of components is dealt with under the assumption that $F(x)$ is specified up to an unknown parameter.

Conditions are established that guarantee the maximal speed of convergence of the average cost to the optimum.

1. FORMULATION OF THE PROBLEM

Let us imagine an infinite stock of components such that the failed components can be immediately exchanged. The components are specified by their failure times t_i. Thus, the basic probability space is the space of infinite sequences of positive numbers $\{t_1, t_2, \ldots\}$ with product probability measure

$$P = F_0 \times F \times \ldots \times F \times \ldots$$

depending on the age of first component at time 0. $F(x)$ is the *distribution function of failure times*. We assume

$$\bar{F}(x) = 1 - F(x) = \exp\left\{-\int_0^x g(y)\,dy\right\},$$

where the failure rate $g(x)$ is a continuous function on $[0, \infty)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}x} F(x) = f(x) = g(x) \exp\left\{-\int_0^x g(y)\,\mathrm{d}y\right\}.$$

We distinguish *service replacements* after failure (type 1) and *preventive replacements* (type 2). The latter consists in replacing an operating component when its operation time (age) reaches a critical value. The replacements are performed according to a policy ω that prescribes the way, in which the stock of components, i.e. the sample point $\{t_1, t_2, \ldots\}$, is processed. The choice of the replacement age depends on the past experience. A *replacement policy* is determined by a sequence of functions

$$\omega = \{z_{n+1}(s_1, j_1, \ldots, s_n\, j_n),\ n = 0, 1, \ldots\},$$

where $z_1 > 0$ is a constant, $z_{n+1}\ (s_1, j_1, \ldots, s_n, j_n) > 0$ is the critical age for the $(n+1)$-st component given the history

$$s_1, j_1, \ldots, s_n, j_n\,.$$

s_k, $k = 1, 2, \ldots, n$ is the time of k-th replacement, and $j_k = 1$ for replacement after failure, $j_k = 2$ for preventive replacement.

Denote by τ_n the actual replacement times, $n = 1, 2, \ldots$, and by l_n the labels marking the types of the replacements. Then the following recursive relations hold under policy ω

$$\tau_0 = 0$$

$$\tau_{n+1} = \tau_n + t_{n+1} \wedge z_{n+1}(\tau_1, l_1, \ldots, \tau_n, l_n),$$

$$l_{n+1} = \begin{cases} 1 & \text{if } t_{n+1} < z_{n+1}(\tau_1, l_1, \ldots, \tau_n, l_n), \\ 2 & \text{if } t_{n+1} \geqq z_{n+1}(\tau_1, l_1, \ldots, \tau_n, l_n), \end{cases}$$

$$n = 1, 2, \ldots.$$

Policy ω is admissible iff

$$\lim_{n\to\infty} \tau_n \to \infty \quad \text{a.s. (almost surely)}.$$

This excludes the probability of infinitely many replacements in finite time. Further let

$$N_t^j = \sum_{n=1}^{\infty} \chi_{\{\tau_n \leqq t, l_n = j\}}, \quad j = 1, 2,$$

be the total number of replacements of the type j up to time t, c_1 the cost of the service replacement, c_2 the cost of the preventive replacement, and C_t the cost accumulated up to time t. We assume $c_1 > c_2 > 0$. We have

$$C_t = c_1 N_t^1 + c_2 N_t^2 .$$

Next we define the *critical age* Z_t and the *virtual age* X_t of the components at time t:

$$Z_t = z_1 , \quad \text{if} \quad 0 \leqq t \leqq \tau_1$$

$$Z_t = z_{n+1}(\tau_1, l_1, \ldots, \tau_n, l_n) , \quad \text{if} \quad \tau_n < t \leqq \tau_{n+1} , \quad n = 1, 2, \ldots .$$

$$X_t = X_0 + t , \quad \text{if} \quad 0 \leqq t \leqq \tau_1$$

$$X_t = t - \tau_n , \quad \text{if} \quad \tau_n < t \leqq \tau_{n+1} , \quad n = 1, 2, \ldots .$$

The component is replaced when X_t reaches Z_t. The *average cost per unit time* corresponding to the policy with constant critical age $x \in (0, \infty)$ equals

$$\Theta(x) = (c_1 F(x) + c_2 \bar{F}(x)) / \int_0^\infty \bar{F}(y)\, \mathrm{d}y . \tag{1.1}$$

The denominator in (1.1) is the mean time between replacements. The replacements are of type 1 and 2 with probabilities $F(x)$ and $\bar{F}(x)$, respectively. We assume the existence of a critical age $d \in (0, \infty)$, for which

$$\Theta = \Theta(d) \leqq \Theta(x) , \quad x \in (0, \infty) .$$

The search for an *optimal policy* consists in the sequential improvement of replacement policies. This property will be meant here in the asymptotic sense, namely that

$$\lim_{t \to \infty} Z_t = d \quad \text{a.s.}$$

2. SOME AUXILIARY ASSERTIONS

We consider the process of failures $\{N_t^1, t \geqq 0\}$ under an arbitrary admissible policy ω. Let $\mathscr{F}_t$ be σ-algebra induced by the process of replacements up to time t, i.e.

$$\mathscr{F}_t = \sigma a((N_s^1, N_s^2) , \quad s \in [0, t]) .$$

Lemma 1.

$$M_t = N_t^1 - \int_0^t g(X_s)\, \mathrm{d}s , \quad t \geqq 0$$

is a martingale with respect to $\{\mathscr{F}_t, t \geqq 0\}$.

Proof. We have to prove for arbitrary $T > t$ that

$$\mathrm{E}\{M_T - M_t \mid \mathscr{F}_t\} = \mathrm{E}\left\{N_T^1 - N_t^1 - \int_t^T g(X_s)\,\mathrm{d}s \mid \mathscr{F}_t\right\} = 0\,.$$

Investigate first

$$\mathrm{E}\{N_{t+\Delta}^1 - N_t^1 \mid \mathscr{F}_t\} \quad \text{for} \quad \Delta \to 0_+\,.$$

$\mathscr{F}_t$ gives the history of the process up to time t, i.e. $s_1, j_1, \ldots, s_n, j_n$, where s_n is the last replacement time not exceeding t. The corresponding replacement age fulfils the inequality

$$z_{n+1}(s_1, j_1, \ldots, s_n, j_n) > t - s_n\,,$$

where $t - s_n$ is the age of the component in time t, more exactly

$$t - s_n = X_t^+ = \lim_{s \to t_+} X_s\,.$$

Assume $t - s_n + \Delta < z_{n+1}$. Then

$$P(N_{t+\Delta}^1 - N_t^1 = 0 \mid \mathscr{F}_t) = 1 - \frac{F(X_t^+ + \Delta) - F(X_t^+)}{\bar{F}(X_t^+)} =$$

$$= 1 - \frac{f(X_t^+)\,\Delta + o(\Delta)}{\bar{F}(X_t^+)} = 1 - g(X_t^+)\,\Delta + o(\Delta)\,,$$

$$P(N_{t+\Delta}^1 - N_t^1 \geqq 1 \mid \mathscr{F}_t) = g(X_t^+)\,\Delta + o(\Delta)\,,$$

$$P(N_{t+\Delta}^1 - N_t^1 \geqq 2 \mid \mathscr{F}_t) = \int_0^\Delta F(\Delta - y)\,.\,f(X_t^+ + y)\,\mathrm{d}y = o(\Delta)\,.$$

Hence,

$$\mathrm{E}\{N_{t+\Delta}^1 - N_t^1 \mid \mathscr{F}_t\} = g(X_t^+)\,\Delta + o(\Delta)\,.$$

Further,

$$\mathrm{E}\left\{\int_t^{t+\Delta} g(X_s)\,\mathrm{d}s \mid \mathscr{F}_t\right\} = \left(\int_0^\Delta g(X_t^+ + y)\,\mathrm{d}y\right).\,P(N_{t+\Delta}^1 - N_t^1 = 0 \mid \mathscr{F}_t) +$$

$$+ \mathrm{E}\left\{\int_t^{t+\Delta} g(X_s)\,\mathrm{d}s \mid N_{t+\Delta}^1 - N_t^1 > 0, \mathscr{F}_t\right\}.\,P(N_{t+\Delta}^1 - N_t^1 > 0 \mid \mathscr{F}_t) =$$

$$= g(X_t^+)\,\Delta + o(\Delta)\,.$$

Consequently,

$$\mathrm{E}\{M_{t+\Delta} - M_t \mid \mathscr{F}_t\} = o(\Delta)\,.$$

Set

$$\Delta = \frac{T - t}{2^n}\,.$$

 Then

$$\mathrm{E}\{M_T - M_t \mid \mathscr{F}_t\} = \mathrm{E}\{_{t+\Delta 2^n} - M_t \mid \mathscr{F}_t\} =$$

$$= \sum_{k=0}^{2^n-1} \mathrm{E}\{M_{t+(k+1)\Delta} - M_{t+k\Delta} \mid \mathscr{F}_t\} =$$

$$= \mathrm{E}\{\sum \mathrm{E}\{M_{t+(k+1)\Delta} - M_{t+k\Delta} \mid \mathscr{F}_{t+k\Delta}\} \mid \mathscr{F}_t\} =$$

$$= \mathrm{E}\{\sum_k o(\Delta) \mid \mathscr{F}_t\} \to 0 \,, \quad n \to \infty \,.$$ □

The following lemma is proved by similar considerations.

Lemma 2. Let $a(x)$ be a continuous function on $[0, \infty)$. Then

$$A_t = \int_0^t a(X_s)\,(\mathrm{d}N_s^1 - g(X_s)\,\mathrm{d}s)\,, \quad t \geqq 0\,, \tag{2.1}$$

is a martingale with respect to $\{\mathscr{F}_t, t \geqq 0\}$.

Lemma 3. Let in (2.1) $a(X_t)$, $t \geqq 0$ be bounded a.s. Then $\{A_t, t \geqq 0\}$ fulfils the strong law of large numbers, i.e.

$$\lim_{t \to \infty} t^{-1} A_t = 0 \quad \text{a.s.}$$

If in addition

$$\lim_{t \to \infty} t^{-1} \int_0^t a(X_s)^2\, g(X_s)\,\mathrm{d}s = \varrho^2 > 0 \quad \text{a.s.,} \tag{2.2}$$

then $\{A_t, t \geqq 0\}$ fulfils the law of iterated logarithm, i.e.

$$\overline{\lim_{t \to \infty}} \pm A_t/\sqrt{(2t \log\log t)} = \varrho \quad \text{a.s.}$$

The proofs of the statements of Lemma 3 are similar to those of Lemma 3 and Lemma 5 in [2].

3. MAIN THEOREM

In the sequel we consider the parametric situation, when the distribution function of the failure times is specified up to an unknown parameter α, ranging in an interval A. Hence, the failure rate is $g(x, \alpha)$. We denote by α_0 the true value of α. We are going to derive conditions under which the maximum likelihood estimate of α_0 is strongly consistent.

First we have to find the likelihood function. Let be given an arbitrary admissible policy ω. Consider the observation $(s_1, j_1, \ldots, s_n, j_n)$ of the replacements during time interval $[0, t]$. For knowing the policy this complete observation can be reconstructed from the observed failure times

(3.1) $$(\varphi_1, \varphi_2, \ldots, \varphi_k)\,.$$

Set

$$G_0(x) = \int_0^x g_0(y)\,\mathrm{d}y\,, \quad \bar{F}_0(x) = \mathrm{e}^{-G_0(x)}\,,$$

where $g_0(x) = g(x, \alpha_0)$. Take $\alpha \in A$ and denote

$$\frac{g_1(x)}{g_0(x)} = \frac{g(x, \alpha)}{g(x, \alpha_0)} = l(x)\,, \quad 0 \leqq l(x) < \infty\,,$$

in what follows subscript 0 refers to parameter value α_0, subscript 1 to value α. Let $l(x)$ be continuous on $[0, \infty)$. The time of first failure has probability density

$${}^i r_1(\varphi) = h_i(\varphi)\,.\,\exp\left\{-\int_0^\varphi h_i\,\mathrm{d}s\right\},$$

where $h_i(s) = g_i(X_s)$, $i = 0, 1$. Similarly, the moment of j-th failure has conditional density

$${}^i r_j(\varphi \mid \varphi_1, \ldots, \varphi_{j-1}) = h_i(\varphi \mid \varphi_1, \ldots, \varphi_{j-1})\,.\,\exp\left\{-\int_{\varphi_{j-1}}^\varphi h_i\,\mathrm{d}s\right\},$$

where $h_i(\varphi \mid \varphi_1, \ldots, \varphi_{j-1}) = g_i(X_\varphi)$, $i = 0, 1$, $j = 2, 3, \ldots, k$. Observation (3.1) on $[0, t]$ means that $\varphi_{k+1} > t$. Conditional probability of this event is

$$\exp\left\{-\int_{\varphi_k}^t g_i(X_s)\,\mathrm{d}s\right\}, \quad i = 0, 1\,.$$

Denote by P_0, P_1 the respective probability distributions of observation. The density of P_1 with respect to P_0 equals

$$\frac{\mathrm{d}P_1}{\mathrm{d}P_0} = p(s_1, j_1, \ldots, s_n, j_n; t) =$$

$$= \frac{g_1(X_{\varphi_1})\,.\exp\left\{-\int_0^{\varphi_1} g_1\,\mathrm{d}s\right\}\ldots g_1(X_{\varphi_k})\,.\exp\left\{-\int_{\varphi_{k-1}}^{\varphi_k} g_1\,\mathrm{d}s\right\}.\,\exp\left\{-\int_{\varphi_k}^t g_1\,\mathrm{d}s\right\}}{g_0(X_{\varphi_1})\,.\exp\left\{-\int_0^{\varphi_1} g_0\,\mathrm{d}s\right\}\ldots g_0(X_{\varphi_k})\,.\exp\left\{-\int_{\varphi_{k-1}}^{\varphi_k} g_0\,\mathrm{d}s\right\}.\,\exp\left\{-\int_{\varphi_k}^t g_0\,\mathrm{d}s\right\}} =$$

$$= l(X_{\varphi_1})\,.\,l(X_{\varphi_2})\ldots l(X_{\varphi_k})\,.\,\exp\left\{\int_0^t (g_0 - g_1)\,\mathrm{d}s\right\}.$$

Hence,

$$\log p = \int_0^t \log\frac{g_1(X_s)}{g_0(X_s)}\,\mathrm{d}N_s^1 - \int_0^t (g_1 - g_0)\,\mathrm{d}s\,.$$

 Denote

$$L_t(\alpha) = \int_0^t \log g(X_s)\, dN_s^1 - \int_0^t g(X_s)\, ds\,.$$

Then

$$L_t(\alpha) - L_t(\alpha_0) = \int_0^t \log \frac{g}{g_0}\, dN_s^1 - \int_0^t (g - g_0)\, ds\,, \tag{3.2}$$

where α_0 is the true value of the parameter α, and $L_t(\alpha)$ is the log-likelihood function. The maximum likelihood estimate of α_0 based on the observation up to time t, $\hat{\alpha}_t$ satisfies

$$L_t(\hat{\alpha}_t) = \sup_{\alpha\in A} L_t(\alpha)\,.$$

Example. Let $g(x, \alpha) = \alpha\,.\,x^p$, where $p \geqq 0$ is known. Denote by

$$Y_1^1,\, Y_2^1,\, \ldots,\, Y_{N_t^1}^1$$

the service replacement ages and by

$$Y_1^2,\, Y_2^2,\, \ldots,\, Y_{N_t^2}^2$$

the preventive replacement ages. Then the log-likelihood function

$$L_t(\alpha) = N_t^1 \log \alpha + p \sum_{k=1}^{N_t^1} \log Y_k^1 -$$

$$- \frac{\alpha}{p+1} \Big(\sum_{k=1}^{N_t^1} (Y_k^1)^{p+1} + \sum_{k=1}^{N_t^2} (Y_k^2)^{p+1} + X_t^{p+1}\Big)\,.$$

Hence,

$$\hat{\alpha}_t = (p+1)\, N_t^1\,.\Big(\sum_{k=1}^{N_t^1} (Y_k^1)^{p+1} + \sum_{k=1}^{N_t^2} (Y_k^2)^{p+1} + X_t^{p+1}\Big)^{-1}\,.$$

Theorem 1. Let the following conditions be fulfilled:

(1) A is a closed bounded interval.

(2) $g(x, \alpha)$ is continuous in (x, α).

(3) $\log (g/g_0)$ is continuous for all α.

(4) $|g(x, \alpha) - g(x, \alpha')| \leqq k_1(|\alpha - \alpha'|)\, k_2(x)$,
$|\log g(x, \alpha) - \log g(x, \alpha')| \leqq k_1(|\alpha - \alpha'|)\, k_2(x)$, $\alpha, \alpha' \in A$, where $\lim_{x\to 0+} k_1(x) = 0$,

$$\int_0^\infty k_2(x)\,.\,(1 + g_0(x))\, e^{-G_0(x)}\, dx < \infty\,.$$

(5) The replacement policy is such that $Z_t \geqq \Delta > 0$, $t \geqq 0$, and Z_t, $t \geqq 0$ is bounded a.s.

(6) The set $\{x : g(x, \alpha) \neq 0\}$ is independent of α, and for each $\alpha \in A$, $\alpha \neq \alpha_0$, $g(x, \alpha) \neq g(x, \alpha_0)$ for some $x \in [0, \Delta]$.

Then

(3.3) $$\hat{\alpha}_t \to \alpha_0 \quad \text{a.s.} \quad \text{as} \quad t \to \infty .$$

Assuming (1)–(6) we first prove two lemmas.

Lemma 4.

(3.4) $$\overline{\lim_{t\to\infty}}\, t^{-1}(L_t(\alpha) - L_t(\alpha_0)) \leqq \lambda(\alpha) \quad \text{a.s.},$$

where

$$\lambda(\alpha) = \int_0^{\Delta} \left(\log \frac{g}{g_0} - \frac{g}{g_0} + 1\right) g_0\, \bar{F}_0\, \mathrm{d}y \Big/ \int_0^{\infty} \bar{F}_0\, \mathrm{d}y ,$$

and $\lambda(\alpha) < 0$ for $\alpha \neq \alpha_0$.

Proof. Under the replacement policy with constant critical age x we get from (3.2)

(3.5) $$\lim_{t\to\infty} t^{-1}(L_t(\alpha) - L_t(\alpha_0)) =$$

$$= \int_0^{x} \left(g_0 \log \frac{g}{g_0} - (g - g_0)\right) \bar{F}_0\, \mathrm{d}y \Big/ \int_0^{x} \bar{F}_0\, \mathrm{d}y .$$

Moreover, $\log(g/g_0) - (g/g_0) + 1 \leqq 0$, with equality only for $g = g_0$. If $x \geqq \Delta > 0$, then the right hand side of (3.5) will be at most equal to

$$\int_0^{\Delta} \left(\log \frac{g}{g_0} - \frac{g}{g_0} + 1\right) g_0\, \bar{F}_0\, \mathrm{d}y \Big/ \int_0^{\infty} \bar{F}_0\, \mathrm{d}y = \lambda(\alpha) .$$

From (6) we get $\lambda(\alpha) < 0$, $\alpha \neq \alpha_0$.

The proof of (3.4) will be done in two steps:

1. Find a potential $v(y)$ for which

$$S_t = v(X_t^+) - v(X_0) + \int_0^t \log \frac{g}{g_0}\, \mathrm{d}N_s^1 - \int_0^t (g - g_0)\, \mathrm{d}s - \lambda t , \quad t \geqq 0 ,$$

is a supermartingale, i.e.

(3.6) $$\mathrm{E}\{S_{t+\mathrm{d}t} - S_t \mid X_t = y\} \leqq 0$$

under arbitrary admissible policy satisfying (5). We distinguish two cases:

(i) $y < \Delta$. The only decision at time t is not to do a preventive replacement. Thus,

$$\mathrm{E}\{S_{t+\mathrm{d}t} - S_t \mid X_t = y\} =$$
$$= (1 - g_0(y)\,\mathrm{d}t)\,[v(y + \mathrm{d}t) - v(y) - (g - g_0)\,\mathrm{d}t - \lambda\,\mathrm{d}t] +$$
$$+ g_0(y)\,\mathrm{d}t \left[\log \frac{g}{g_0} + v(0) - v(y)\right],$$

where $v(y + \mathrm{d}t) = v(y) + v'(y)\,\mathrm{d}t$, and we set $v(0) = 0$. Consequently, (3.6) implies

$$\text{(3.7)} \qquad \mathrm{E}\{S_{t+\mathrm{d}t} - S_t \mid X_t = y\} =$$
$$= \left[v' - \lambda - g_0 v - g_0\left(\frac{g}{g_0} - \log\frac{g}{g_0} + 1\right)\right]\mathrm{d}t \leqq 0\,.$$

(ii) $y \geqq \Delta$. (3.7) must hold if the decision means no preventive replacement. If a preventive replacement is made at time t, then (3.6) is

$$\text{(3.8)} \qquad \mathrm{E}\{S_{t+\mathrm{d}t} - S_t \mid X_t = y\} = -v(y) + O(dt) \leqq 0\,.$$

Inequality (3.7) holds, if

$$\text{(3.9)} \qquad v' - \lambda - (g - g_0) - g_0\left(v - \log\frac{g}{g_0}\right) = 0\,, \quad y \leqq \Delta\,,$$

$$\text{(3.10)} \qquad v' - \lambda - g_0 v = 0\,, \quad y > \Delta\,.$$

By solving of (3.9) with initial condition $v(0) = 0$ we get

$$v(y) = \mathrm{e}^{G_0(y)}\int_0^y\left(g - g_0 + \lambda - g_0\log\frac{g}{g_0}\right)\mathrm{e}^{-G_0}\,\mathrm{d}s\,, \quad y \leqq \Delta\,.$$

It is not difficult to verify that

$$v(\Delta) = -\lambda\,\mathrm{e}^{G_0(\Delta)}\int_\Delta^\infty \mathrm{e}^{-G_0}\,\mathrm{d}s\,.$$

Denote

$$v(y) = -\lambda\,\mathrm{e}^{G_0(y)}\int_y^\infty \mathrm{e}^{-G_0}\,\mathrm{d}s\,, \quad y \geqq \Delta\,.$$

Then the potential $v(y)$ fulfils (3.10), and $v(y) \geqq 0$. Thus, inequality (3.8) is fulfilled, too.

2. Next we show that

$$\text{(3.11)} \qquad M_t = S_t + \int_0^t \chi_{\{X_s \geqq \Delta\}}\left(g - g_0 - g_0\log\frac{g}{g_0}\right)\mathrm{d}s + \int_0^t v(X_s)\,\mathrm{d}N_s^2$$

is a martingale.

From (3.9) and (3.10) we get

$$v(X_t^+) - v(X_0) = \int_0^t v'(X_s)\,\mathrm{d}s - \int_0^t v(X_s)\,\mathrm{d}(N_s^1 + N_s^2) =$$
$$= \int_0^t \chi_{\{X_s < \Delta\}}\left[\lambda + (g - g_0) + g_0\left(v - \log\frac{g}{g_0}\right)\right]\mathrm{d}s +$$
$$+ \int_0^t \chi_{\{X_s \geqq \Delta\}}\left[\lambda + g_0 v\right]\mathrm{d}s - \int_0^t v(X_s)\,\mathrm{d}(N_s^1 + N_s^2)\,.$$

Thus,

$$M_t = \int_0^t \chi_{\{X_s<\Delta\}} \left[\lambda + (g - g_0) + g_0 \left(v - \log \frac{g}{g_0}\right)\right] \mathrm{d}s +$$

$$+ \int_0^t \chi_{\{X_s \geqq \Delta\}} (\lambda + g_0 v) \, \mathrm{d}s - \int_0^t v(X_s) \, \mathrm{d}N_s^1 + \int_0^t \log \frac{g}{g_0} \, \mathrm{d}N_s^1 - \int_0^t (g - g_0) \, \mathrm{d}s - \lambda t +$$

$$+ \int_0^t \chi_{\{X_s \geqq \Delta\}} \left(g - g_0 - g_0 \log \frac{g}{g_0}\right) \mathrm{d}s = \int_0^t \left(\log \frac{g}{g_0} - v(X_s)\right) . \, (\mathrm{d}N_s^1 - g_0 \, \mathrm{d}s) .$$

From Assumptions (3), (5) and by Lemmas 2, 3 M_t, $t \geqq 0$, is a martingale that fulfils the law of large numbers, i.e.

$$\lim_{t\to\infty} t^{-1} M_t = 0 \quad \text{a.s.}$$

From (3.11) then follows

$$\overline{\lim_{t\to\infty}} \, t^{-1} S_t \leqq 0 \quad \text{a.s.,} \tag{3.12}$$

and (3.12) implies (3.4). □

Lemma 5.

$$\overline{\lim_{t\to\infty}} \, t^{-1} \left[\int_0^t k_2(X_s) \, (\mathrm{d}N_s^1 + \mathrm{d}s)\right] \leqq \varkappa , \tag{3.13}$$

where

$$\varkappa = \int_0^\infty k_2(1 + g_0) \, \bar{F}_0 \, \mathrm{d}y \Big/ \int_0^\Delta \bar{F}_0 \, \mathrm{d}y .$$

The proof is similar to that of Lemma 4. We find a potential $\bar{v}(y)$ for which

$$\bar{S}_t = \bar{v}(X_t^+) - \bar{v}(X_0) + \int_0^t k_2(x) \, (\mathrm{d}N_s^1 + \mathrm{d}s) - \varkappa t$$

is a supermartingale. The solution is

$$\bar{v}(y) = \mathrm{e}^{G_0(y)} \int_0^\infty k_2(1 + g_0) \, \mathrm{e}^{-G_0} \, \mathrm{d}s , \quad y > 0 .$$

Then we prove that

$$\bar{M}_t = \bar{S}_t + \varkappa \int_0^t \chi_{\{X_s \geqq \Delta\}} \, \mathrm{d}s + \int_0^t \bar{v}(X_s) \, \mathrm{d}N_s^2 =$$

$$= \int_0^t (k_2(X_s) - \bar{v}(X_s)) \, (\mathrm{d}N_s^1 - g_0(X_s) \, \mathrm{d}s) .$$

Hence, $\bar{M}_t$, $t \geqq 0$ is a martingale. From the law of large numbers we derive (3.13). □

Proof of Theorem 1. Assume that (3.3) is not true, i.e. there exists an $\varepsilon > 0$ such that for

$$B = \{\overline{\lim_{t\to\infty}} |\hat{\alpha}_t - \alpha_0| > \varepsilon\}$$

we have $P(B) > 0$. Define

$$-\delta = \sup_{|\alpha-\alpha_0|\geqq\varepsilon} \lambda(\alpha).$$

From Assumption (6) and from (3.4) we get for $|\alpha - \alpha_0| \geqq \varepsilon$

$$\overline{\lim_{t\to\infty}}\, t^{-1}(L_t(\alpha) - L_t(\alpha_0)) \leqq \lambda(\alpha) \leqq -\delta \quad \text{a.s.}$$

From Assumption (4) follows

$$t^{-1}(L_t(\alpha) - L_t(\alpha')) \leqq k_1(|\alpha - \alpha'|)\, t^{-1} \int_0^t k_2(X_s)\,(\mathrm{d}N_s^1 + \mathrm{d}s)\,,$$

where by (3.13)

$$\overline{\lim_{t\to\infty}}\, t^{-1} \int_0^t k_2(X_s)\,(\mathrm{d}N_s^1 + \mathrm{d}s) \leqq \varkappa \quad \text{a.s.}$$

Thus, a.s. for t sufficiently large

$$t^{-1}|L_t(\alpha) - L_t(\alpha')| \leqq \frac{\varkappa}{2}\, k_1(|\alpha - \alpha'|)\,, \quad \alpha, \alpha' \in A\,.$$

This relation implies that for almost every trajectory functions $t^{-1} L_t(\alpha)$ are uniformly continuous with respect to t. Consequently, with probability one holds for sufficiently large t

$$t^{-1}(L_t(\alpha) - L_t(\alpha_0)) \leqq -\delta/2\,, \quad |\alpha - \alpha_0| \geqq \varepsilon\,. \tag{3.14}$$

(3.14) implies $|\hat{\alpha}_t - \alpha_0| < \varepsilon$, which contradicts to the assumption $P(B) > 0$. Thus, the Theorem 1 is established. □

Remark. It is not difficult to see that Assumption (4) can be omitted if $Z_t \leqq$ const., $t \geqq 0$. On the other hand a strengthening of the integrability condition in (4) enables us to omit the assumption that Z_t, $t \geqq 0$, is bounded.

4. INSERTION OF PARAMETER ESTIMATE INTO THE OPTIMAL POLICY

Denote by $d(\alpha)$ the optimal constant replacement age under the failure rate $g(x, \alpha)$. Assume (1)–(3) and (6) of Theorem 1 with (5) replaced by

(5′) $d(\alpha) \geqq \Delta > 0$, $\alpha \in A$, and $d(\alpha)$ is continuously differentiable.

Further let hold:

(7) α_0 is an inner point of A.

(8) $\log g(x, \alpha)$ has the first and second derivative with respect to α continuous in (x, α).

(9) $$\int_0^{\Delta} \left(\frac{g_0'}{g_0}\right)^2 f_0(x)\,\mathrm{d}x > 0\,.$$

Consider the replacement policy

(4.1) $$Z_t = d(\hat{\alpha}_t)\,, \quad t \geqq 0\,,$$

where $\hat{\alpha}_t$ is the estimate of α_0 by the maximum likelihood method. From Theorem 1 follows

(4.2) $$Z_t \to d = d(\alpha_0) \quad \text{a.s.} \quad \text{as} \quad t \to \infty\,.$$

Next we present a more precise statement about the convergence of $\hat{\alpha}_t$ to α_0.

Theorem 2. Let conditions (1)–(3), (5')–(9) hold with (4.1). Then

(4.3) $$\overline{\lim_{t\to\infty}}\, |\hat{\alpha}_t - \alpha_0| \sqrt{(t/\log\log t)} = \sqrt{2}/\varrho \quad \text{a.s.,}$$

where
$$\varrho^2 = \int_0^d \left(\frac{g_0'}{g_0}\right)^2 f_0\,\mathrm{d}x \Big/ \int_0^d \bar{F}_0\,\mathrm{d}x > 0\,.$$

Proof. In virtue of Assumption (7) $\hat{\alpha}_t$ is an inner point of A for sufficiently large t. Since $L_t(\hat{\alpha}_t)$ is the maximum of the log-likelihood function $L_t(\alpha)$, we have

$$0 = L_t'(\hat{\alpha}_t) = L_t'(\alpha_0) - (\hat{\alpha}_t - \alpha_0)\, L_t''(\alpha_0) + \int_{\alpha_0}^{\hat{\alpha}_t} |L_t''(a) - L_t''(\alpha_0)|\,\mathrm{d}a\,.$$

Hence,

(4.4) $$-\frac{L_t'(\alpha_0)}{t} = (\hat{\alpha}_t - \alpha_0)\frac{L_t''(\alpha_0)}{t} + \int_{\alpha_0}^{\hat{\alpha}_t} \frac{L_t''(a) - L_t''(\alpha_0)}{t}\,\mathrm{d}a\,.$$

It is clear that
$$L_t(\alpha_0) = \int_0^t \frac{g_0'}{g_0}(\mathrm{d}N_s^1 - g_0\,\mathrm{d}s)$$
is a martingale.

Using (4.2) it can be shown that

(4.5) $$\lim_{t\to\infty} t^{-1} \int_0^t \left(\frac{g_0'}{g_0}\right)^2 g_0\,\mathrm{d}s = \varrho^2 > 0 \quad \text{a.s.}$$

(See the proof of Theorem 3 in [2]).

 (4.5) corresponds to (2.2) of Lemma 3. Consequently martingale $L'_t(\alpha_0)$, $t \geqq 0$, fulfils the law of the iterated logarithm:

$$\overline{\lim_{t\to\infty}} \pm L'_t(\alpha_0)/\sqrt{(2t \log\log t)} = \varrho\,. \tag{4.6}$$

Investigate next

$$L''_t(\alpha_0) = \int_0^t \frac{g''_0 g_0 - (g'_0)^2}{g_0^2}\,(\mathrm{d}N_s^1 - g_0\,\mathrm{d}s) - \int_0^t \left(\frac{g'_0}{g_0}\right)^2 g_0\,\mathrm{d}s\,.$$

The first term on the right is a martingale that fulfils the law of large numbers. Thus, with regard to (4.5)

$$\lim_{t\to\infty} t^{-1}\, L''_t(\alpha_0) = -\varrho^2 \quad \text{a.s.} \tag{4.7}$$

Concerning the last term in (4.4) note that from Assumption (4) and from the boundedness of Z_t, $t \geqq 0$ follows that $t^{-1}\, L''_t(\alpha)$ is uniformly continuous with respect to t. Thus,

$$\int_{\alpha_0}^{\hat{\alpha}_t} \frac{L''_t(a) - L''_t(\alpha_0)}{t}\,\mathrm{d}a = o(\hat{\alpha}_t - \alpha_0)\,, \quad \text{as} \quad t\to\infty\,. \tag{4.8}$$

From (4.4) and (4.8) we get

$$-\frac{L'_t(\alpha_0)}{t} = (\hat{\alpha}_t - \alpha_0)\,\frac{L''_t(\alpha_0)}{t} + o(\hat{\alpha}_t - \alpha_0)\,.$$

Hence by (4.6) and (4.7)

$$\varrho = \overline{\lim_{t\to\infty}}\, \frac{|L'_t(\alpha_0)|}{\sqrt{(2t\log\log t)}} = \overline{\lim_{t\to\infty}}\, \frac{t|\hat{\alpha}_t - \alpha_0|}{\sqrt{(2t\log\log t)}}\,\varrho^2\,.$$

This establishes (4.3). □

From Assumption (5′) and (4.3) imply

$$\overline{\lim_{t\to\infty}}\, |Z_t - d|\,\sqrt{(t/\log\log t)} = \overline{\lim_{t\to\infty}}\, |d'(\alpha_0)|\,.\,|\hat{\alpha}_t - \alpha_0|\,\sqrt{(t/\log\log t)} =$$
$$= \sqrt{(2)}\,|d'(\alpha_0)|/\varrho\,.$$

By Theorem 1 in [2] this relation guarantees the best attainable convergence of the average cost $t^{-1}C_t$ to the optimum $\Theta(d)$. We formulate the result as a corollary.

Corollary. Under the assumption of Theorem 2,

$$\overline{\lim_{t\to\infty}} \pm (C_t - \Theta(d)\,.\,t)/\sqrt{(2t\log\log t)} = \sigma \quad \text{a.s.}$$

where

$$\sigma^2 = \int_0^d (c_1 - w)^2 f_0 \, \mathrm{d}y \Big/ \int_0^d \bar{F}_0 \, \mathrm{d}y \, ,$$

with $w(y) = \left(-c_1 F_0(y) + \Theta(d) \int_0^y \bar{F}_0(x) \, \mathrm{d}x\right) / \bar{F}_0(x)$.

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RNDr. Valéria Menyhértová, katedra matematickej analýzy PF UPJŠ (P. J. Šafárik University — Department of Math. Analysis), Komenského 14, 041 54 Košice. Czechoslovakia.