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Kybernetika, Vol. 32 (1996), No. 6, 615--624

Persistent URL: http://dml.cz/dmlcz/125363

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THE INVARIANT POLYNOMIAL ASSIGNMENT
PROBLEM FOR LINEAR PERIODIC
DISCRETE–TIME SYSTEMS

LEOPOLDO JETTO AND SAURO LONGHI

This paper considers the problem of assigning the closed loop invariant polynomials of a feedback control system, where the plant is a linear, discrete-time, periodic system. By a matrix algebraic approach, necessary and sufficient conditions for problem solvability are established and a parameterization of all periodic output controllers assigning the desired invariant polynomials is given.

1. INTRODUCTION

Various classes of processes, such as periodically time-varying networks and filters (for example switched-capacitors circuits and multirate digital filters), chemical processes, multirate sampled-data systems, can be modeled through a linear periodic system (see, e.g., [2, 13] and references therein). Moreover, the study of linear periodic systems can be helpful even for the stabilization and control of time-invariant linear systems through a periodic controller [1, 8, 18, 19, 21, 27], and for the stabilization and control of a class of bilinear systems [10, 11, 12].

In the discrete-time case, a control theory is developing with the help of algebraic and geometric techniques and contributions on several control problem have been given, including eigenvalue assignment, state and output dead-beat control, disturbance decoupling, model matching, adaptive control, robust control and optimal $H_2/H_\infty$ control (see, e.g., [3, 5, 7, 13, 15, 17, 22, 25, 26]).

The aim of this paper is to analyze the invariant polynomial assignment problem for the class of discrete-time linear periodic systems. This problem generalizes the characteristic polynomial assignment, which, for the same class of systems, was solved by a geometric approach in [5, 15, 17, 22]. For time-invariant plants, the invariant polynomial assignment was considered in [19, 20, 23, 27].

The paper is organized in the following way. In Section 2 preliminary definitions and results are given. The problem considered in this paper is formally stated in Section 3, and conditions for its solvability are constructively established in Section 4.

\(^1\) Work supported by the Ministero dell’Università e della Ricerca Scientifica.
2. PRELIMINARY RESULTS

Consider the $\omega$-periodic discrete-time system $E$ described by

$$x(k + 1) = A(k)x(k) + B(k)u(k), \quad (2.1)$$
$$y(k) = C(k)x(t), \quad (2.2)$$

where $k \in \mathbb{Z}$, $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^p$ is the input, $y(k) \in \mathbb{R}^q$ is the output and $A(\cdot), B(\cdot), C(\cdot)$ are periodic matrices of period $\omega$ (briefly, $\omega$-periodic). Denote also by $\Phi(k, k_0)$, $k \geq k_0$, the transition matrix associated with $A(\cdot)$.

It is well-known that, for any initial time $k_0 \in \mathbb{Z}$, the output response of system $E$ for $k > k_0$, to given initial state $x(k_0)$ and control function $u(\cdot)$, can be obtained through the time-invariant associated system of $E$ at time $k_0$, denoted by $E_{a\omega}(k_0)$ [24]. $E_{a\omega}(k)$ is represented by

$$x_k(h + 1) = E_kx_k(h) + J_ku_k(h) \quad (2.3)$$
$$y_k(h) = L_kx_k(h) + M_ku_k(h) \quad (2.4)$$

where $E_k := \Phi(\omega + k, k)$, $J_k := \Phi(\omega + k, i + k)B(i - 1 + k)$, $i = 1, \ldots, \omega$, $L_k := [(L_{k})_1, \ldots, (L_{k})_\omega]$, $(L_{k})_i := C(i - 1 + k)\Phi(i - 1 + k, k)$, $i = 1, \ldots, \omega$, $M_k := [(M_{k})_{ij} \in \mathbb{R}^{\times p}$, $i, j = 1, \ldots, \omega$, with $(M_{k})_{ij} := C(i - 1 + k)\Phi(i - 1 + k, j + k)B(j - 1 + k)$, if $i > j$, and $(M_{k})_{ij} := 0$, if $i \leq j$.

In fact, if $x_k(0) = x(k)$ and $u_k(h) := [u'(\omega + k) \ u'(\omega + k + 1) \ldots u'(\omega + k + \omega - 1)]'$ for all $h \in \mathbb{Z}^+$, then $x_k(h) = x(k + h\omega)$ and $y_k(h) = [y'(\omega + k) \ y'(\omega + k + 1) \ldots y'(\omega + k + \omega - 1)]'$ for all $h \in \mathbb{Z}^+$. The notion of associated system at time $k$ allows one to analyze structural and stability properties and pole-zero-structures of periodic systems [2, 4, 14]. For example, the subspace of reachable (unobservable) states of system $\Sigma$ at time $k$ is readily seen to coincide with that of system $E_{a\omega}(k)$ if it is expressed in terms of matrices $E_k, J_k, L_k$ and $M_k$ [14]. Obviously, $E_{a\omega}(k + \omega) = E_{a\omega}(k)$ for all integer $k$. A simple test for the reachability (observability) of system $\Sigma$ at time $k$ was also introduced in [16] making use of the following block-diagonal matrices:

$$A_k := \text{blockdiag}\{A(k), A(k + 1), \ldots, A(\omega - 1 + k)\}, \quad (2.5)$$
$$B_k := \text{blockdiag}\{B(k), B(k + 1), \ldots, B(\omega - 1 + k)\}, \quad (2.6)$$
$$C_k := \text{blockdiag}\{C(k), C(k + 1), \ldots, C(\omega - 1 + k)\}, \quad (2.7)$$
$$R_k(\lambda) := \begin{bmatrix} 0 & I_{(\omega - 1)n} \\ \lambda I_n & 0 \end{bmatrix}, \quad \lambda \in \mathbb{C}, \quad (2.8)$$

where $I_n$ denotes the identity matrix of dimension $n$.

**Lemma 2.1.** [16] System $\Sigma$ is reachable (observable) at time $k$ if and only if the following matrix

$$[A_k - R_k(\lambda) \ B_k] \quad ([A'_k - R'_k(\lambda) \ B'_k]')$$

has full row-rank (column-rank) for all $\lambda \in \mathbb{C}$, or equivalently for all the eigenvalues of $E_k$.  


The notions of invariant zero, transmission zero and pole of the \( \omega \)-periodic system \( \Sigma \) at time \( k \) are defined with reference to the following \( \omega q \times \omega p \) matrix

\[
W_k(d) = L_k d (I_n - dE_k)^{-1} J_k + M_k,
\]

where \( d := z^{-1} \) is the backward shift operator. The rational matrix \( W_k(d) \) is the transfer matrix of the associated system of \( \Sigma \) at time \( k \) and is called the associated transfer matrix of \( \Sigma \) at time \( k \). A complete analysis of pole-zero structure of system \( \Sigma \) is reported in [14] and [16] making use of the associated transfer matrix characterized with the forward shift operator \( z \). The following result, that follows from Lemma 2.1 in [14], shows the dependence of \( W_k(d) \) with respect to the initial time \( k \).

**Lemma 2.2.** For any integer \( k \) it holds that:

\[
W_{k+1}(d) = \begin{bmatrix} d I_q & I_{q(\omega - 1)} \\ d^{-1} I_q & 0 \end{bmatrix} W_k(d) \begin{bmatrix} d I_p \\ I_{p(\omega - 1)} \end{bmatrix}.
\]

As a consequence of this result the rank \( m \) of \( W_k(d) \) is independent of time \( k \) (see, e.g., [14] for a similar result with the forward shift operator \( z \)).

The transfer matrix \( W_k(d) \) can be factored as

\[
W_k(d) = A^{-1}_k(d) B_k(d) = \overline{B}_k(d) A^{-1}_k(d),
\]

where \( A_k(d) \) and \( B_k(d) \) are relatively left prime (rlp) polynomial matrices and \( \overline{A}_k(d) \) and \( \overline{B}_k(d) \) are relatively right prime (rrp) polynomial matrices.

Analogously to the time-invariant case [23], the invariant polynomials of \( I_n - dE_k \) are called the invariant polynomials of \( \Sigma \) at time \( k \). As shown in [14, 16], the product of these polynomials characterizes the stability properties of \( \Sigma \).

Under the hypothesis of reachability and observability of \( \Sigma \) at time \( k \), the invariant polynomials of \( \Sigma \) at time \( k \) are associate of the invariant polynomials of the Smith forms of \( A_k(d) \) and \( \overline{A}_k(d) \) [23].

Denote by \( \chi(q, p, \omega) \) the class of \( \omega q \times \omega p \) rational matrices

\[
W(d) = \begin{bmatrix} W_{11}(d) & W_{12}(d) & \cdots & W_{1\omega}(d) \\ W_{21}(d) & W_{22}(d) & \cdots & W_{2\omega}(d) \\ \vdots & \vdots & \ddots & \vdots \\ W_{\omega 1}(d) & W_{\omega 2}(d) & \cdots & W_{\omega\omega}(d) \end{bmatrix}, \quad W_{ij}(d) \in \mathbb{C}^{q \times p}, \quad i, j = 1, \ldots, \omega,
\]

with \( W_{ij}(0) = 0, \ i < j, \ i, j = 1, \ldots, \omega \). The class \( \chi(q, p, \omega) \) characterizes the transfer matrices of \( \omega \)-periodic systems. In fact, the causality of \( \omega \)-periodic system \( \Sigma \) implies that the associated transfer matrix of \( \Sigma \) at time \( k \) belongs to the class \( \chi(q, p, \omega) \) for all \( k \in \mathbb{Z} \) [6]. Then, the causality of \( \Sigma \) implies that the roots of the invariant polynomials of \( \Sigma \) at time \( k \) are different from zero for all integers \( k \). This in turn implies that matrices \( A_k(0) \) and \( \overline{A}_k(0) \) are nonsingular. Foregoing considerations and Lemma 2.2 allow us to prove the following result.

**Lemma 2.3.** The invariant polynomials of \( \Sigma \) at time \( k \) are independent of \( k \).
Remark 2.1. The choice of the backward shift operator \( d = z^{-1} \) allowed us to prove the independence of pole structure of \( \Sigma \) of time \( k \). The same result does not hold if the forward operator \( z \) is used \([16]\). In particular in \([14]\) it is shown that the structure of null poles may depend on \( k \).

Moreover, \( \chi(q, p, \omega) \) characterizes also the class of rational matrices that can be realized by an \( \omega \)-periodic system of the form (2.1), (2.2). The solution of the minimal realization problem for the periodic case is described by a system reachable and observable at any time whose matrices have generally time-varying dimensions. In general, the subspaces of reachable states and/or observable states may have time-varying dimensions. Therefore, it is natural, in order to consistently solve the minimal realization problem, to allow for state-space description having time-varying dimensions. The possibility of computing a “quasi” minimal (reachable and observable at least in one time) uniform (fixed-dimension) realization is also available. Efficient algorithms for the computation of minimal or quasi minimal realization of a given transfer matrix are introduced in \([6]\) and \([9]\).

Remark 2.2. Note that, given a transfer matrix \( H(d) = D^{-1}(d) N(d) = \overline{N}(d) \overline{D}^{-1}(d) \in \mathbb{C}_{\omega}^{q \times p} \) with \( D(d) \) and \( N(d) \) rlp polynomial matrices and \( \overline{D}(d) \) and \( \overline{N}(d) \) rrp polynomial matrices and both \( D(0) \) and \( \overline{D}(0) \) nonsingular, then a sufficient condition for \( H(d) \) belong to the class \( \chi(q, p, \omega) \) is that \( \overline{N}(0) = 0 \) and \( N(0) = 0 \).

3. CONTROL SYSTEM STRUCTURE AND PROBLEM STATEMENT

Assume that system \( \Sigma \) is minimal (reachable and observable at all times), and consider an \( \omega \)-periodic minimal controller \( \Sigma_G \) for system \( \Sigma \) acting in the feedback control structure of Figure 1 and described by

\[
\begin{align*}
x_G(k + 1) &= A_G(k)x_G(k) + B_G(k)e_2(k), \\
y_2(k) &= C_G(k)x_G(k) + D_G(k)e_2(k),
\end{align*}
\]  
(3.1)

where \( x_G(k) \in \mathbb{R}^{n_G(k)} \) is the state, with \( n_G(k + \omega) = n_G(k) \), and

\[
\begin{align*}
e_1(k) := u_1(k) - y_2(k), \\
e_2(k) := u_2(k) + y_1(k),
\end{align*}
\]  
(3.3)

with \( y_1(k) = y(k) \) (the output of \( \Sigma \)), \( e_1(k) = u(k) \) (the input of \( \Sigma \)) and \( u_1(k) \) and \( u_2(k) \) external inputs.

The \( \omega p \times \omega q \) associated transfer matrix of \( \Sigma_G \) at time \( k \) is expressed by

\[
W_G^G(d) = L_k^G d(I_{n_G(k)} - dE_k^G)^{-1}J_k^G + M_k^G,
\]  
(3.5)

where matrices \( L_k^G \in \mathbb{R}^{\omega p \times n_G(k)} \), \( E_k^G \in \mathbb{R}^{n_G(k) \times n_G(k)} \), \( J_k^G \in \mathbb{R}^{n_G(k) \times \omega q} \) and \( M_k^G \in \mathbb{R}^{\omega p \times \omega q} \) are defined as matrices \( L_k, E_k, J_k \) and \( M_k \) with matrices \( A(\cdot), B(\cdot) \) and \( C(\cdot) \) substituted by matrices \( A_G(\cdot), B_G(\cdot), C_G(\cdot) \) respectively and with \( (M_k^G)_{ii} = D_G(i - 1 + k), \ i = 1, \ldots, \omega. \)
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\[ u(k) e_1(k) \]

\[ y(k) \]

\[ \Sigma \]

\[ \Sigma_G \]

\[ e_2(k) + u(k) \]

Fig. 1. The feedback control structure.

Causality of system \( \Sigma_G \) implies that \( W_k^G(d) \) belongs to the class \( \chi(p, q, \omega) \).

Let \( W_k^G(d) \) be factored as

\[
W_k^G(d) = P_k^{-1}(d) Q_k(d) = \overline{Q}_k(d) \overline{P}_k^{-1}(d)
\]

(3.6)

where \( P_k(d) \) and \( Q_k(d) \) are rlp polynomial matrices and \( \overline{P}_k(d) \) and \( \overline{Q}_k(d) \) are rrp polynomial matrices. The problem considered in this paper is formally stated as follows.

**Problem 3.1.** Given an \( \omega \)-periodic system \( \Sigma \) reachable and observable at all times, and \( m \) causal polynomials \( s_1(d), s_2(d), \ldots, s_m(d) \) such that \( s_{i+1}(d) \) divides \( s_i(d) \), find a minimally realized \( \omega \)-periodic controller \( \Sigma_G \) described by (3.1), (3.2) and acting in the feedback system of Figure 1, such that the closed loop system \( \Sigma_{f_b} \) be minimally realized and its invariant polynomials be associated of \( s_i(d), i = 1, 2, \ldots, m \).

4. PROBLEM SOLUTION

Denote by \( \Sigma_{f_b} \) the \( \omega \)-periodic system reported in Figure 1 and described by (2.1), (2.2), (3.1), (3.2), (3.3) and (3.4) with input \( u(k) \) and output \( y(k) \) of \( \Sigma \) equal to \( e_1(k) \) and \( y_1(k) \), respectively.

Define:

\[
v(k) := [u'_1(k) \ u'_2(k)]', \ w_1(k) := [y'_1(k) \ e'_1(k)]', \ w_2(k) := [y'_2(k) \ e'_2(k)]',\]

(4.1)

the \( \omega \)-periodic feedback system \( \Sigma_{f_b} \) is described by the following equations:

\[
\begin{bmatrix}
  x(k+1) \\
  x_G(k+1)
\end{bmatrix} = \begin{bmatrix}
  A(k) - B(k) D_G(k) C(k) & -B(k) C_G(k) \\
  B_G(k) C(k) & A_G(k)
\end{bmatrix} \begin{bmatrix}
  x(k) \\
  x_G(k)
\end{bmatrix} \\
  + \begin{bmatrix}
  B(k) & -B(k) D_G(k) \\
  0 & B_G(k)
\end{bmatrix} v(k),
\]

(4.2)

\[
w_1(k) = \begin{bmatrix}
  C(k) & 0 \\
  -D_G(k) C(k) & -C_G(k)
\end{bmatrix} \begin{bmatrix}
  x(k) \\
  x_G(k)
\end{bmatrix} + \begin{bmatrix}
  0 & 0 \\
  I & -D_G(k)
\end{bmatrix} v(k),
\]

(4.3)

\[
w_2(k) = \begin{bmatrix}
  D_G(k) C(k) & C_G(k) \\
  C(k) & 0
\end{bmatrix} \begin{bmatrix}
  x(k) \\
  x_G(k)
\end{bmatrix} + \begin{bmatrix}
  0 & D_G(k) \\
  0 & I
\end{bmatrix} v(k).
\]

(4.4)
Denote with $W_k^1(d)$ and $W_k^2(d)$ the associated transfer matrices at time $k$ of the $\omega$-periodic feedback system $\Sigma_{f_b}$ relating input $v(\cdot)$ with outputs $w_1(\cdot)$ and $w_2(\cdot)$, respectively.

Introducing the lifted representations of inputs and outputs of $\Sigma_{f_b}$:

\[ u_k^1(h) := [u'_1(k + h\omega) u'_1(k + 1 + h\omega) \cdots u'_1(k + \omega - 1 + h\omega)]', \quad (4.5) \]
\[ u_k^2(h) := [u'_2(k + h\omega) u'_2(k + 1 + h\omega) \cdots u'_2(k + \omega - 1 + h\omega)]', \quad (4.6) \]
\[ v_k(h) := [v'(k + h\omega) v'(k + 1 + h\omega) \cdots v'(k + \omega - 1 + h\omega)]', \quad (4.7) \]
\[ y_k^1(h) := [y'_1(k + h\omega) y'_1(k + 1 + h\omega) \cdots y'_1(k + \omega - 1 + h\omega)]', \quad (4.8) \]
\[ e_k^1(h) := [e'_1(k + h\omega) e'_1(k + 1 + h\omega) \cdots e'_1(k + \omega - 1 + h\omega)]', \quad (4.9) \]
\[ w_k^1(h) := [w'_1(k + h\omega) w'_1(k + 1 + h\omega) \cdots w'_1(k + \omega - 1 + h\omega)]', \quad (4.10) \]
\[ y_k^2(h) := [y'_2(k + h\omega) y'_2(k + 1 + h\omega) \cdots y'_2(k + \omega - 1 + h\omega)]', \quad (4.11) \]
\[ e_k^2(h) := [e'_2(k + h\omega) e'_2(k + 1 + h\omega) \cdots e'_2(k + \omega - 1 + h\omega)]', \quad (4.12) \]
\[ w_k^2(h) := [w'_2(k + h\omega) w'_2(k + 1 + h\omega) \cdots w'_2(k + \omega - 1 + h\omega)]', \quad (4.13) \]

it can be verified the existence of appropriate unimodular matrices $U_a$ and $U_b$ such that the following relations are satisfied:

\[ \begin{bmatrix} u_k^1(h) \\ u_k^2(h) \end{bmatrix} = U_a v_k(h), \quad (4.14) \]
\[ \begin{bmatrix} y_k^1(h) \\ e_k^1(h) \end{bmatrix} = U_b w_k^1(h), \quad (4.15) \]
\[ \begin{bmatrix} y_k^2(h) \\ e_k^2(h) \end{bmatrix} = U_a w_k^2(h). \quad (4.16) \]

Then, the associated transfer matrices $W_k^1(d)$ and $W_k^2(d)$ of $\Sigma_{f_b}$ at time $k$ satisfy the following relations:

\[ W_k^1(d) = U_b^{-1} \left[ \begin{array}{cc} W_k y_1 u_1^1(d) & W_k y_1 u_2^1(d) \\ W_k e_1 u_1^1(d) & W_k e_1 u_2^1(d) \end{array} \right] U_a, \quad (4.17) \]
\[ W_k^2(d) = U_a^{-1} \left[ \begin{array}{cc} W_k y_2 u_1^1(d) & W_k y_2 u_2^2(d) \\ W_k e_2 u_1^1(d) & W_k e_2 u_2^2(d) \end{array} \right] U_a, \quad (4.18) \]

where $W_k y_{j,i}^u(d)$ and $W_k e_{j,i}^u(d)$ denote the associated transfer matrices at time $k$ of the $\omega$-periodic feedback system $\Sigma_{f_b}$ relating input $u_j(\cdot)$, $j = 1, 2$ with output $y_i(\cdot)$, $e_i(\cdot)$ $i = 1, 2$, respectively.

Denoting as

\[ F_k^1(d) = P_k(d) \overline{A}_k(d) + Q_k(d) \overline{B}_k(d), \quad (4.19) \]
\[ F_k^2(d) = A_k(d) \overline{P}_k(d) + B_k(d) \overline{Q}_k(d), \quad (4.20) \]

and arguing as in [23] it can be shown that

\[ W_k^1(d) = U_b^{-1} \left[ \begin{array}{c} \overline{B}_k(d) \\ \overline{A}_k(d) \end{array} \right] (F_k^1(d))^{-1} \left[ \begin{array}{c} P_k(d) \\ -Q_k(d) \end{array} \right] U_a, \quad (4.21) \]
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\[ W_k^2(d) = U_{a}^{-1} \begin{bmatrix} -Q_k(d) \\ P_k(d) \end{bmatrix} (F_k^2(d))^{-1} \begin{bmatrix} B_k(d) & A_k(d) \end{bmatrix} U_{a}. \] (4.22)

We are now in a position to prove the following main theorem.

**Theorem 4.1** Problem 3.1 admits a solution if and only if \( m < \min(\omega_p, \omega_q) \).

**Proof.** Necessity. Under the hypothesis on reachability and observability at all times of the \( \omega \)-periodic systems \( \Sigma \) and \( \Sigma_G \), by Lemma 2.1 applied to \( \Sigma_{fb} \) it can be shown that the \( \omega \)-periodic system \( \Sigma_{fb} \) is reachable at all times and observable through the outputs \( w_1(\cdot) \) and \( w_2(\cdot) \) at all times. Then (4.2) and (4.3) constitute a minimal realization of transfer matrix \( W_1^2(d) \) and (4.2) and (4.4) constitute a minimal realization of transfer matrix \( W_k^2(d) \). Moreover, for each time \( k \), the nonunit invariant polynomials of the \( (\omega_p \times \omega_p) \) polynomial matrix \( F_k^1(d) \) are associated of the nonunit invariant polynomials of the \( (\omega_q \times \omega_q) \) polynomial matrix \( F_k^2(d) \) and both are associated of the nonunit invariant polynomials at time \( k \) of the \( \omega \)-periodic feedback system \( \Sigma_{fb} \) [23]. This implies that the number \( m \) of the invariant polynomials at time \( k \) of the \( \omega \)-periodic feedback system \( \Sigma_{fb} \) can not be larger than \( m < \min(\omega_p, \omega_q) \).

Sufficiency. As \( A_k(d) \) and \( B_k(d) \) are \( rlp \) and \( \overline{A}_k(d) \) and \( \overline{B}_k(d) \) are \( rrp \), equations (4.19) and (4.20) can be solved for arbitrary \( F_k^1(d) \) and \( F_k^2(d) \). Hence, if \( m \leq \min(\omega_p, \omega_q) \), the \( s_i(d), i = 1, \ldots, m \) can be assigned to \( \Sigma_{fb} \) as invariant polynomials choosing \( F_k^1(d) \) and \( F_k^2(d) \) as polynomial matrices whose nonunit invariant polynomial are associate (two polynomials are called associate if their ratio is a scalar [23]) of the \( s_i(d), i = 1, \ldots, m \) and then to solve (4.19) or (4.20) with respect to the pairs \( (P_k(d), Q_k(d)) \) or \( (\overline{P}_k(d), \overline{Q}_k(d)) \) respectively. Moreover, as the invariant polynomials of \( \Sigma_{fb} \) are independent of \( k \), the solutions of (4.19) and (4.20) can be found for arbitrary \( k \).

For an arbitrary integer \( k \), all the solutions \( P_k(d) \) and \( Q_k(d) \) of (4.19) are given by
\[
\begin{bmatrix} P_k(d) \\ Q_k(d) \end{bmatrix} = [ F_k^1(d) & T_k(d) ] U_k(d)
\] (4.23)
where \( U_k(d) \) is the unimodular matrix given by
\[
U_k(d) = \begin{bmatrix} G_k(d) & H_k(d) \\ -B_k(d) & A_k(d) \end{bmatrix},
\]
\( G_k(d) \) and \( H_k(d) \) are polynomial matrices such that
\[ G_k(d)\overline{A}_k(d) + H_k(d)\overline{B}_k(d) = I_{\omega_p}, \]
and \( T_k(d) \) is an arbitrary polynomial matrix. For the solution (4.23) be adequate for Problem 3.1, \( T_k(d) \) must be such that

4a) \( P_k(d) \) and \( Q_k(d) \) are \( rlp \),

4b) \( P_k^{-1}(d)Q_k(d) \in \chi(p, q, \omega) \).

Analogously, for an arbitrary integer \( k \), all the solutions of (4.20) are given by
\[
\begin{bmatrix} \overline{P}_k(d) \\ \overline{Q}_k(d) \end{bmatrix} = \overline{U}_k(d) \begin{bmatrix} F_k^2(d) \\ T_k(d) \end{bmatrix},
\] (4.24)
where \( \overline{U}_k(d) \) is the unimodular matrix given by

\[
\overline{U}_k(d) = \begin{bmatrix}
\overline{G}_k(d) & -\overline{B}_k(d) \\
\overline{H}_k(d) & \overline{A}_k(d)
\end{bmatrix},
\]

\( \overline{G}_k(d) \) and \( \overline{H}_k(d) \) are polynomial matrices such that

\[
A_k(d)\overline{G}_k(d) + B_k(d)\overline{H}_k(d) = I_{\omega q},
\]

and \( \overline{T}_k(d) \) is an arbitrary polynomial matrix. For the solution (4.24) to be adequate to Problem 3.1, \( \overline{T}_k(d) \) must be such that:

- 4a) \( \overline{P}_k(d) \) and \( \overline{Q}_k(d) \) are rrp,
- 4b) \( \overline{Q}_k(d)\overline{P}_k(d)^{-1} \in \chi(p, q, \omega). \)

It remains to show that matrices and \( \overline{T}_k(d) \) such that the pairs \((\overline{P}_k(d), \overline{Q}_k(d))\) and \((\overline{P}_k(d), \overline{Q}_k(d))\) satisfy properties 4a, 4b and 4b, 4b respectively, can always be found.

With reference to solutions (4.24), matrix \( \overline{T}_k(d) \) can be found as follows. By the causality of \( \Sigma \), \( A_k(0) \) is non-singular, so that left primeness of \( A_k(d) \) and \( B_k(d) \) implies left primeness of \( A_k(d) \) and \( dB_k(d) \). This in turn implies that the equation

\[
A_k(d)\overline{P}_k^a(d) + dB_k(d)\overline{Q}_k^a(d) = F_k^2(d),
\]

can be solved with respect to \( \overline{P}_k^a(d) \) and \( \overline{Q}_k^a(d) \) for any \( F_k^2(d) \). For an arbitrary integer \( k \) the general solution of (4.25) is

\[
\begin{bmatrix}
\overline{P}_k^a(d) \\
\overline{Q}_k^a(d)
\end{bmatrix} = \overline{U}_k^a(d) \begin{bmatrix}
F_k^2(d) \\
\overline{T}_k^a(d)
\end{bmatrix},
\]

(4.26)

where \( \overline{U}_k^a(d) \) is a unimodular matrix given by

\[
\overline{U}_k^a(d) = \begin{bmatrix}
\overline{G}_k^a(d) & -d\overline{B}_k(d) \\
\overline{H}_k(d) & \overline{A}_k^a(d)
\end{bmatrix},
\]

\( \overline{G}_k^a(d) \) and \( \overline{H}_k^a(d) \) are polynomial matrices satisfying

\[
A_k(d)\overline{G}_k^a(d) + dB_k(d)\overline{H}_k^a(d) = I_{\omega q},
\]

(4.27)

and \( \overline{T}_k^a(d) \) is an arbitrary polynomial matrix. The unimodularity of \( \overline{U}_k^a(d) \) implies that if \( \overline{T}_k^a(d) \) is chosen right coprime with \( F_k^2(d) \), also \( \overline{P}_k^a(d) \) and \( \overline{Q}_k^a(d) \) are right coprime. Taking into account that by the causality of \( \Sigma_{fb} \) and (4.25), \( \overline{P}_k^a(0) \) is nonsingular, one has that also \( \overline{P}_k^a(d) \) and \( d\overline{Q}_k^a(d) \) are right coprime. So one can put \( \overline{G}_k^a(d) = \overline{G}_k^a(d), \overline{H}_k^a(d) = d\overline{H}_k^a(d), \overline{T}_k^a(d) = d\overline{T}_k^a(d) \) one has that the pair \((\overline{P}_k(d), \overline{Q}_k(d))\) given by

\[
\begin{align*}
\overline{P}_k(d) &= \overline{P}_k^a(d) = \overline{G}_k^a(d)F_k^2(d) - \overline{B}_k(d)\overline{T}_k^a(d), \\
\overline{Q}_k(d) &= d\overline{Q}_k^a(d) = \overline{H}_k^a(d)F_k^2(d) + \overline{A}_k^a(d)\overline{T}_k^a(d),
\end{align*}
\]

(4.28, 4.29)

defines a class of solutions (4.24) satisfying 4a and 4b (see Remark 2.2).
By arguing in a similar way, one has that the pair
\begin{align}
P_k(d) &= F_k^1(d)G_k(d) - T_k(d)B_k(d), \\
Q_k(d) &= F_k^1(d)H_k(d) + T_k(d)A_k(d),
\end{align}
where \( G_k(d) = G_k^a(d) \), \( H_k(d) = dH_k^a(d) \) with \( G_k^a(d) \) and \( H_k^a(d) \) such that
\[ G_k^a(d)A_k(d) + H_k^a(d)dB_k(d) = I_{\omega_p}, \]
and where \( T_k(d) = dT_k^a(d) \), \( T_k^a(d) \) being any polynomial matrix left prime with \( F_k^1(d) \), defines a class of solutions of (4.19) satisfying 4a and 4b (see Remark 2.2). Hence, under the assumption \( m < \min(\omega_p, \omega_q) \), the existence of solutions of Problem 3.1 has been constructively established.

5. CONCLUSIONS

In this paper the pole placement problem for linear discrete-time periodic systems has been considered. This problem has been formulated in the more general context of the invariant polynomial assignment, whence pole placement follows as a particular case. Necessary and sufficient conditions for problem solvability have been given in Theorem 3.1. The sufficiency proof of this theorem gives a parameterization of all controllers solving the problem in terms of causal transfer matrices that are minimally realizable with a periodic state-space representation. The proof has been performed in two steps. First, the set of all admissible solutions has been formally defined, then a procedure to effectively construct an admissible solution has been provided.

(Received February 14, 1996.)

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