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A CHARACTERIZATION OF SEPARABLE UTILITY FUNCTIONS

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The purpose of this paper is to characterize, by a certain set of properties, the separable utility functions among all possible utility functions depending on product quantities and "attractions". The main tool used is the branching property, analogous to the property of the same name in information theory. A new term is found, in addition to the separable part, which may be meaningful for utility functions.

1. A CHARACTERIZATION OF SEPARABLE UTILITY FUNCTIONS

In a recent paper, Beckmann and Funke [2] have proposed a parametrization of the utility function by a set of variables called "attractions". These variables represent the properties of a product which might be affected by marketing activities other than setting prices, e.g., advertising and packaging.

The proposed utility function of a household is of the form $u = u(\mathbf{a}, \mathbf{x})$, where $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is the vector of product attractions and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ the vector of product quantities. Beckmann and Funke have *a priori* restricted the class of utility functions to be considered to that of separable functions

$$u = \sum_{i=1}^{n} \Phi(a_i, x_i)\,.$$

In this paper, a characterization of separable utility functions is given.

Let us begin by supposing that the attractions a_i lie in some subset S of the real numbers R; similarly, $x_i \in T \subset R$, for $i = 1, 2, \ldots, n$. Moreover, we assume that the various attractions can be combined in S under a binary operation $\cdot$. Similarly, the measurements x_i of quantity are scaled so that they can be combined within T through a binary operation $*$.

Certainly the operations $\cdot$ and $*$ should be commutative and associative. For example, the associativity follows because the aggregate attraction of a combination of three products with attractions a_1, a_2, and a_3 should be the same whether it is computed as $(a_1 a_2) a_3$ or as $a_1(a_2 a_3)$, and similarly for the product quantities. Thus $(S, \cdot)$ and $(T, *)$ are commutative semigroups, and we assume that they have identity elements 1 and e, repectively.

Now suppose that u is *branching*, viz.,

$$u(a_1, x_1; a_2, x_2; \ldots; a_n, x_n) = \tag{1.1}$$

$$= u(a_1, x_1; \ldots; a_{i-1}, x_{i-1}; a_i a_{i+1}, x_i * x_{i+1}; 1, e; a_{i+2}, x_{i+2}; \ldots; a_n, x_n) +$$

$$+ \Delta_i(a_i, x_i; a_{i+1}, x_{i+1}), \quad \text{for} \quad i = 1, 2, \ldots, n-1 .$$

That is, if two products of a set of n under consideration are combined, the resulting difference in utility depends only on the quantities and attractions of those two products. For a discussion of the branching property (1.1) in the context of information theory, see Aczél/Daróczy [1]. The pair $(1, e)$ in (1.1) serves only as a placeholder to make the presentation simpler. Property (1.1) will be used to characterize separable utility functions.

The characterization depends on the structures of the monoids (i.e., semigroups with identity) $(S, \cdot)$ and $(T, *)$. In fact, we assume that these monoids are in the class $\mathbf{S}$ defined as follows.

Definition. A monoid $(S, \cdot)$ is said to belong to class $\mathbf{S}$ if $(S, \cdot)$ is commutative, and if all solutions of the functional equation

$$\Delta(a, b) + \Delta(ab, c) = \Delta(a, bc) + \Delta(b, c) , \tag{1.2}$$

for $a, b, c \in S$ and $\Delta : S^2 \to R$, can be put in the form

$$\Delta(a, b) = \delta(a) + \delta(b) - \delta(ab) + \Psi(a, b) \tag{1.3}$$

for an arbitrary map $\delta : S \to R$ and a map $\Psi : S^2 \to R$ which is *antisymmetric*

$$\Psi(a, b) = -\Psi(b, a) , \quad \forall a, b \in S , \tag{1.4}$$

and *bi-additive*. Additivity in the second variable, for example, means

$$\Psi(a, bc) = \Psi(a, b) + \Psi(a, c) , \quad \forall a, b, c \in S .$$

This definition appears strange, but the following example and theorem show that there are some familiar objects in $\mathbf{S}$.

Example. Let $S = [0, 1]$ and $ab := \min(a, b)$ for all $a, b \in S$. And let $(T, *) = (R^+, +)$ = the nonnegative real numbers under addition. By Theorem 1.1 (below), both are in $\mathbf{S}$. Now equation (1.1) governs the change in utility which occurs when two products, parametrized by (a_i, x_i) and (a_{i+1}, x_{i+1}), are replaced by a single product in quantity $x_i + x_{i+1}$ and with attractiveness equal to the minimum of the attractions of the two original products.

Theorem 1.1. $\mathcal{S}$ contains all commutative monoids among the following: idempotent monoids, monoids with zero, threads, groups, cancellative w-threads, near-threads, and $\{g \in G \mid g > 0\}$ for any ordered group G.

In our example above, $(S, \cdot)$ is an idempotent commutative monoid, and $(T, *)$ is a cancellative w-thread. (A w-thread is a connected, totally ordered topological semigroup. See [3] for other definitions and proof of Theorem 1.1.)

In the next section, a functional equation central to our purpose will be derived from (1.1). Section 3 contains the main results, which are based on the solution of the functional equation. This equation is solved in Section 4, at the end of which some extensions and generalizations are mentioned.

2. REDUCTION TO A FUNCTIONAL EQUATION

By repeated use of (1.1), we get

$$\begin{aligned}
&u(a_1, x_1; \ldots; a_{i-1}, x_{i-1}; a_i a_{i+1} a_{i+2}, x_i * x_{i+1} * x_{i+2};\\
&\quad 1, e; 1, e; a_{i+3}, x_{i+3}; \ldots; a_n, x_n) +\\
&\quad + \Delta_i(a_i, x_i; a_{i+1}a_{i+2}, x_{i+1} * x_{i+2}) + \Delta_{i+1}(a_{i+1}, x_{i+1}; a_{i+2}, x_{i+2}) =\\
&\quad = u(a_1, x_1; \ldots; a_i, x_i; a_{i+1}a_{i+2}, x_{i+1} * x_{i+2}; 1, e; a_{i+3}, x_{i+3}; \ldots; a_n, x_n) +\\
&\quad + \Delta_{i+1}(a_{i+1}, x_{i+1}; a_{i+2}, x_{i+2}) = u(a_1, x_1; \ldots; a_n, x_n) =\\
&\quad = u(a_1, x_1; \ldots; a_{i-1}, x_{i-1}; a_i a_{i+1}, x_i * x_{i+1}; 1, e; a_{i+2}, x_{i+2}; \ldots; a_n, x_n) +\\
&\quad + \Delta_i(a_i, x_i; a_{i+1}, x_{i+1}) =\\
&\quad = u(a_1, x_1; \ldots; a_{i-1}, x_{i-1}; a_i a_{i+1}, x_i * x_{i+1}; a_{i+2}, x_{i+2}; 1, e;\\
&\qquad a_{i+3}, x_{i+3}; \ldots; a_n, x_n) + \Delta_{i+1}(1, e; a_{i+2}, x_{i+2}) + \Delta_i(a_i, x_i; a_{i+1}, x_{i+1}) =\\
&\quad = u(a_1, x_1; \ldots; a_{i-1}, x_{i-1}; a_i a_{i+1} a_{i+2}, x_i * x_{i+1} * x_{i+2}; 1, e; 1, e;\\
&\qquad a_{i+3}, x_{i+3}; \ldots; a_n, x_n) + \Delta_i(a_i a_{i+1}, x_i * x_{i+1}; a_{i+2}, x_{i+2}) +\\
&\quad + \Delta_{i+1}(1, e; a_{i+2}, x_{i+2}) + \Delta_i(a_i, x_i; a_{i+1}, x_{i+1}).
\end{aligned}$$

Comparing extremes of this line of equations, we get

$$\begin{aligned}
(2.1)\qquad &\Delta_i(a, x; bc, y * z) + \Delta_{i+1}(b, y; c, z) =\\
&= \Delta_i(ab, x * y; c, z) + \Delta_{i+1}(1, e; c, z) + \Delta_i(a, x; b, y),
\end{aligned}$$

for all $a, b, c \in S$; $x, y, z \in T$; and $i = 1, 2, \ldots, n-2$.

With $(a, x) = (1, e)$, equation (2.1) yields

$$\begin{aligned}
(2.2)\qquad &\Delta_{i+1}(b, y; c, z) =\\
&= \Delta_i(b, y; c, z) + \Delta_{i+1}(1, e; c, z) + \Delta_i(1, e; b, y) - \Delta_i(1, e; bc, y * z).
\end{aligned}$$

By this relation, (2.1) becomes

$$\Delta_i(a, x; bc, y * z) - \Delta_i(1, e; bc, y * z) + \Delta_i(b, y; c, z) =$$
$$= \Delta_i(ab, x * y; c, z) + \Delta_i(a, x; b, y) - \Delta_i(1, e; b, y),$$

for $i = 1, 2, \ldots, n - 1$ (we get it for $i = n - 1$ by letting $i = n - 2$ in (2.2) and (2.1) and eliminating all Δ_{n-2} terms between them.) Fixing $i = 1$ temporarily and defining $F : (S \times T)^2 \to R$ by

$$F(a, x; b, y) := \Delta_1(a, x; b, y) - \Delta_1(1, e; b, y), \tag{2.3}$$

we have

$$F(a, x; bc, y * z) + F(b, y; c, z) = F(ab, x * y; c, z) + F(a, x; b, y). \tag{2.4}$$

3. MAIN RESULTS

In Section 4, we shall prove the following result.

Theorem 3.1. The general solution of (2.4), for $(S, \cdot)$ and $(T, *)$ in **S**, is given by

$$F(a, x; b, y) = \varphi(a, x) + \varphi(b, y) - \varphi(ab, x * y) + \Psi(a, x; b, y), \tag{3.1}$$

for all $(a, b) \in S^2$ and $(x, y) \in T^2$, where $\varphi : S \times T \to R$ is arbitrary and Ψ is both *antisymmetric* and *bi-additive* in the following sense.

$$\Psi(a, x; b, y) = -\Psi(b, y; a, x), \tag{3.2}$$
$$\Psi(a_1a_2, x_1 * x_2; b, y) = \Psi(a_1, x_1; b, y) + \Psi(a_2, x_2; b, y).$$

Using this result, we can find the form of the Δ_i's and of u. By (2.3) and (3.1), Δ_1 has the representation

$$\Delta_1(a, x; b, y) = \varphi_1(a, x) + \varphi_2(b, y) - \varphi_1(ab, x * y) + \Psi(a, x; b, y), \tag{3.3}$$

where $\varphi_1(a, x) := \varphi(a, x)$ and $\varphi_2(b, y) := \varphi(b, y) + \Delta_1(1, e; b, y)$ for the arbitrary function $\Delta_1(1, e; b, y)$. Now, starting with $i = 2$, we can define Δ_i recursively in terms of Δ_{i-1}.

Starting with equation (3.3), using (2.2) successively for $i = 1, 2, \ldots, n - 2$, we find that

$$\Delta_i(a, x; b, y) = \varphi_i(a, x) + \varphi_{i+1}(b, y) - \varphi_i(ab, x * y) + \Psi(a, x; b, y), \tag{3.4}$$

where $\varphi_{i+1}(b, y) := \varphi_i(b, y) + \Delta_i(1, e; b, y)$ for the arbitrary function $\Delta_i(1, e; b, y)$.

Now, by (3.4) and the bi-additivity of Ψ, equation (1.1) becomes

$$u(a_1, x_1; \ldots; a_n, x_n) = u(a_1, x_1; \ldots; a_{n-2}, x_{n-2}; a_{n-1}a_n, x_{n-1} * x_n; 1, e) + \\ + \varphi_{n-1}(a_{n-1}, x_{n-1}) + \varphi_n(a_n, x_n) - \varphi_{n-1}(a_{n-1}a_n, x_{n-1} * x_n) + \\ + \Psi(a_{n-1}, x_{n-1}; a_n, x_n) = \ldots = u(a_1 \cdot \ldots \cdot a_n, x_1 * \ldots * x_n; 1, e; \ldots, 1, e) + \\ + \sum_{i=1}^{n} \varphi_i(a_i, x_i) - \varphi_1(a_1 \cdot \ldots \cdot a_n, x_1 * \ldots * x_n) + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Psi(a_i, x_i; a_j, x_j).$$

With $\varphi_0(a, x) := u(a, x; 1, e; \ldots; 1, e) - \varphi_1(a, x)$, we have therefore

$$u(a_1, x_1; \ldots; a_n, x_n) = \sum_{i=1}^{n} \varphi_i(a_i, x_i) + \varphi_0(a_1 \cdot \ldots \cdot a_n, x_1 * \ldots * x_n) + \\ + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Psi(a_i, x_i; a_j, x_j). \tag{3.5}$$

Based on Theorem 3.1, the following is now established.

Theorem 3.2. Let $(S, \cdot) \in \mathbf{S}$, $(T, *) \in \mathbf{S}$. A utility function $u = u(\mathbf{a}, \mathbf{x})$ has the (1.1) branching property over S and T, if and only if there exist maps $\varphi_i : S \times T \to R$ $(i = 0, 1, \ldots, n)$ and an antisymmetric bi-additive map $\Psi : (S \times T)^2 \to R$ representing u through (3.5).

From Theorem 3.2, we can derive the following result, which says that u is "almost" separable. The $\bar{\varphi}_0$ term below may also be useful for measuring utility, and gives some added flexibility.

Corollary. If, in addition to being (1.1) branching, u is also symmetric in the pairs (a_i, x_i), then

$$u(\mathbf{a}, \mathbf{x}) = \sum_{i=1}^{n} \varphi(a_i, x_i) + \bar{\varphi}_0(a_1 \cdot \ldots \cdot a_n, x_1 * \ldots * x_n). \tag{3.6}$$

Since the utility should not depend on whether a product is called the first or the jth, symmetry in the pairs (a_i, x_i) is a desirable property. Furthermore, the result (3.6) reduces to separability if $\bar{\varphi}_0$ is constant, since it can then be incorporated into the terms $\varphi(a_i, x_i)$.

Proof of Corollary. First, interchange two pairs of arguments, say (a_k, x_k) and (a_{k+1}, x_{k+1}), in (3.5). By hypothesis, the utility is unchanged, so

$$\Psi(a_{k+1}, x_{k+1}; a_k, x_k) + \varphi_k(a_{k+1}, x_{k+1}) + \varphi_{k+1}(a_k, x_k) = \\ = \Psi(a_k, x_k; a_{k+1}, x_{k+1}) + \varphi_k(a_k, x_k) + \varphi_{k+1}(a_{k+1}, x_{k+1}). \tag{3.7}$$

Substituting $x_{k+1} = e$, $a_{k+1} = 1$ and using the bi-additivity of Ψ, $(\Psi(a, x; 1, e) = \Psi(1, e; a, x) = 0)$, we find that

$$\varphi_{k+1} = \varphi_k + \text{constant}.$$

Now (3.7) yields

$$\Psi(a_{k+1}, x_{k+1}; a_k, x_k) = \Psi(a_k, x_k; a_{k+1}, x_{k+1}),$$

which gives, by the antisymmetry of Ψ,

$$\Psi \equiv 0.$$

By defining $\varphi := \varphi_1$ and absorbing the constants in $\bar{\varphi}_0$, we have (3.6). □

4. PROOF OF THEOREM 3.1

We first establish the following lemma.

Lemma 4.1. The general solution of (2.4), for $(T, *) \in \mathbf{S}$, is given by

$$(4.1)\qquad F(a, x; b, y) = \eta(a, b) + \tilde{\mu}(a, x) + \tilde{\mu}(b, y) - \tilde{\mu}(ab, x * y) + \\ + \tilde{\Psi}(a, x; b, y),$$

for all $(a, b) \in S^2$ and $(x, y) \in T^2$, where $\tilde{\Psi}$ is (3.2) antisymmetric and bi-additive, and $\eta : S^2 \to R$ satisfies (1.2).

Proof. We begin by fixing $a, b, c \in S$ temporarily, so that

$$(4.2)\qquad F_1(x, y * z) + F_2(y, z) = F_3(x * y, z) + F_4(x, y),$$

where $F_1(x, y) := F(a, x; bc, y)$, $F_2(x, y) := F(b, x; c, y)$, $F_3(x, y) := F(ab, x; c, y)$, and

$$(4.3)\qquad F_4(x, y) := F(a, x; b, y).$$

Since $(T, *) \in \mathbf{S}$, the solution of (4.2) is provided by Theorem 8.1 in [3]. F_4 has the form

$$(4.4)\qquad F_4(x, y) = f(x) + g(y) - h(x * y) + \Psi(x, y),$$

where Ψ is (1.4) antisymmetric and bi-additive.

If a, b, c are freed again, (4.3) and (4.4) give

$$(4.5)\qquad F(a, x; b, y) = f(a, b, x) + g(a, b, y) - h(a, b, x * y) + \Psi(a, b, x, y),$$

where $\Psi(a, b, x, y)$ is antisymmetric and bi-additive in (x, y). Substituting (4.5) into (2.4), we have

$$(4.6)\qquad f(a, bc, x) + g(a, bc, y * z) - h(a, bc, x * y * z) + \\ + \Psi(a, bc, x, y * z) + f(b, c, y) + g(b, c, z) - h(b, c, y * z) + \Psi(b, c, y, z) = \\ = f(ab, c, x * y) + g(ab, c, z) - h(ab, c, x * y * z) + \Psi(ab, c, x * y, z) + \\ + f(a, b, x) + g(a, b, y) - h(a, b, x * y) + \Psi(a, b, x, y),$$

for all $a, b \in S$ and $x, y \in T$.

The three steps that follow sequentially lead to a proof of Lemma 4.1.

Step 1.

$$g(a, b, y) = A(a, b, y) + l(a, b) + \alpha(b, y), \tag{4.7}$$

for some maps l, α, and A satisfying

$$A(a, b, x * y) = A(a, b, x) + A(a, b, y). \tag{4.8}$$

Proof. Notice that the bi-additivity of $\Psi(a, b, x, y)$ in (x, y) implies

$$\Psi(a, b, x, e) = \Psi(a, b, e, y) = 0, \quad \forall a, b \in S;\ x, y \in T. \tag{4.9}$$

By (4.9), (4.6) with $(b, y) = (1, e)$ yields

$$f(1, c, e) + g(1, c, z) - h(1, c, z) = f(a, 1, x) + \\ + g(a, 1, e) - h(a, 1, x) = -\gamma \text{ (constant)}.$$

Thus h satisfies

$$\begin{aligned} h(a, 1, x) &= f(a, 1, x) + g(a, 1, e) + \gamma \\ h(1, c, z) &= f(1, c, e) + g(1, c, z) + \gamma. \end{aligned} \tag{4.10}$$

Using equations (4.10), the bi-additivity of $\Psi(a, b, \cdot, \cdot)$, and $b = 1$ in (4.6), we obtain

$$\begin{aligned} & f(a, c, x) + g(a, c, y * z) + \Psi(a, c, x, y * z) + f(1, c, y) + \\ & + g(1, c, z) - f(1, c, e) - g(1, c, y * z) + \Psi(1, c, y, z) = \\ & = f(a, c, x * y) + g(a, c, z) + \Psi(a, c, x * y, z) + f(a, 1, x) + \\ & + g(a, 1, y) - f(a, 1, x * y) - g(a, 1, e) + \Psi(a, 1, x, y). \end{aligned} \tag{4.11}$$

By (4.9), (4.11) with $z = e$, resp. $x = e$, yields

$$\begin{aligned} & f(a, c, x * y) + f(a, 1, x) + g(a, 1, y) - f(a, 1, x * y) - g(a, 1, e) + \\ & + \Psi(a, 1, x, y) = f(a, c, x) + g(a, c, y) + \Psi(a, c, x, y) + f(1, c, y) + \\ & + g(1, c, e) - f(1, c, e) - g(1, c, y) - g(a, c, e), \end{aligned}$$

$$\begin{aligned} & g(a, c, y * z) + f(1, c, y) + g(1, c, z) - f(1, c, e) - g(1, c, y * z) + \\ & + \Psi(1, c, y, z) = f(a, c, y) + g(a, c, z) + \Psi(a, c, y, z) + f(a, 1, e) + g(a, 1, y) - \\ & - f(a, 1, y) - g(a, 1, e) - f(a, c, e). \end{aligned} \tag{4.12}$$

We can substitute from these last two equations back into (4.11), using the bi-additivity of Ψ again, to get

$$\begin{aligned} f(a, c, y) = {} & g(a, c, y) + f(a, c, e) - g(a, c, e) + f(1, c, y) - g(1, c, y) - \\ & - f(1, c, e) + g(1, c, e) + f(a, 1, y) - g(a, 1, y) - f(a, 1, e) + g(a, 1, e). \end{aligned}$$

Thus we can define maps k_1, δ_1, and δ_2 so that

$$(4.13)\qquad f(a, c, y) = g(a, c, y) + k_1(a, c) + \delta_1(c, y) + \delta_2(a, y)\,,$$

$$(4.14)\qquad \delta_i(1, y) = \delta_i(c, e) = 0 \ (i = 1, 2)\,, \quad \forall c \in S\,, \ y \in T\,.$$

Now, by (4.14) and (4.13), (4.12) yields

$$\begin{aligned} g(a, c, y * z) - g(1, c, y * z) + \Psi(1, c, y, z) = \\ = g(a, c, y) - g(1, c, y) - g(a, c, e) + g(1, c, e) + \\ + g(a, c, z) - g(1, c, z) + \Psi(a, c, y, z)\,. \end{aligned}$$

With respect to an interchange of y and z, the symmetric part of this equation is

$$(4.15)\quad \begin{aligned} g(a, c, y * z) - g(1, c, y * z) = g(a, c, y) - g(1, c, y) - g(a, c, e) + \\ + g(1, c, e) + g(a, c, z) - g(1, c, z) \end{aligned}$$

and the antisymmetric part is $\Psi(a, c, y, z) = \Psi(1, c, y, z)$. From this last equation we deduce that $\Psi'(c, y, z) := \Psi(1, c, y, z)$ defines a map which is antisymmetric and bi-additive in (y, z), and that

$$(4.16)\qquad \Psi'(c, y, z) = \Psi(a, c, y, z)\,.$$

From (4.15) we get (4.8) for the map A defined by

$$A(a, c, y) := g(a, c, y) - g(1, c, y) - g(a, c, e) + g(1, c, e)\,.$$

Hence there are maps l and α such that g is given by (4.7). □

Step 2. There exist maps k, m, μ, Ψ_2, γ such that

$$(4.17)\qquad f(a, b, x) = A(a, b, x) + k(a, b) + m(b, x) + \mu(a, x)\,,$$

$$(4.18)\qquad \Psi_2(x, y) = \Psi(a, b, x, y)\,,$$

$$(4.19)\qquad h(a, b, z) = A(a, b, z) + \alpha(b, z) - \mu(b, z) + \mu(ab, z) + \gamma(a, b)\,,$$

where m is additive in the second variable, i.e.

$$(4.20)\qquad m(a, x * y) = m(a, x) + m(a, y)\,.$$

Proof. By Step 1, (4.13), (4.14), (4.16), and the bi-additivity of $\Psi'(c, y, z)$ in (y, z), (4.11) becomes

$$\begin{aligned} \alpha(c, x) + \delta_1(c, x) - \alpha(1, y) - \alpha(c, e) + \alpha(1, e) + \\ + \alpha(c, y) + \delta_1(c, y) - \alpha(1, y) + \Psi'(c, x, y) = \\ = \alpha(c, x * y) + \delta_1(c, x * y) - \alpha(1, x * y) + \Psi'(1, x, y)\,. \end{aligned}$$

Comparing parts of this equation which are antisymmetric, resp. symmetric, with respect to the interchange of x and y, we find that

$$\Psi'(c, x, y) = \Psi'(1, x, y)$$

is the antisymmetric part. Defining $\Psi_2(x, y) := \Psi'(1, x, y)$, we thus get (by way of (4.16)) equation (4.18).

On the other hand, the symmetric part becomes simply (4.20) if we define $m(a, x) := \alpha(a, x) + \delta_1(a, x) - \alpha(1, x) - \alpha(a, e) + \alpha(1, e)$. Thus, defining $\varepsilon_1(a) := \alpha(a, e)$ and $\varepsilon_2(x) := \alpha(1, x) - \alpha(1, e)$, we have

$$\delta_1(a, x) = m(a, x) - \alpha(a, x) + \varepsilon_1(a) + \varepsilon_2(x),$$

where m satisfies (4.20). This representation for δ_1 can be used with (4.13) and (4.7) to obtain (4.17), where $k(a, b) := l(a, b) + k_1(a, b) + \varepsilon_1(b)$ and $\mu(a, x) := \delta_2(a, x) + \varepsilon_2(x)$.

Finally, we use (4.7), (4.17), and (4.18) (which shows that Ψ_2 is bi-additive) to rewrite (4.6), which becomes

$$\begin{aligned}(4.21)\qquad & A(a, bc, x) + k(a, bc) + m(bc, x) + A(a, bc, y * z) + l(a, bc) + \\ & + \alpha(bc, y * z) - h(a, bc, x * y * z) + A(b, c, y) + k(b, c) + \\ & + m(c, y) + \mu(b, y) + A(b, c, z) + l(b, c) - h(b, c, y * z) = \\ & = A(ab, c, x * y) + k(ab, c) + m(c, x * y) + \mu(ab, x * y) + \\ & + A(ab, c, z) + l(ab, c) - h(ab, c, x * y * z) + A(a, b, x) + \\ & + k(a, b) + m(b, x) + A(a, b, y) + l(a, b) + \alpha(b, y) - h(a, b, x * y).\end{aligned}$$

We shall also use the facts that $A(a, b, e) = 0$ and $m(a, e) = 0$ for all $a, b \in S$, which follow from (4.8), resp. (4.20), with $y = e$. Now, with $c = 1$ and $x = y = e$, by (4.7), (4.10) and (4.17), (4.21) yields a representation (4.19) for h, where $\gamma(a, b) := \mu(b, e) - \alpha(b, e) - \mu(ab, e) + h(a, b, e)$. □

Step 3. Representation (4.1) holds for F.

Proof. First define a map $\beta : S \times T \to R$ by

$$(4.22)\qquad \beta(a, x) := \alpha(a, x) - \mu(a, x).$$

Then, by (4.8), substitution of (4.19) into (4.21) gives

$$\begin{aligned}(4.23)\qquad & \eta(a, bc) + m(bc, x) + \beta(bc, y * z) - \beta(bc, x * y * z) + \\ & + \eta(b, c) + m(c, y) - \beta(c, y * z) = \\ & = \eta(ab, c) + m(c, x * y) - \beta(c, x * y * z) + \eta(a, b) + \\ & + m(b, x) + \beta(b, y) - \beta(b, x * y),\end{aligned}$$

where η is defined by

$$(4.24)\qquad \eta(a, b) := k(a, b) + l(a, b) - \gamma(a, b).$$

Now define a map $B : S \times T \to R$ by

$$B(a, x) := -\beta(a, x) + m(a, x) + \beta(a, e), \tag{4.25}$$

so that (4.23) can be written (using (4.20))

$$\begin{aligned} \eta(a, bc) + \eta(b, c) - B(bc, y * z) + B(bc, x * y * z) + B(c, y * z) = \\ = \eta(ab, c) + \eta(a, b) + B(c, x * y * z) - B(b, y) + B(b, x * y). \end{aligned} \tag{4.26}$$

With $x = y = z = e$, (4.26) gives (1.2) for η, as required, which in turn reduces (4.26) to

$$\begin{aligned} -B(bc, y * z) + B(bc, x * y * z) + B(c, y * z) = \\ = B(c, x * y * z) - B(b, y) + B(b, x * y). \end{aligned} \tag{4.27}$$

With $b = c = 1$ and $y = e$, (4.27) gives $B(1, e) = B(1, x) =$ constant. Moreover, (4.25) implies $B(a, e) = 0$, since $m(a, e) = 0$ by (4.20). So (4.27) with $c = 1$ and $y = e$ yields

$$-B(b, z) + B(b, x * z) = B(b, x), \tag{4.28}$$

i.e. B is additive in the second variable. Furthermore, (4.28) transforms (4.27) into

$$B(bc, x) = B(c, x) + B(b, x),$$

i.e. B is also additive in the first variable.

Now using, in order, (4.5), (4.17), (4.7), (4.19), (4.18), (4.8), (4.24), (4.20), (4.22), (4.25), and (4.28), we obtain

$$\begin{aligned} F(a, x; b, y) &= f(a, b, x) + g(a, b, y) - h(a, b, x * y) + \Psi(a, b, x, y) = \\ &= \eta(a, b) + m(b, x * y) - m(b, y) + \mu(a, x) + \alpha(b, y) - \alpha(b, x * y) + \\ &\quad + \mu(b, x * y) - \mu(ab, x * y) + \Psi_2(x, y) = \\ &= \eta(a, b) + m(b, x * y) - m(b, y) + \mu(a, x) + \mu(b, y) + \beta(b, y) - \\ &\quad - \beta(b, x * y) - \mu(ab, x * y) + \Psi_2(x, y) = \\ &= \eta(a, b) + \mu(a, x) + \mu(b, y) - \mu(ab, x * y) + \Psi_2(x, y) + \\ &\quad + B(b, x * y) - B(b, y) = \\ &= \eta(a, b) + \mu(a, x) + \mu(b, y) - \mu(ab, x * y) + \Psi_2(x, y) + B(b, x), \end{aligned}$$

where Ψ_2 is (1.4) antisymmetric and bi-additive, and B is bi-additive.

Finally, we define $\tilde{\Psi}$ and $\tilde{\mu}$ by

$$\begin{aligned} \tilde{\Psi}(a, x; b, y) &:= \Psi_2(x, y) + \tfrac{1}{2} B(b, x) - \tfrac{1}{2} B(a, y), \\ \tilde{\mu}(a, x) &:= \mu(a, x) - \tfrac{1}{2} B(a, x). \end{aligned} \tag{4.29}$$

Then

$$F(a, x; b, y) = \eta(a, b) + \tilde{\mu}(a, x) + \tfrac{1}{2} B(a, x) + \tilde{\mu}(b, y) + \\ + \tfrac{1}{2} B(b, y) - \tilde{\mu}(ab, x * y) - \tfrac{1}{2} B(ab, x * y) + \\ + \tilde{\Psi}(a, x; b, y) + \tfrac{1}{2} B(a, y) + \tfrac{1}{2} B(b, x) = \\ = \eta(a, b) + \tilde{\mu}(a, x) + \tilde{\mu}(b, y) - \tilde{\mu}(ab, x * y) + \tilde{\Psi}(a, x; b, y),$$

by the bi-additivity of B. Moreover, $\tilde{\Psi}$ is (3.2) antisymmetric and bi-additive in the required sense, by definition (4.29) and the known properties of Ψ_2 and B.

Conversely, it is easy to check that any F given by (4.1), where η satisfies (1.2) and $\tilde{\Psi}$ is bi-additive, is a solution of (2.4). □

Proof of Theorem 3.1. By Lemma 4.1, F must have a representation of the form (4.1), where $\eta : S^2 \to R$ satisfies (1.2). Since $(S, \cdot) \in \mathbf{S}$ by hypothesis, η must have the form (cf. (1.3))

$$\eta(a, b) = \delta(a) + \delta(b) - \delta(ab) + \Psi_1(a, b)$$

for some $\delta : S \to R$ and antisymmetric, bi-additive $\Psi_1 : S^2 \to R$. Now with $\varphi : S \times T \to R$ defined by $\varphi(a, x) := \delta(a) + \tilde{\mu}(a, x)$ for all $(a, x) \in S \times T$, and $\Psi : (S \times T)^2 \to R$ defined by $\Psi(a, x; b, y) := \tilde{\Psi}(a, x; b, y) + \Psi_1(a, b)$ for all $(a, x; b, y) \in (S \times T)^2$, (4.1) becomes (3.1).

Conversely, any F of the form (3.1), where Ψ is (3.2) antisymmetric and bi-additive, satisfies (2.4). □

Remarks. These results can be extended in several directions. For instance, R may be replaced by any divisible abelian group G as the range of all functions. (See [3], Remark 7.2; also [4].)

In addition, Ng [5] has shown that the symmetric solutions of (1.2), where the operation $\cdot$ is addition, for $(a, b, c) \in D_3 := \{(p, q, r) \mid p, q, r \geqq 0;\ p + q + r \leqq 1\}$ have the form $\Delta(a, b) = \delta(a) + \delta(b) - \delta(a + b)$. This result can be extended as in [3], Corollary 5.2, to show that without symmetry, Δ has the form (1.3). Therefore, Theorems 3.1 and 3.2 and Lemma 4.1 remain true if $(S, \cdot, S^2, S^n)$ and/or $(T, *, T^2, T^n)$ are/is replaced by $([0, 1]$, addition, $D_2 := \{(p, q) \mid p, q \geqq 0;\ p + q \leqq 1\}$, $\Gamma_n := \{(p_1, p_2, \ldots, p_n) \mid p_i \geqq 0;\ i = 1, 2, \ldots, n;\ \sum_i p_i = 1\})$.

Moreover, Lemma 4.1 remains true if S and S^2 are replaced by, respectively, $S_1 \times S_2 \times \ldots \times S_n$ and $S_{12} \times S_{22} \times \ldots \times S_{n2}$, where for each i, S_i and S_{i2} are either S and S^2, for some commutative monoid $S \in \mathbf{S}$, or $[0, 1]$ and D_2; and the operation $\cdot$ on $S_1 \times S_2 \times \ldots \times S_n$ is defined componentwise.

Finally, the result of Lemma 4.1 can be extended to other (than probabilistic) "restricted" domains. For instance, one may consider equation (2.4) for $(a, b, c) \in \Omega_3$ and $(x, y, z) \in D_3$ (defined above), where Ω_n $(n = 2, 3, \ldots)$ is defined by

$$\Omega_n := \{(a_1, a_2, \ldots, a_n) \mid a_i \cap a_j = \emptyset \text{ for } i \neq j;\ a_i \in B;\ i, j = 1, 2, \ldots, n\}$$

for some ring B of sets and where $ab := a \cup b$ and $x * y := x + y$. Notice that, in our pro of of Lemma 4.1, nowhere was any property of S or S^2 used which would be violate d if a, b, c were required to be *disjoint* sets in B. Thus we have proved the following.

Lemma 4.2. The general solution of

$$F(a, x; b \cup c, y + z) + F(b, y; c, z) = F(a \cup b, x + y; c, z) + F(a, x; b, y),$$

for all $(a, b, c) \in \Omega_3$ and $(x, y, z) \in D_3$, is given by

$$F(a, x; b, y) = \eta(a, b) + \tilde{\mu}(a, x) + \tilde{\mu}(b, y) - \tilde{\mu}(a \cup b, x + y) + \tilde{\Psi}(a, x; b, y),$$

for all $(a, b) \in \Omega_2$ and $(x, y) \in D_2$, where $\tilde{\Psi}$ is antisymmetric and bi-additive (again, in the pairs (a, x), (b, y)), and η satisfies

$$\eta(a, b) + \eta(a \cup b, c) = \eta(a, b \cup c) + \eta(b, c),$$

for all $(a, b, c) \in \Omega_3$.

This result can be applied to the characterization of "inset" measures of information (see [6]) and will be so applied in a subsequent paper [7].

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REFERENCES

[1] J. Aczél, Z. Daróczy: On Measures of Information and Their Characterizations. Academic Press, New York 1975.

[2] M. J. Beckmann, U. H. Funke: Product attraction, advertising, and sales: Towards a utility model of market behavior. Z. Oper. Res. *22* (1978), 1—11.

[3] B. R. Ebanks: Branching measures of information on strings. Canad. Math. Bull. *22* (1979), 4, 433—448.

[4] B. Jessen, J. Karpf, A. Thorup: Some functional equations in groups and rings. Math. Scand. *22* (1968), 257—265.

[5] C. T. Ng: Representation for measures of information with the branching property. Inform. Contr. *25* (1974), 45—56.

[6] J. Aczél, Z. Daróczy: A mixed theory of information. I: Symmetric, recursive and measurable entropies of randomized systems of events. Rev. Fr. Automat. Inform. Rech. Opér., Inform. Téor. *12* (1978), 149—155.

[7] B. R. Ebanks: Symmetric, β-recursive inset entropies (to appear).

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