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GENERALIZATION OF DISCRIMINATION-RATE THEOREMS OF CHERNOFF AND STEIN

IGOR VAJDA

We consider a simple hypothesis and alternative about an abstract random observation parameterized by $\lambda$ from a directed set $A$. The asymptotics over $A$ is evaluated for mixed errors of the Bayes tests and second kind errors of the Neyman-Pearson tests. Similar asymptotics has been first evaluated by H. Chernoff and Ch. Stein when $A$ is the set of naturals and the observation consists of the first $\lambda$ terms of a sequence of i.i.d. r.v.'s (cf. [3], [4]). Extensions to the observation consisting of segments of random sequences, processes and fields have been studied by many authors. We show that the extension essentially depends on the existence of an asymptotic Rényi distance of the hypothesis and alternative and that this distance explicitly describes the discrimination rates attained by the Bayes and Neyman-Pearson tests. We do not use the theory of large deviations — all results are deduced from several simple properties of Rényi distances established by Liese and Vajda in [20] and from elementary inequalities established for Rényi distances by Kraft and Plachky and Vajda (cf. [15], [31]).

1. INTRODUCTION

Let $A$ be a directed set containing a dominating sequence $(\lambda_n | n = 1, 2, \ldots)$. This means that $A$ is ordered and contains $\lambda_1 \leq \lambda_2 \leq \ldots$ such that every finite subset of $A$ is dominated by at least one $\lambda_n$. We consider for every $\lambda \in A$ a measurable space $(\mathcal{X}_\lambda, \mathcal{A}_\lambda)$ and probability measures $P_\lambda, Q_\lambda, R_\lambda$ on it. We assume that on basis of a random observation $X_\lambda$ from $(\mathcal{X}_\lambda, \mathcal{A}_\lambda, R_\lambda)$, one is testing the hypothesis $H_\lambda$: $R_\lambda = P_\lambda$ against the alternative $K_\lambda$: $R_\lambda = Q_\lambda$. By a test corresponding to $\lambda \in A$ we mean $\mathcal{A}_\lambda$-measurable real valued function $\phi: \mathcal{X}_\lambda \to [0, 1]$. By $\Phi_\lambda$ we denote the set of all such functions. For each test $\phi \in \Phi_\lambda$ we define probability of error of first and second kind

$$ z(\phi) = \int \phi \, dP_\lambda, \quad \beta(\phi) = \int (1 - \phi) \, dQ_\lambda. \quad (1.1) $$

Put finally

$$ \Phi_\lambda(\alpha) = \{ \phi \in \Phi_\lambda | z(\phi) \leq \alpha \}, \quad 0 < \alpha < 1. $$

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Hereafter we consider fixed $b > 0$ and $0 < \alpha < 1$. We define optimum $\alpha$-tests $\varphi_\lambda$, $\lambda \in A$, by the condition

$$\beta(\varphi_\lambda) = \inf_{\varphi \in \Phi_\lambda} \beta(\varphi)$$

(1.2)

and Bayes $b$-tests $\psi_\lambda$, $\lambda \in A$, by the condition

$$\alpha(\psi_\lambda) + b \beta(\psi_\lambda) = \inf_{\varphi \in \Phi_\lambda} \left[ \alpha(\varphi) + b \beta(\varphi) \right].$$

(1.3)

In the terminology of Lehman [18], $\varphi_\lambda$ is a most powerful test of level at most $\alpha$. It follows from his version of the Neyman-Pearson lemma that such a test exists and that it is a likelihood ratio test of level $\alpha$ for $0 < \alpha \leq 1 - P_{\lambda,\alpha}(\mathcal{F}_\lambda)$, where $P_{\lambda,\alpha}$ is the singular part of $P_\lambda$ with respect to $Q_\lambda$. (If $1 - P_{\lambda,\alpha}(\mathcal{F}_\lambda) < \alpha < 1$ then $\varphi_\lambda$ is of level less than $\alpha$ and $\beta(\varphi_\lambda) = 0$.) Further, it is well known that $\psi_\lambda$ is a Bayes test in the sense of Wald [33] for the zero-one loss function and for the prior probabilities $1/(1 + b)$, $b/(1 + b)$. It is also easy to verify that every likelihood ratio test for $P_\lambda$, $Q_\lambda$ with the critical value $b$ is $\psi_\lambda$.

In this paper we evaluate the asymptotics of $\beta(\varphi_\lambda)$ and of

$$\alpha(\psi_\lambda) = \alpha(\psi_\lambda) + b \beta(\psi_\lambda)$$

(1.4)

for $\lambda$ growing above all bounds in $A$. This asymptotics will be given in terms of distances of probability measures $P_\lambda$, $Q_\lambda$ considered in the next section.

The following fact is frequently used in the sequel.

**Lemma 1.1.** For every real valued function $\gamma$ defined on $A$ and for every dominating sequence $(\lambda_n \mid n = 1, 2, \ldots)$, $\lim_{n \to \infty} \gamma(\lambda_n)$ exists iff $\lim_{A} \gamma(\lambda_n)$ exists, in which case the two limits are equal.

**Proof.** Clear in one direction. The opposite direction is easily proved by contradiction. □

2. DISTANCES OF HELLMINGER, KULLBACK-LEIBLER AND RÉNYI

Let $P$, $Q$ be probability measures on a measurable space $(\mathcal{X}, \mathcal{A})$ with Radon-Nikodym densities $p$, $q$ with respect to a dominating $\sigma$-finite measure $\mu$, and let us consider $\alpha \geq 0$. In accordance with [20] we define *Hellinger integrals*

$$H(a) = \int_{(p/q \geq 0)} (p/q)^a \, dQ$$

if $a > 1$ and $P \ll Q$  

(2.1)

otherwise.

We see that if $P \ll Q$ then $H(a)$ is nothing but the bilateral Laplace transform

$$H(a) = \int_{-\infty}^{\infty} \exp \{ay\} \, dF(y)$$

(2.2)

of the distribution function

$$F(y) = Q(\{pq > 0, \ln (p/q) < y\}) = \begin{cases} \frac{1}{y} & \text{if } y \geq 1, \\ 0 & \text{otherwise} \end{cases}$$

(2.3)

which is complete (i.e. $F(\infty) = 1$) if also $Q \ll P$. This fact has interesting analytical
consequences. These are not however considered in this paper which is based on inequalities and convexity rather than on derivatives and analyticity.

**Lemma 2.1.** It holds

\[ H(0) = 1 - Q_s(x), \quad H(1) = 1 - P_s(x) \]

where \( P_s, Q_s \) are singular parts of \( P, Q \) with respect to \( Q, P \). Further, \( H(a) \) is logarithmically convex, i.e. the function

\[ \varrho(a) = \ln H(a) \]

is convex. Finally, \( \varrho(a) \) is continuous on \([0, 1]\) and

\[ \lim_{a \uparrow 1} \frac{\varrho(a)}{a - 1} = I(P \parallel Q) \]

where \( I(P \parallel Q) \) is the \( I \)-divergence defined by

\[ I(P \parallel Q) = \begin{cases} \int_{\{p > 0\}} \ln \frac{p}{q} \, dP & \text{if } P \not\ll Q \\ \infty & \text{otherwise} \end{cases} \]

**Proof.** The first assertion is clear from (2.1). The second assertion is clear from (2.1) if \( P \perp Q \) (singularity). For nonsingular \( P, Q \) it follows from H"older's inequality

\[ H(\alpha a_1 + (1 - \alpha) a_2) \leq H(\alpha a_1) H^{1 - \alpha}(a_2) \]

for all \( 0 < \alpha < 1 \) and all \( a_1, a_2 \geq 0 \) so that the second assertion is clear from (2.4). The last assertion follows from (2.14) in [20].

The relation (2.5) motivates our definition of the Rényi distance (cf. [2], [30]):

\[ R(a) = \frac{\varrho(a)}{I(P \parallel Q)} \]

if \( a \neq 1 \)

\[ \frac{a - 1}{I(P \parallel Q)} \]

if \( a = 1 \).

**Lemma 2.2.** The Rényi distance takes on values from \([0, \infty]\) and is nondecreasing. It is null at some point from \((0, 1)\) iff \( P = Q \) in which case it is null everywhere. It is infinite at some point from \((0, 1)\) iff \( P \perp Q \) in which case it is infinite everywhere. It is continuous on \([0, a_0]\) where \( a_0 = \sup \{ a > 0 \mid R(a) < \infty \} \) and where it holds \( a_0 \geq 1 \) unless \( P \perp Q \).

**Proof.** All stated properties follow directly from (2.11), (2.14) and (2.15) in [20].

Let us consider nonsingular \( P, Q \) and \( 0 < a < 1 \). It holds \( H(a) > 0 \) and we define probability measures \( P_a \ll \mu \) by the Radon-Nikodym derivatives

\[ P_a = \frac{p^a q^{1-a}}{H(a)} \]

It is clear that \( P_a \ll Q \). It follows from here and from (2.1) that for every \( 0 < t < 1/a \)
the Hellinger integral
\[ H(t \mid P_a, Q) = \int_{\{p_{a} > 0\}} \left( \frac{p_{a}}{q} \right)^{t} dQ, \tag{2.9} \]
which coincides with (2.1) for a, P replaced by t, P_a, satisfies the relation

\[ H(t \mid P_a, Q) = \frac{H(at)}{H'(a)}. \tag{2.10} \]

This relation together with the next two auxiliary results are needed in Section 6 only.

**Lemma 2.3.** Let \( \varphi \) be a likelihood ratio test of \( H: P \) against \( K: Q \) with the critical value 1 and \( \varphi_a \) a likelihood ratio test of \( H_a: P_a \) against \( K: Q \) with the critical value \( 1/H(a) \) for some \( 0 < a < 1 \). Then the two tests coincide \( Q \) - a.s.

**Proof.** Clear from (2.8).

The following assertion holds in particular for \( \xi(a) = \ln H(a) \), or for \( \xi(a) = H(a) \) which are convex on \((0, 1)\).

**Lemma 2.4.** Let \( \xi: (0, 1) \to \mathbb{R} \) be convex. Then there exist right- and left-hand derivatives \( \xi'_+(a), \xi'_-(a) \), \( 0 < a < 1 \), and they are continuous from the right and from the left, respectively.

**Proof.** Let us consider \( 0 < a < s < t \). The existence of
\[ \xi'_+(a) = \lim_{s \downarrow a} \frac{\xi(s) - \xi(a)}{s - a} \]
is the well known consequence of Jensen inequality. Denote by \( \xi^{+}_+(a) \) the limit of \( \xi^{+}_+(s) \) for \( s \downarrow a \). The well known monotonicity of the right-hand derivative implies
\[ \xi'_+(a) \leq \xi^{+}_+(s) \leq \frac{\xi(t) - \xi(s)}{t - s}. \]
For \( s \downarrow a \) we obtain
\[ \xi'_+(a) \leq \xi^{+}_+(a) \leq \frac{\xi(t) - \xi(a)}{t - a}. \]
Sending \( t \downarrow a \) we obtain from here the desired identity \( \xi^{+}_+(a) = \xi'_+(a) \).

3. EXAMPLES

In this section we present several specifications of testing models of Section 1, and evaluate the distances of Section 2 for probability distributions \( P_x, Q_x \) considered in these specifications. In the rest of this paper we write
\[ H_x(a) = H(a \mid P_x, Q_x), \varphi_x(a) = \varphi(a \mid P_x, Q_x), R_x(a) = R(a \mid P_x, Q_x), \tag{3.1} \]
i.e. the distances \( H(a), g(a), R(a) \) of measures \( P = P_\lambda, Q = Q_\lambda \) are denoted by \( H_\lambda(a), g_\lambda(a), R_\lambda(a) \) for \( \lambda \in \Lambda \).

The most important class of asymptotic testing models of Section 1 is represented by martingale models. In these models there is given a measurable space \((\mathcal{X}, \mathcal{A})\) and a generalized increasing sequence of sub-\(\sigma\)-algebras \(\mathcal{A}_\lambda \subset \mathcal{A}_\mu\) for \(\lambda \leq \mu\) and the union of all \(\mathcal{A}_\lambda\) generates \(\mathcal{A}\) (in symbols this is denoted by \(\mathcal{A}_\lambda \uparrow \mathcal{A}\)). Further, there are given two probability measures \(P, Q\) on \((\mathcal{X}, \mathcal{A})\) with densities \(p, q\) with respect to a dominating probability measure \(\mu\). If we denote by \(P_\lambda, Q_\lambda, \mu_\lambda\) restriction of \(P, Q, \mu\) on \(\mathcal{A}_\lambda\) then the quadruples \((\mathcal{X}_\lambda, \mathcal{A}_\lambda, P_\lambda, Q_\lambda)\) of Section 1 are assumed to be defined by

\[
(\mathcal{X}_\lambda, \mathcal{A}_\lambda, P_\lambda, Q_\lambda) = (\mathcal{X}, \mathcal{A}_\lambda, \mu_\lambda).
\]

(3.2)

Since \(P_\lambda, Q_\lambda\) are dominated by \(\mu_\lambda\), we may consider the corresponding Radon-Nikodym derivatives \(p_\lambda, q_\lambda\). These derivatives are satisfying the conditional-expectation relations

\[
p_\lambda = E_\mu(p \mid \mathcal{A}_\lambda), \quad q_\lambda = E_\mu(q \mid \mathcal{A}_\lambda),
\]

(3.3)
i.e. they are martingales for \((\mathcal{A}_\lambda \mid \lambda \in \Lambda)\). This fact considerably simplifies evaluation of distances \(H_\lambda(a), g_\lambda(a), R_\lambda(a)\) as well as of their limits over \(\Lambda\).

The Hellinger integrals have been evaluated and applied in the martingale models of sequences and processes by many authors, e.g. by Kakutani [12], Chernoff [3], [4], Koopmans [14], Newman [23], [24], Perez [28], Newman and Stuck [25], Oosterhoff and van Zwet [26], Méméin and Shiryaev [21], Liese [19] and Kolomiets [13]. The distance \(R_\lambda(1)\) has been studied in the martingale models of sequences and processes e.g. by Perez [27], Hájek [8, 9], Pinsker [29], Kullback et al. [16] and in the martingale models of random fields e.g. by Künsch [17]. The distances \(R_\lambda(a)\) for martingale models of sequences and processes have been studied by Liese and Vajda [20] and Vajda [32], and for martingale models of random fields by Janžura [11].

**Example 3.1.** Let us consider the martingale model of stochastic processes on the time domain \([0, \infty)\), i.e. let \(\Lambda = [0, \infty)\) with the natural ordering and let us consider probability measures \(P, Q\) induced by two random processes \(X = (X(t) \mid 0 \leq t < \infty), Y = (Y(t) \mid 0 \leq t < \infty)\) on the Kolmogorov Borel-lines product \((\mathcal{X}, \mathcal{A})\). The model of Section 1 is completely specified for every \(0 \leq \lambda < \infty\) by probability measures \(P_\lambda, Q_\lambda\) induced by processes \(X_\lambda = (X(t) \mid 0 \leq t \leq \lambda), Y_\lambda = (Y(t) \mid 0 \leq t \leq \lambda)\) on the sub-\(\sigma\)-algebra \(\mathcal{A}_\lambda \subset \mathcal{A}\) corresponding to the time-subdomain \([0, \lambda] \subset [0, \infty)\).

Let us assume that it holds \(X = m_1 + Z, Y = m_2 + Z\), where \(m_1, m_2 : [0, \infty) \to \mathbb{R}\) are continuous such that \(m = m_1 - m_2\) satisfies the relations \(m(0) = 0\) and

\[
m(t) - m(s) = \int_s^t m_*(y) \, dy, \quad 0 \leq s < t < \infty,
\]

for \(m_*\) measurable and bounded on each bounded subdomain, and where \(Z = (Z(t) \mid 0 \leq t < \infty)\) is a diffusion process satisfying the stochastic differential
equation
\[ dZ(t) = -A(t) Z(t) \, dt + B(t) \, dW(t) \]
with the initial condition \( Z(0) = N(\mu, \sigma^2_0) \) for \( \mu \in \mathbb{R}, \sigma^2_0 > 0 \). In the equation it is assumed that \( A: [0, \infty) \to \mathbb{R}, B: [0, \infty) \to (0, \infty) \) are continuous and that \( W = W(t) \mid 0 \leq t < \infty \) is the standard Wiener process. We shall show that it holds for every \( 0 \leq \lambda < \infty \)
\[ R_\lambda(a) = \frac{a}{2} \int_0^\lambda \eta^2(t) \, dt. \] (3.4)
where
\[ \eta(t) = -\frac{A(t) m(t) + m_*(t)}{B(t)}, \quad 0 \leq t < \infty. \] (3.5)

By Theorem 3 on p. 90 of [7], it holds \( P_\lambda \ll Q_\lambda \) and the Radon-Nikodym derivative \( \frac{dP_\lambda}{dQ_\lambda} \) is given by
\[ \frac{dP_\lambda}{dQ_\lambda} = \exp \left\{ \int_0^\lambda \eta(t) \, dW(t) - \frac{1}{2} \int_0^\lambda \eta^2(t) \, dt \right\}. \]

Thus it follows from (2.1)
\[ H_\lambda(a) = \int \left( \frac{dP_\lambda}{dQ_\lambda} \right)^a \, dQ_\lambda = \exp \left\{ -\frac{a}{2} \int_0^\lambda \eta^2(t) \, dt \right\} \exp \left\{ a \int_0^\lambda \eta(t) \, dW(t) \right\}. \]
The exponent behind the expectation has the normal density \( \phi(y/\sigma)/\sigma \) for \n\[ \sigma^2 = a^2 \int_0^\lambda \eta^2(t) \, dt. \]
Thus the expectation is equal to
\[ \int_{-\infty}^\infty \exp \left\{ \frac{y}{\sigma} \phi \left( \frac{y}{\sigma} \right) \right\} \, dy = \exp \left\{ \frac{a^2}{2} \int_0^\lambda \eta^2(t) \, dt \right\}. \]
Therefore
\[ H_\lambda(a) = \exp \left\{ \frac{a(a-1)}{2} \int_0^\lambda \eta^2(t) \, dt \right\}, \]
and the rest follows from (2.4), (2.7) and from the continuity of \( R_\lambda(a) \) from the left at 1 (see Lemma 2.2).

**Example 3.2.** In order to illustrate the relevance of results that follow in testing of hypotheses about random fields let us consider \( A = \{1, 2, \ldots\} \times \{1, 2, \ldots\} \) ordered by the condition
\[ \lambda_1 = (\lambda'_1, \lambda''_1) \leq \lambda_2 = (\lambda'_2, \lambda''_2) \iff \lambda'_1 \leq \lambda'_2 \quad \text{and} \quad \lambda''_1 \leq \lambda''_2. \]
Obviously, \( (\lambda_n = (\lambda'_n, \lambda''_n) \mid n = 1, 2, \ldots) \) is dominating in \( A \) iff
\[ \lim_{n \to \infty} \min \{\lambda'_n, \lambda''_n\} = \infty. \]
For example, \( (\lambda_n = (n, n) \mid n = 1, 2, \ldots) \) is dominating in \( A \).
Let us consider a martingale model of binary random field. In this case $\mathcal{X}$ is the set of all binary arrays $x = (\xi_{ij} | (i, j) \in A)$. For every $\lambda = (\lambda', \lambda'') \in A$ we consider the algebra $\mathcal{A}_\lambda$ of subsets of $\mathcal{X}$ generated by the sub-arrays $x_\lambda = (\xi_{ij} | i = 1, \ldots, \lambda', j = 1, \ldots, \lambda'')$. It suffices to define probability distributions $P_\lambda, Q_\lambda$ on the sets $\mathcal{X}_\lambda$ of all arrays $x_\lambda$ consistent in the usual Kolmogorov sense for different $\lambda_1, \lambda_2 \in A$, $\lambda_1 \leq \lambda_2$. To this end decompose $\mathcal{X}_\lambda$ into the singleton $\mathcal{X}_\lambda^0$ containing the zero array $x_\lambda$ consisting of pure 0's, the set $\mathcal{X}_\lambda^1$ containing all arrays $x_\lambda$ with $\xi_{11} = 1$, and the set $\mathcal{X}_\lambda^2 = \mathcal{X}_\lambda - (\mathcal{X}_\lambda^0 \cup \mathcal{X}_\lambda^1)$. Put
\[ P_\lambda(\mathcal{X}_\lambda^0) = 1 - \alpha \quad (\text{cf. Sec. 1}), \quad P_\lambda(\mathcal{X}_\lambda^2) = 0, \]
and assume that $P_\lambda$ is uniform on $\mathcal{X}_\lambda^1$ and $Q_\lambda$ is uniform on the whole $\mathcal{X}_\lambda$. The second assumption implies that the $Q_\lambda$-probability of all arrays $x_\lambda$ is
\[ \beta_\lambda = \frac{1}{2^{\lambda' + \lambda''}} \quad \text{for} \quad \lambda = (\lambda', \lambda''). \]
It is easy to see that $Q_\lambda(\mathcal{X}_\lambda^1) = \frac{1}{2}$. By (2.1)
\[ H_\lambda(a) = \sum_{i=0}^{2} \left( \frac{P_\lambda(\mathcal{X}_\lambda^i)}{Q_\lambda(\mathcal{X}_\lambda^i)} \right)^a Q_\lambda(\mathcal{X}_\lambda^i) = \left( \frac{1 - \alpha}{\beta_\lambda} \right)^a \frac{\alpha}{\beta_\lambda} + (2\alpha)^a \frac{1}{2} + 0 \]
and by (2.6) analogically
\[ R_\lambda(1) = (1 - \alpha) \ln \frac{1 - \alpha}{\beta_\lambda} + \alpha \ln 2\alpha. \]
Hence, by (2.4) and (2.7)
\[ \lim_{a} R_\lambda(a) = \begin{cases} \ln 2 + \frac{a}{1 - a} \ln \frac{1}{\alpha} & \text{if} \quad 0 \leq a < 1 \\ \infty & \text{if} \quad a \geq 1 \end{cases} \quad (3.6) \]
where the asymptotics in case $a \geq 1$ may be expressed as follows
\[ R_\lambda(a) \approx \begin{cases} (1 - \alpha) \ln \frac{1}{\beta_\lambda} & \text{if} \quad a = 1 \\ \ln \frac{1}{\beta_\lambda} & \text{if} \quad a > 1. \end{cases} \quad (3.7) \]

4. $R$-DISTANCE

In this section we define an asymptotic distance for the hypothesis and alternative
\[ H = (H_\lambda | \lambda \in A) \quad \text{and} \quad K = (K_\lambda | \lambda \in A) \]
considered in Section 1. This distance is based on the Rényi distance studied in Sections 2 and 3.

Let us consider $a_0 > 0$ and a neighborhood $A \subset [0, \infty)$ of $a_0$. We shall say that the asymptotic distance of $H$ and $K$ in the neighborhood $A$ of $a_0$ exists if $R_\lambda(a_0)$ is positive and unboundedly increasing on $A$ and if for every $a \in A$ there exists in the
topological space \([0, \infty]\) the limit
\[
\bar{R}(a) = \lim_{\lambda \to \infty} \frac{R_{\lambda}(a)}{R_{\lambda}(a_0)}.
\] 
(4.1)

The function \(\bar{R}(a), a \in A\), is then considered to be the asymptotic distance — it is called briefly \(\bar{R}\)-distance.

If there exists the \(\bar{R}\)-distance in the neighborhood \((0, 1)\) of \(a_0 = \frac{1}{2}\) then we define
\[
\bar{g}(a) = (a - 1) \bar{R}(a), \quad 0 < a < 1.
\] 
(4.2)

It follows from the relations \((a - 1) R_{\lambda}(a) = \varrho_{\lambda}(a), R_{\lambda}(\frac{1}{2}) = -2 \varrho_{\lambda}(\frac{1}{2})\) that the \(\bar{R}\)-distance in the neighborhood \((0, 1)\) of \(\frac{1}{2}\) exists iff \(-\varrho_{\lambda}(\frac{1}{2})\) is positive and unboundedly increasing on \(\lambda\) and there exist in the topolgical space \([0, \infty]\) the limits
\[
\lim_{\lambda \to \infty} \frac{-\varrho_{\lambda}(\frac{1}{2})}{2 \varrho_{\lambda}(\frac{1}{2})}, \quad 0 < a < 1.
\] 
(4.3)

In the positive case the limits (4.1) and (4.3) are related by (4.2).

**Lemma 4.1.** The function defined by (4.3) is finite, negative, continuous and convex on its domain of definition \((0, 1)\).

**Proof.** By Lemma 2.1, the function behind the limit in (4.3) is nonpositive, continuous and convex on \((0,1)\). Consequently \(\bar{g}(a)\) is nonpositive and convex on \((0,1)\). Since it is finite at \(\frac{1}{2}\), it is finite and continuous on \((0,1)\). The negativity of \(\bar{g}(a)\) follows from the fact that the nonnegative convex function behind the limit in (4.3) cannot exceed the piecewise linear function \(f(a)\) the graph of which is passing through the points \((0,0), (\frac{1}{2}, -2), (1,0)\) of the \((a,f)\)-plane.

**Example 4.1.** Let us consider the diffusion processes of Example 3.1 and let the integral figuring in (3.4) satisfy the condition
\[
\lim_{\lambda \to \infty} \int_0^\lambda \eta^2(t) \, dt = \infty.
\]

Then the \(\bar{R}\)-distance exists in the neighborhood \(A = [0, \infty)\) of an arbitrary point \(0 < a_0 < \infty\) and is linear on \(A\),
\[
\bar{R}(a) = a/a_0.
\] 
(4.4)

The function (4.3) takes on the form
\[
\bar{g}(a) = -2a(1 - a), \quad 0 < a < 1.
\] 
(4.5)

**Example 4.2.** Let us consider the random fields of Example 3.2. It follows from (3.6), (3.7) that the \(\bar{R}\)-distance exists in no nontrivial neighborhood of \(0 < a_0 < 1\).

The \(\bar{R}\)-distance in the neighborhood \(A = [0, \infty)\) of \(a_0 = 1\) is given by
\[
\bar{R}(a) = \begin{cases} 
0 & \text{if } 0 \leq a < 1 \\
1 & \text{if } a = 1 \\
\frac{1}{1 - a} & \text{if } a > 1.
\end{cases}
\] 
(4.6)
5. GENERALIZED THEOREM OF STEIN

Let us consider the testing model of Section 1, in particular optimum \( \alpha \)-tests \( \varphi_\lambda \) and corresponding second kind error probabilities \( \beta(\varphi_\lambda), \lambda \in \Lambda \). Results of this and of the following section are to the great extent based on the following fact proved as Theorem 1 in [15]: For every \( 0 < \alpha < 1 < \tau \) it holds

\[
(1 - \alpha)^{\tau/(\tau - 1)} \exp \left\{ -R_\lambda(\tau) \right\} \leq \beta(\varphi_\lambda) \leq (1 - \alpha) (\alpha/\tau)^{(1 - \alpha)} \exp \left\{ -R_\lambda(\alpha) \right\}. \tag{5.1}
\]

**Theorem 5.1.** Let the \( \bar{R} \)-distance exist in an open neighborhood \( A \subset [0, \infty) \) of \( a_0 = 1 \) and let \( \bar{R}(a) \) be continuous at \( a = 1 \). Then

\[
\lim_{a} \left( \beta(\varphi_\lambda) \right)^{1/R_\lambda(1)} = \exp \left\{ -1 \right\}. \tag{5.2}
\]

**Proof.** Let \( (\lambda_n \mid n = 1, \ldots) \) be a dominating sequence in \( A \). It follows from the right-hand inequality in (5.1) and from (4.1) that it holds for every \( a, \tau \in A, a < 1 < \tau \), (with \( \varphi_{\lambda_n}, R_{\lambda_n} \) written instead of \( \varphi_\lambda, R_\lambda \))

\[
\limsup_{n \to \infty} \left( \beta(\varphi_{\lambda_n}) \right)^{1/R_{\lambda_n}(1)} \leq \exp \left\{ -\bar{R}(a) \right\},
\]

\[
\limsup_{n \to \infty} \left( \beta(\varphi_{\lambda_n}) \right)^{1/R_{\lambda_n}(1)} \geq \exp \left\{ -\bar{R}(\tau) \right\}.
\]

Hence, by the continuity of \( \bar{R}(a) \) at \( a = 1 \) and by the identity \( \bar{R}(1) = 1 \),

\[
\limsup_{n \to \infty} \left( \beta(\varphi_{\lambda_n}) \right)^{1/R_{\lambda_n}(1)} \geq \exp \left\{ -1 \right\}.
\]

The rest is clear from Lemma 1.1. \( \square \)

**Example 5.1.** For diffusion processes of Example 4.1 the assumptions of Theorem 5.1 hold for \( A = [0, \infty) \) so that in this case (5.2) holds for \( R_\lambda(1) \) equal to half of the integral figuring in (3.4).

**Example 5.2.** We shall show that (5.2) need not be true without the continuity of \( \bar{R}(a) \) at \( a = 1 \). This is an argument justifying also the assumption of existence of the \( \bar{R} \)-distance in Theorem 5.1.

Let us consider the random fields of Example 3.2. It follows from the definition of \( P_\lambda, Q_\lambda \) that if \( 0 < z < \frac{1}{2} \) then the optimum \( \alpha \)-test \( \varphi_\lambda \) is nonrandomized, equal for every \( \lambda = (\lambda', \lambda'') \in \Lambda \) to the characteristic function of \( \mathcal{X}_\lambda - \mathcal{X}_\lambda^0 \). It is clear that \( \varphi_\lambda = P_\lambda(\mathcal{X}_\lambda - \mathcal{X}_\lambda^0) = 1 - (1 - z) = z \). It is also clear that \( \beta(\varphi_\lambda) = Q_\lambda(\mathcal{X}_\lambda^0) = \beta_\lambda \). Therefore it follows from (3.7) that in this case

\[
\lim_{a} \left( \beta(\varphi_\lambda) \right)^{1/R_\lambda(1)} = \exp \left\{ -1/(1 - z) \right\}
\]

so that (5.2) is not true. By (4.6), \( \bar{R}(a) \) in this case exists in the neighborhood \([0, \infty)\) of 1, but it is discontinuous at 1.

A less general version of Theorem 5.1 appeared in [13] (cf. also the inner references
to further more special versions). The proof presented here is essentially simpler. Another less general version appeared also as Theorem 11.19 in [32].

Now we present a variant of Theorem 5.1 with a weaker continuity assumption.

**Theorem 5.2.** Let us consider $0 < \alpha_h < 1$ decreasing on $A$ to zero and let $\varphi_h$ be the optimum $\alpha_h$-test. If

$$\lim_{A} (\alpha_h)^{1/R_h(1)} = 1$$

and if there is $0 < s < 1$ such that the $R$-distance exists in the neighborhood $(s, 1]$ of 1 and is continuous from left at 1, then

$$\lim_{A} (\beta(\varphi_h))^{1/R_h(1)} = \exp \{-1\}.$$  

**Proof.** Analogically as in the proof of Theorem 5.1 we prove that for every $s < a < 1$

$$\limsup_{n \to \infty} (\beta(\varphi_n))^{1/R_n(1)} \leq \exp \{-R(a)\}$$

so that, by continuity,

$$\limsup_{n \to \infty} (\beta(\varphi_n))^{1/R_n(1)} \leq \exp \{-1\}.$$  

To prove the converse for $\liminf$ take first into account that the family of probabilities \(\{(\varphi_h(x), 1 - \varphi_h(x)) \mid x \in \mathcal{A}\}\) defines a Markov kernel from \((\mathcal{A}, \mathcal{A})\) to the binary set \(\{0, 1\}\). The images of measures $P_h$, $Q_h$ defined by this kernel on \(\{0, 1\}\) are $(\alpha(\varphi_h), 1 - \alpha(\varphi_h)), (1 - \beta(\varphi_h), \beta(\varphi_h))$. Therefore Theorem (1.24) of [20] yields the inequality

$$R_h(1) \geq \alpha(\varphi_h) \ln \frac{\alpha(\varphi_h)}{1 - \beta(\varphi_h)} + (1 - \alpha(\varphi_h)) \ln \frac{1 - \alpha(\varphi_h)}{\beta(\varphi_h)}$$

where $0 \ln (0/\beta) = 0$ for $\beta \geq 0$ and $\alpha \ln (\alpha/\beta) = \infty$ for $\alpha > 0$. Since $R_h(1)$ in (4.1) is assumed to be finite, it holds $P_h \ll Q_h$. Thus the singular part $P_h, \mathcal{A}$ considered in Section 1 below (1.3) is zero and, consequently, $\alpha(\varphi_h) = \alpha_h$. Since $0 < \alpha_h < 1$, it follows from the above inequality $0 < \beta(\varphi_h) < 1$. Therefore the above inequality yields

$$R_h(1) \geq \alpha_h \ln \alpha_h + (1 - \alpha_h) \ln (1 - \alpha_h) - (1 - \alpha_h) \ln \beta(\varphi_h).$$

The desired inequality

$$\liminf_{n \to \infty} (\beta(\varphi_n))^{1/R_n(1)} \geq \exp \{-1\}$$

follows obviously from here. \qed

**Example 5.3.** It follows from the formula for $R_h(1)$ in Example 3.2 that if $\alpha = \alpha_h$ is satisfying (5.3) then (5.4) holds in spite of that the $R$-distance (4.6) is discontinuous at $a = 1$.

To illustrate the applicability of Theorem 5.2 let us take the Borel line for $((\mathcal{A}, \mathcal{A}), \mathcal{A})$,
\( \lambda \in \Lambda \) arbitrary, and consider on \( \mathbb{R} \) the densities

\[
P_\lambda(x) = \frac{1}{2} \exp\left\{-|x|\right\}, \quad q_\lambda(x) = \frac{1}{\sigma(\lambda)} \varphi\left(\frac{x}{\sigma(\lambda)}\right),
\]

where \( \varphi \) is the standard normal density and \( \sigma: \Lambda \to (0, \infty) \) is increasing to \( \infty \). After some calculations one obtains that the \( R \)-distance in the neighborhood \( [0, \infty) \) of arbitrary \( 0 < a_0 \leq 1 \) is given by

\[
R(a) = \begin{cases} 
0 & \text{if } a = 0 \\
-1 & \text{if } 0 < a \leq 1 \\
\infty & \text{if } a > 1.
\end{cases}
\]

(Note that a martingale model with independent observations satisfying (5.5) is easily implanted here.) We see that Theorem 5.1 is not able to say anything about the asymptotics of \( \beta(\varphi_\lambda) \). Theorem 5.2. says that if (5.3) holds then also (5.4) holds with

\[
R_\lambda(1) = \frac{\ln \pi \sigma^2(\lambda)}{2} + \frac{1}{\sigma^2(\lambda)} - 1,
\]

i.e. in fact with \( R_\lambda(1) = \ln \sigma(\lambda) \). The formula (5.6) is obtained by a routine integration after substitution to (2.6).

Let us note that the case where the \( R \)-distance considered in Theorem 5.2 exists and is null for \( 0 \leq a < 1 \) (and consequently discontinuous at 1 from left) has been studied by Janssen [10]. He restricted the attention to special models with sequences of i.i.d. observations.

Let us also note that if \( \Lambda \) is the set of natural numbers, i.e. if we restrict ourselves to the statistical observation of segments of random sequences, then one easily obtains from the Shannon-McMillan-Breiman theorem (cf. [1]) that, in the model of Theorem 5.1,

\[
\lim_{A} \beta(\varphi_\lambda)^{1/R_\lambda(1)} \leq \exp \{ -1 \}
\]

provided the limit of \( R_\lambda(1)/\lambda \) is non-zero and finite (cf. [28]). The assumption here is weaker than in Theorem 5.1, but the assertion is weaker too.

6. GENERALIZED THEOREM OF CHERNOFF

In this section we consider the testing model of Section 1, in particular a Bayes \( b \)-test \( \psi_\lambda \) and its mixed error \( \sigma(\psi_\lambda) \). It is easy to see that, up to a multiplicative factor not depending on \( \lambda \), \( \sigma(\psi_\lambda) \) is the Bayes risk (in the sense of Wald [33]) in a Bayes testing model with the hypothesis \( H_\lambda \) and alternative \( K_\lambda \).

In the following theorem we consider the negative function \( \tilde{\varrho}(a) \) defined on \( (0, 1) \) by (4.2) under the assumption that the \( R \)-distance exists in the neighborhood \( (0, 1) \) of \( \frac{1}{2} \). We also assume that the subset argmin \( \tilde{\varrho}(a) \subset (0, 1) \) is nonempty (Examples 4.1
or 5.3 presents the situations where this assumption is or is not satisfied respectively). By Lemma 4.1, \( \tilde{g}(a) \) is continuous and convex. It follows from here that argmin \( \tilde{g}(a) \) is an interval. The closure of this interval we denote by \([a_-, a_+]\) where \( 0 \leq a_+ \leq a_+ \leq 1 \). It also follows from here that the right- and left-hand derivatives \( \tilde{g}'_+(a) \) and \( \tilde{g}'_-(a) \) exist on \((0, 1)\).

**Theorem 6.1.** Let the \( \tilde{R} \)-distance exists in the neighborhood \((0, 1)\) of \( \frac{1}{2} \) and let either \( a_+ < 1 \) and \( \tilde{g}'_+(a_+) = 0 \) or \( a_- > 0 \) and \( \tilde{g}'_-(a_-) = 0 \). Then for any \( a_0 \in \text{argmin} \tilde{g}(a) \)

\[
\lim_{a} \left( \varepsilon(\psi_a) \right)^{1/R(a^{1/2})} = \exp \left\{ \tilde{g}(a_0) \right\}.
\]

**Proof.** Let us denote by \( \varepsilon_1(\psi_a) \) the value of \( \varepsilon(\psi_a) \) for \( b = 1 \) (cf. (1.4)). By Theorem 2 in [31]

\[
\frac{2 \min \{1, b\}}{1 + b} \varepsilon_1(\psi_a) \leq \varepsilon(\psi_a) \leq \frac{2 \max \{1, b\}}{1 + b} \varepsilon_1(\psi_a).
\]

Thus it suffices to prove (6.1) for \( b = 1 \). Further, by Theorem 1 ibid., the mixed error \( a(\phi) + b \beta(\phi) \) of every Bayes \( b \)-test of \( H: P \) against \( K: Q \) is bounded above for every \( a \in (0, 1) \) by \( b^{1-a} H(a) \) where \( H(a) \) is given by (2.1). Therefore

\[
\varepsilon(\psi_a) \leq b^{1-a} \exp \left\{ (a - 1) R_x(a) \right\}, \quad a \in (0, 1).
\]

Let \( (\lambda_n) \mid n = 1, 2, \ldots \) be a dominating sequence in \( \Lambda \) and let us write in the sequel \( \psi_n, R_n(a) \) instead of \( \psi_{\lambda_n}, R_{\lambda_n}(a) \). The last inequality together with (4.1) (where \( a_0 \) is replaced by \( \frac{1}{2} \)) and (4.2) imply

\[
\limsup_{n \to \infty} \left( \varepsilon(\psi_n) \right)^{1/R_n(a^{1/2})} \leq \exp \left\{ \tilde{g}(a) \right\}, \quad a \in (0, 1).
\]

In particular,

\[
\limsup_{n \to \infty} \left( \varepsilon(\psi_n) \right)^{1/R_n(a^{1/2})} \leq \exp \left\{ \tilde{g}(a_0) \right\}
\]

for \( a_0 \in \text{argmin} \tilde{g}(a) \). Thus in view of Lemma 1.1 it suffices to prove for \( b = 1 \) that there exists \( a_0 \in \text{argmin} \tilde{g}(a) \) such that

\[
\liminf_{n \to \infty} \left( \varepsilon(\psi_n) \right)^{1/R_n(a^{1/2})} \geq \exp \left\{ \tilde{g}(a_0) \right\}.
\]

Let us consider \( b = 1 \) and let \( a_+ < 1 \) and \( \tilde{g}'_+(a_+) = 0 \). In this case \( a_+ \in \text{argmin} \tilde{g}(a) \). We shall prove that (6.3) holds for \( a_0 = a_+ \) (if \( a_- > 0 \) and \( \tilde{g}'_-(a_-) = 0 \) then one can prove analogically that (6.3) holds for \( a_0 = a_- \)). Let us consider \( a \in (a_+, 1), \ t \in (0, 1), \tau \in (1, 1/a) \), and \( \lambda \in \Lambda \). The existence of \( \tilde{R} \)-distance implies that \( R_\lambda(\frac{1}{2}) \) is finite, i.e. that \( P_\lambda, Q_\lambda \) are nonsingular. Therefore \( H_\lambda(a) > 0 \) and there exists the measure \( P_{\lambda,a} \) defined by (2.8) for \( (P, Q) = (P_\lambda, Q_\lambda) \). Let \( \varphi_a \) be a likelihood ratio test of \( H_\lambda(a) : P_{\lambda,a} \) against \( K_\lambda : Q_\lambda \), with the critical value \( 1/H_\lambda(a) \) and let

\[
\alpha(\varphi_a) = \int \varphi_a \ dP_{\lambda,a}, \quad \beta(\varphi_a) = \int (1 - \varphi_a) \ dQ_\lambda,
\]

be its errors. In view of what has been said in Section 1, \( \varphi_a \) is a Bayes \( 1/H_\lambda(a) \)-test.
Hence, using the inequality which (6.2) is based on, we obtain
\[ x(\varphi_\alpha) + (1/H_\lambda(a)) \beta(\varphi_\alpha) \leq (1/H_\lambda(a))^{1-\tau} H(t \mid P_{\lambda,\alpha}, Q_\lambda) \quad (\text{cf. (2.9)}) \]
\[ = H_\lambda(at)/H_\lambda(a) \quad (\text{cf. (2.10)}). \]

It follows from here
\[ x(\varphi_\alpha) \leq \frac{H_\lambda(at)}{H_\lambda(a)}. \tag{6.4} \]

Further, we know from Section 1 that \( \psi_\lambda \) is a likelihood ratio test of \( H_\lambda : P_\lambda \) against \( K_\lambda : Q_\lambda \) with the critical value \( b = 1 \). Therefore it follows from Lemma 2.3 that \( \psi_\lambda \) and \( \varphi_\alpha \) coincide \( Q_\lambda \) – a.s. so that
\[ \beta(\psi_\lambda) = \beta(\varphi_\alpha). \]

Let us consider at last an optimum \( x(\varphi_\alpha) \)-level test of \( H_\lambda : P_{\lambda,\alpha} \) against \( K_\lambda : Q_\lambda \).

We denote this test by \( \varphi_\alpha^* \). By definition in Section 1,
\[ \beta(\varphi_\alpha^*) = \int (1 - \varphi_\alpha^*) \, dQ_\lambda \leq \beta(\varphi_\alpha). \]

Hence it follows from the left-hand part of (5.1) and from (6.4)
\[ \beta(\psi_\lambda) \geq \beta(\varphi_\alpha^*) \geq \left( \frac{(1 - a(\varphi_\alpha)^*)}{H(\tau \mid P_{\lambda,\alpha}, Q_\lambda)} \right)^{1/(\tau - 1)} \quad (\text{cf. (2.9)}) \]
\[ \geq \left( 1 - \frac{H_\lambda(at)}{H_\lambda(a)} \right)^{\tau/(\tau - 1)} \left( \frac{H_\lambda^*(a)}{H_\lambda(at)} \right)^{1/(\tau - 1)} \quad (\text{cf. (2.10)}). \tag{6.5} \]

Putting \( t = a_+^*/a \) and taking into account that \( x(\psi_\lambda) \geq \beta(\psi_\lambda) \) we get from (6.5)
\[ x(\psi_\lambda) \geq H_\lambda(a) \left( 1 - \frac{H_\lambda(a_+)}{H_\lambda(a)} \right)^{\tau/(\tau - 1)} \left( \frac{H_\lambda(a)}{H_\lambda(at)} \right)^{1/(\tau - 1)} = \]
\[ = \exp \{ (a - 1) R_\lambda(a) - A_\lambda(a, \tau) - B_\lambda(a, \tau) \} \]
where
\[ A_\lambda(a, \tau) = -\frac{\tau}{\tau - 1} \ln \left( 1 - \frac{H_\lambda(a_+)}{H_\lambda(a)} \right), \]
\[ B_\lambda(a, \tau) = -\frac{1}{\tau - 1} \left[ (a\tau - 1) R_\lambda(at) - (a - 1) R_\lambda(a) \right]. \]

The easily verified relation
\[ \lim_{\lambda} \left( \frac{H_\lambda(a_+)}{H_\lambda(a)} \right)^{1/(\lambda \tau \lambda^{1/2} \lambda)} = \exp \{ \bar{\alpha}(a_+) - \bar{\alpha}(a) \} \quad (\text{cf. (4.3), (2.4)}) \]

together with the obvious inequality \( \bar{\alpha}(a_+) < \bar{\alpha}(a) \) implies that
\[ \lim_{\lambda} A_\lambda(a, \tau) = 0. \]
This and the following easily verified fact
\[
\lim_{\lambda \to \infty} \frac{B_{\lambda}(a_0, \tau)}{R_{\lambda}(\frac{1}{2})} = a \frac{\tilde{q}(\tau - a)}{\tau - a}
\]

imply that in the above introduced notation
\[
\lim_{n \to \infty} \inf \left\{ \varepsilon \left( e_{\psi_{\lambda}} \right) \right\}^{1/R_{\lambda}(1/2)} \geq \exp \left\{ \tilde{q}(a) - a \tilde{q}_+'(a) \right\}.
\]

Here \( \tilde{q}(a) \) is continuous by Lemma 4.1 and \( \tilde{q}_+(a) \) is continuous from the right by Lemma 4.1 and 2.4. Thus the right-hand side of last inequality tends for \( a \downarrow a_+ \) to \( \tilde{q}(a_+) - a_+ \tilde{q}_+(a_+) = \tilde{q}(a_+) \). Therefore (6.3) holds for \( a_+ = a_+ \).

It is clear that if the function \( \tilde{q} \) exists and has a unique point of minima \( a_\# \) at which it is differentiable then all assumptions of Theorem 6.1 hold. Using the results (2.2), (2.3), one can establish differentiability of \( \tilde{q} \) on \((0, 1)\) under wide assumptions about \((P_x, Q_x), x \in A\).

**Example 6.1.** Let us consider the diffusion processes of Example 4.1. By (4.5), argmin \( \tilde{q}(a) = \{ \frac{1}{2} \} \) and all assumptions of Theorem 6.1 hold. Therefore Bayes tests \( \psi_{\lambda}, \lambda \in [0, \infty) \), satisfy the asymptotic relation
\[
\lim_{\lambda \to \infty} \left( \varepsilon \left( e_{\psi_{\lambda}} \right) \right)^{1/R_{\lambda}(1/2)} = \exp \left\{ -\frac{1}{2} \right\}
\]
where \( R_{\lambda}(\frac{1}{2}) \) is given by (3.4) for \( a = \frac{1}{2} \).

**Example 6.2.** The random fields model of Example 3.2 is not satisfying the assumptions of Theorem 6.1. By what is said in Example 4.2, the \( \bar{R} \)-distance required in the present section does not exist. The model of Example 5.3 has the desired \( \bar{R} \)-distance but it is constant on \((0, 1)\), equal to 1. Therefore the function
\[
\tilde{q}(a) = a - 1, \quad 0 < a < 1,
\]
has argmin \( \tilde{q}(a) \) empty and nothing can be said here about Bayes tests.

Let us note that results similar to Theorem 6.1 can also be deduced from known general large deviation theorems (e.g. Ellis [5] or Theorem VII.6.1. in Ellis [6]). Similar approach was in fact used already in the fundamental paper of Chernoff [3]. The result of Chernoff valid for sequences of i.i.d. observations has been extended to Markov chains and processes by Koopmans [14], Nemetz [22], and Newman and Stuck [25].

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