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ON THE CONVERGENCE OF A TIME-VARIANT LINEAR DIFFERENTIAL EQUATION ARISING IN IDENTIFICATION[1]

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This paper discusses the asymptotic stability for a well-known time-variant system by means of the direct method of Liapunov. The system exhibits a positive time-invariant Liapunov function with negative semi-definite derivative. The paper focuses on the extra conditions needed in order to guarantee asymptotic stability. The proposed criterion is compared with the results available in the literature.

1. INTRODUCTION

For autonomous differential equations there exists a well developed Liapunov function approach for the examination of asymptotic stability. In particular Lasalle's principle is an important tool in cases where the derivative of the Liapunov function along solutions is negative semi-definite. Unfortunately, a complete extension of this principle to nonautonomous differential equations seems unlikely te become available. Basically this is explained by the particular properties exhibited by the limit sets of autonomous differential equations as opposed to nonautonomous equations. In fact, a crucial step in the proof of Lasalle's principle (or closely related formulations like Barbashin's theorem) [3] is based on the fact that for autonomous differential equations limit sets are invariant under the flow. For periodic differential equations this property also holds true (with an appropriate definition of the notion of invariance) and so does Lasalle's principle. Extensions to asymptotically constant or asymptotically periodic systems, as well as to almost periodic systems have been developed in the literature, in general leading to weaker statements. For differential equations with more general types of time-variance no definite statements can be made; however, there are theorems capturing some features of the invariance principle [3].

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Time-varying equations arise quite naturally in different applications. We will concentrate on the following equation

$$\dot{x} = -m(t)m(t)^T x \tag{1}$$

with $x(t)$ and $m(t)$ n-dimensional. This equation or related versions arise in identification and control problems and have been studied extensively ([2], [5], [6]). Usually $x(t)$ represents the parameter error that is driven to zero, based on an observation of the error $m^T(t)\,x(t)$.

The equation arises also in other contexts such as pattern recognition, associative memory, and in many questions of numerical mathematics where e.g. algorithms are to be constructed converging to solutions of linear algebraic equations, or in computing pseudo-inverses. It has also been studied in the context of the so-called novelty detector, introduced by Kohonen, where $x(t)$ represents the "weights" or the "memory". The change of weights is then brought about by the product of the output $m^T x$ with the so-called input $m(t)$, (this is a particular case of the adaptive laws encountered in (linear) neural networks).

It is perhaps worthwhile to notice that the linear differential equation (1) has $(n-1)$ eigenvalues equal to zero, and one eigenvalue $-m^T(t)\,m(t)$. The stability study when $m(t)$ is constant based on the Liapunov function $V(x) = x^T x$ is quite trivial but not interesting from the point of view of applications. With $m(t)$ periodic, (asymptotic) stability can be investigated quite directly with the help of the Liapunov function $V(x) = x^T x$ and Lasalle's invariance principle. These results are well known. Notice also that for any $m(t)$ a Liapunov study quickly leads to stability of the origin.

A set of stability results is related to the notion of persistency of excitation of $m(t)$. Let $m(t)$ be a regulated function. It is called persistently exciting if there exists $T > 0$ such that $\forall\, s$

$$\alpha I \le \int_s^{s+T} m(t)\,m(t)^T \mathrm{d}t \le \beta I \tag{2}$$

with $\alpha > 0$ and $\beta > 0$.

It has been shown ([5], [2]) that this is a necessary and sufficient condition for *exponential* stability, (trajectories are bounded above and below by exponentials). The importance of this result is that it accomodates a wide class of signals, beyond the (almost) periodicity constraint of Lasalle's principle, still securing asymptotic stability.
Several remarks are in order. Notice that T is independent of s. Can this not be relaxed while still guaranteeing (a weaker form of) asymptotic stability? With Anderson ([2]) one notices that if the lower bound fails, there may or may not be convergence and when there is convergence, it will not be exponential. What are the implications of relaxing the upper bound?

When examining the proofs of exponential stability as they appear in the literature, it is not entirely clear how they can be altered so as to accomodate the remarks raised above. Basically we present two results. One is related to relaxing the upper limit of the integral sign; the other discusses the lower and upper bounds of the relation defining the persistency of excitation relation.

2. MAIN THEOREM

This section discusses the main contribution of the paper. It is shown how the upper limit of the integral sign in (2) can be relaxed without destroying the asymptotic stability of (1), being aware that by [2] exponential stability is not retained.

The proof of our main theorem depends on results developed in [1] where asymptotic stability results for time-variant systems have been developed based on a Liapunov approach. For convenience we first state a result from [1] as needed for our purposes.

Consider the linear time-variant system

$$\dot{x} = A(t)\,x. \tag{3}$$

Assume there exists a positive definite quadratic form $V(x) := x^T P x$ such that

$$\dot{V}(x,t) = x^T(A^T(t)\,P + PA(t))\,x =: x^T Q(t)\,x \leq 0.$$

Let $x(t;p,t_i)$, $t \geq t_i$ denote the forward solution of (3) with initial condition p at t_i. For this class of linear systems we have the following theorem.

Theorem 1. If $A(t)$ is measurable and bounded and there is *no* sequence $t_i \to \infty$ such that (with $p \neq 0$)

$$\int_0^\infty \dot{V}(x(t_i+s;p,t_i),\, t_i+s)\,\mathrm{d}s \to 0 \quad \text{for } t_i \to \infty$$

then $\dot{x} = A(t)\,x$ is asymptotically stable.

We are now ready to state our main result.

Theorem 2. Let $m(t)$ be regulated and bounded. A sufficient condition for asymptotic stability of (1) is that for each t there exists a $T(t) > t$ such that

$$\alpha I \leq \int_t^{T(t)} m(\tau)m(\tau)^T d\tau \leq \beta I \tag{4}$$

with $\beta \geq \alpha > 0$.

Proof. First notice that $V(x) := x^T x/2$ is a positive definite Liapunov equation for (1) with negative semidefinite derivative. This implies stability. Notice also that since $V(x) \equiv \|x\|^2/2$ where $\|.\|$ represents the Euclidean norm, Theorem 1 can be reformulated as follows ($x(t_i+t;p,t_i)$ from now on represents a solution of (1)):

The system (1) is asymptotically stable if there is *no* sequence $t_i \to \infty$ such that (with $p \neq 0$)

$$\lim_{t\to\infty} \|x(t_i + t; p, t_i)\|^2 \to \|p\|^2 \quad \text{for } t_i \to \infty. \tag{5}$$

Assume that (1) is not asymptotically stable. There exists then a point p^* and a sequence (t_i^*) such that

$$\lim_{t\to\infty} \|x(t_i^* + t; p^*, t_i^*)\|^2 \to \|p^*\|^2 \quad \text{for } t_i^* \to \infty. \tag{6}$$

From (6) and since $\dot{V} \le 0$ it is immediate that for all $t > t_i^*$

$$\|x(t_i^* + t; p^*, t_i^*)\|^2 \to \|p^*\|^2 \quad \text{for } t_i^* \to \infty. \tag{7}$$

For notational convenience, we denote in the following $x(t; p^*, t_i^*)$ by $x(t; t_i^*)$. The formulas (6) and (7) respectively may be reformulated as follows: there exists a point p^* and a sequence (t_i^*) such that

$$\int_{t_i^*}^{\infty} x^T(\tau; t_i^*) m(\tau)\, m^T(\tau) x(\tau; t_i^*)\, \mathrm{d}\tau \to 0 \quad \text{for } t_i^* \to \infty \tag{8}$$

and therefore also such that for each $t > t_i^*$

$$\int_{t_i^*}^{t} x^T(\tau; t_i^*) m(\tau)\, m^T(\tau) x(\tau; t_i^*)\, \mathrm{d}\tau \to 0 \quad \text{for } t_i^* \to \infty. \tag{9}$$

Consider now the expression

$$\max_{t\in[t_i^*, T(t_i^*)]} \|x(t; t_i^*) - p^*\|^2$$

and assume the maximum be taken for t_i'. We will show first that

$$\|x(t_i'; t_i^*) - p^*\|^2 \to 0 \text{ when } t_i^* \to \infty. \tag{10}$$

In order to do so consider the following identity

$$\|x(t_i'; t_i^*)\|^2 - \|p^*\|^2 = \|x(t_i'; t_i^*) - p^*\|^2 - 2\int_{t_i^*}^{t_i'} p^{*T} m(\tau)\, m^T(\tau)\, x(\tau; t_i^*)\, \mathrm{d}\tau \tag{11}$$

with the left hand side tending to 0 as t_i^* tends to ∞ because of (7).
Therefore as $t_i^* \to \infty$

$$\|x(t_i'; t_i^*) - p^*\|^2 \to 2\int_{t_i^*}^{t_i'} p^{*T} m(\tau)\, m^T(\tau)\, x(\tau; t_i^*)\, \mathrm{d}\tau. \tag{12}$$

By the inequality of Cauchy–Schwartz the modulus of the right hand side of (12) satisfies the following inequalities

$$\left|\int_{t_i^*}^{t_i'} p^{*T} m(\tau)\, m^T(\tau)\, x(\tau; t_i^*)\, \mathrm{d}\tau\right| \le$$

$$\le \int_{t_i^*}^{t_i'} p^{*T} m(\tau)\, m^T(\tau)\, p^* \cdot \int_{t_i^*}^{t_i'} x^T(\tau; t_i^*)\, m(\tau)\, m^T(\tau)\, x(\tau; t_i^*)\, \mathrm{d}\tau \le$$

$$\le \beta \|p^{*T} p^*\| \cdot \int_{t_i^*}^{\infty} x^T(\tau; t_i^*)\, m(\tau)\, m^T(\tau)\, x(\tau; t_i^*)\, \mathrm{d}\tau \tag{13}$$

invoking (4) in the last inequality. Notice also that by (8) the last integral tends to zero for t_i^* tending to ∞. This ends the proof of (10).
Consider now

$$\int_{t_i^*}^{T(t_i^*)} (m^T(\tau)\, (x(\tau; t_i^*) - p^*))^2\, \mathrm{d}\tau$$

$$= \int_{t_i^*}^{T(t_i^*)} (m^T(\tau)\, x(\tau; t_i^*))^2\, \mathrm{d}\tau + \int_{t_i^*}^{T(t_i^*)} (m^T(\tau)\, p^*)^2\, \mathrm{d}\tau - 2 \int_{t_i^*}^{T(t_i^*)} x^T(\tau; t_i^*) m(\tau)\, m^T(\tau)\, p^*\, \mathrm{d}\tau$$

$$\ge \int_{t_i^*}^{T(t_i^*)} (m^T(\tau)\, p^*)^2\, \mathrm{d}\tau - 2 \int_{t_i^*}^{T(t_i^*)} x^T(\tau; t_i^*) m(\tau)\, m^T(\tau)\, p^*\, \mathrm{d}\tau. \tag{14}$$

The first term of the right-hand side in (14) is bounded below by $\alpha \|p^*\|^2$. By the inequality of Cauchy–Schwartz the modulus of the second term is smaller than

$$\int_{t_i^*}^{T(t_i^*)} (x^T(\tau; t_i^*)\, m(\tau))^2\, \mathrm{d}\tau \cdot \int_{t_i^*}^{T(t_i^*)} (p^{*T} m(\tau))^2\, \mathrm{d}\tau$$

and therefore smaller than

$$\int_{t_i^*}^{\infty} (x^T(\tau; t_i^*)\, m(\tau))^2\, \mathrm{d}\tau \cdot \beta \|p^*\|^2$$

which tends to zero by (8). We conclude that for t_i^* large enough the following is true

$$\int_{t_i^*}^{T(t_i^*)} ((x(\tau; t_i^*) - p^*)^T m(\tau))^2\, \mathrm{d}\tau \ge \alpha/2 \|p^*\|^2. \tag{15}$$

We will now invoke (10), (15) and (4) to establish a contradiction. Notice first that the left hand side of (15) is less than or equal than

$$\int_{t_i^*}^{T(t_i^*)} \|x(\tau;t_i^*) - p^*\|^2 \cdot \|m(\tau)\|^2 \, d\tau$$

This again is less than or equal than

$$\max_{t\in[t_i^*,T(t_i^*)]} \|x(t;t_i^*) - p^*\|^2 \cdot \int_{t_i^*}^{T(t_i^*)} m^T(\tau))\, m(\tau)\, d\tau$$

which itself is equal to

$$\|x(t_i';t_i^*) - p^*\|^2 \cdot \left(\sum_k \int_{t_i^*}^{T(t_i^*)} m_k^2(\tau)\, d\tau \right).$$

Because of (10), inequality (15) can only be satisfied if

$$\sum_k \int_{t_i^*}^{T(t_i^*)} m_k^2(\tau)\, d\tau \to \infty \quad \text{for } t_i^* \to \infty.$$

But this would imply that for some k^*

$$\int_{t_i^*}^{T(t_i^*)} m_{k^*}^2(\tau)\, d\tau \to \infty \quad \text{for } t_i^* \to \infty$$

which leads to a contradiction by the right inequality featuring in (4). □

The reader is reminded that $m(t)$ is bounded. This assumption had to be introduced in the development of Theorem 1. We have not been able to relax it. Apart from this condition Theorem 2 represents a generalization of the stability results concerning (1) featuring in [2, 5]. Indeed the sufficient condition is relaxed in that the integration interval $T(t) - t$ of the integral sign is no longer constant. As an example consider the differential equation $\dot{x} = -(1/t)\,x$ for $t \geq 1$. This differential equation is asymptotically stable as can be verified by direct computation. This follows also from (4) for $T(t) = 2t$ but the equation does not satisfy (2). On the other hand, Condition (4) is not necessary as shown by the following example:

Example 1. Consider

$$m(t)^T = \begin{cases} \begin{pmatrix} 1 & 0 \end{pmatrix} & \text{for } i \leq t < i+1 \\ \begin{pmatrix} 0 & 1/\sqrt{t} \end{pmatrix} & \text{for } i+1 \leq t < i+2. \end{cases}$$

By reordering the time-axis we obtain two asymptotically stable decoupled equations $\dot{x}_1 = -x_1$ and $\dot{x}_2 = -x_2/t$. It is quickly verified that condition (4) can not be satisfied, basically since there is no $T(t)$ that works for *both* these equations.

3. THE ROLE OF THE UPPER BOUND

As far as convergence of the solutions of (1) is concerned, the previous section shows that Condition (2) can be relaxed in the sense that the interval of integration may depend on the initial time. A natural question is whether the lower and the upper bound of the integral can be relaxed. In this section, we discuss the implications of the following condition:

$$\exists\, \alpha > 0 : \forall\, t \geq 0 : \exists\, T(t) > t : \alpha I \leq \int_t^{T(t)} m(\tau)\, m^T(\tau)\, \mathrm{d}\tau. \tag{16}$$

Proposition 1. A equivalent formulation for (16) is given by

$$\forall\, \gamma > 0,\ \forall\, t \geq 0 : \gamma I \leq \int_t^{+\infty} m(\tau)\, m^T(\tau)\, \mathrm{d}\tau. \tag{17}$$

Proof. Let $\gamma > 0$ and suppose that (16) holds. Then, by denoting $T^0(t) = T(t)$ and $T^n(t) = T(T^{n-1}(t))$, we obtain for each integer n:

$$\forall\, t \geq 0 : \int_t^{T^n(t)} m(\tau)\, m^T(\tau)\, \mathrm{d}\tau \geq n\alpha I. \tag{18}$$

(17) is thus satisfied with $n > \gamma/\alpha$.
Conversely, if (17) holds, it is obvious that for an arbitrary $\alpha > 0$, there exists a $T(t,\alpha)$ such that

$$\int_t^{T(t,\alpha)} m(\tau)\, m^T(\tau)\, \mathrm{d}\tau \geq \alpha I. \tag{19}$$

□

The following theorem shows that (16) is necessary for attractivity (and therefore for asymptotic stability) of Equation (1).

Theorem 3. If the origin of (1) is attractive, then (16) holds.

Proof. The proof goes by contradiction. Suppose that (16) does not hold. Let $(\alpha_i)_{i\geq 1}$ be a positive sequence tending to zero as $i \to \infty$. Then for each α_i, there exist a p_i with $\|p_i\| = 1$ and a positive t_i such that

$$\int_{t_i}^{+\infty} (m^T(\tau)\, p_i)^2\, \mathrm{d}\tau \leq \alpha_i\, p_i^T p_i = \alpha_i. \tag{20}$$

Let $x(t;t_i) := x(t;p_i,t_i)$ be a solution of (1). We have the following string of equalities:

$$\forall\, i \geq 1: \int_{t_i}^{+\infty} p_i^T\, m(\tau)\, m^T(\tau)\, x(\tau;t_i)\, \mathrm{d}\tau = \lim_{t\to\infty} p_i^T(p_i - x(t;t_i)) = p_i^T p_i = 1, \quad (21)$$

where the first equality follows from (1), and the second equality follows from attractivity of the origin; similarly we have

$$\forall\, i \geq 1: \int_{t_i}^{+\infty} (m^T(\tau)\, x(\tau;t_i))^2\, \mathrm{d}\tau = \lim_{t\to\infty} (p_i^T p_i - x^T(t;t_i)\, x(t;t_i)) = p_i^T p_i = 1. \quad (22)$$

On the other hand, for each $i \geq 1$, we have by the Cauchy–Schwartz inequality:

$$\left| \int_{t_i}^{+\infty} p_i^T\, m(\tau)\, m^T(\tau)\, x(\tau;t_i)\, \mathrm{d}\tau \right| \leq \int_{t_i}^{+\infty} (m^T(\tau)\, p_i)^2\, \mathrm{d}\tau \int_{t_i}^{+\infty} (m^T(\tau)\, x(\tau;t_i))^2\, \mathrm{d}\tau. \quad (23)$$

Using (20) and (22), we conclude that the right-hand side of (23) tends to zero for $i \to \infty$, which contradicts (21). □

Remark. It is obvious that in the scalar case, Conditions (4) and (16) are equivalent (Let $\beta = \alpha$ and choose $T(t)$ in such a way that $\alpha = \int_t^{T(t)} m^2(t)\, \mathrm{d}t = \beta$). In this particular case, Condition (16) is thus necessary and sufficient for asymptotic stability. This also follows from comparing Condition (17) to the explicit solution of (1).

In general, Condition (16) is not sufficient for asymptotic stability, as illustrated by the following example.

Example 2. Define

$$\begin{aligned} \alpha_0 = \pi/2,\ \alpha_{i+1} &= \alpha_i/2,\ i \geq 0, \\ t_0 = 0,\ t_{i+1} &= t_i + 1/\sin^2 \alpha_i, \\ m_i^T &= (\sin \alpha_i, \cos \alpha_i). \end{aligned}$$

Let $m(\cdot): \mathbb{R}^+ \to \mathbb{R}^2 : m(t) = m_i,\ t_i \leq t < t_{i+1}$.
Condition (17) is obviously satisfied: indeed, let $t_i \geq 0$. If $e_1 := (1,0)$ then for each integer n, $\int_{t_n}^{t_{n+1}} (m^T(\tau)\, e_1)^2\, \mathrm{d}\tau = 1$ which implies that $\int_{t_i}^{\infty} (m^T(\tau)\, e_1)^2\, \mathrm{d}\tau$ is unbounded. The condition is thus fulfilled for $p = \epsilon e_1$, $\epsilon \neq 0$. Now if p is not parallel to e_1, then for n sufficiently large, $(m_n^T p)^2 > (m_n^T e_1)^2$ and as a consequence, $\int_{t_n}^{t_{n+1}} (m^T(\tau)\, p)^2\, \mathrm{d}\tau > 1$. This implies that $\int_{t_i}^{\infty} (m^T(\tau)\, p)^2\, \mathrm{d}\tau$ is unbounded for all $p \neq 0$.
Now it can easily be shown that the above $m(t)$ does not force the convergence of solutions of (1) to the origin. Let $m_1^\perp = (\cos \alpha_1, -\sin \alpha_1)$ and consider the solution starting from $m_1^\perp$ at time $t = t_2$, i.e. $x(t; m_1^\perp, t_2)$. Then $x_1(t_2; m_1^\perp, t_2) = \cos \alpha_1 > 0$ and it is clear by inspection that $x_1(t; m_1^\perp, t_2)$ can only increase for all $t > t_2$ which prevents the convergence of the solution to the origin.

4. CONCLUSION

This paper has studied convergence properties of Equation (1). Exponential convergence of (1) is known to be equivalent to the persistency of excitation of $m(t)$. As far as convergence is concerned, Theorem 2 shows that the period of "excitation" of the system may be non uniform in time. On the contrary, the constants α and β in (4) express the uniformity of the excitation in all directions of the state space. A uniform lower bound α was shown to be necessary for convergence (Theorem 3). This is not true with respect to the upper bound β (Example 1) but convergence may be lost when relaxing the upper inequality in (4) (Example 2). Finally notice that with a minor modification of the proofs, the conclusions of the paper still hold if the $n \times 1$ vector $m(t)$ is replaced by a $n \times p$ matrix, with $p \leq n$.

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