THE POLE PLACEMENT EQUATION – A SURVEY

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We consider the linear equation \( AX + BY = C \) where \( A, B \) and \( C \) are given polynomials from \( K[s] \), the ring of polynomials in the indeterminate \( s \) over a field \( K \), and \( X \) and \( Y \) are unknown polynomials in \( K[s] \).

1. MOTIVATION

The equation \( AX + BY = C \) (1) has found application in several design problems for linear control systems, including the pole placement design. This problem consists in the following: given a plant with real-rational proper transfer function

\[
P(s) = \frac{B(s)}{A(s)},
\]

where \( A \) and \( B \) are coprime polynomials, one seeks to determine a dynamic output feedback controller with a real-rational proper transfer function, say

\[
Q(s) = -\frac{Y(s)}{X(s)}
\]

such that the closed-loop system has prespecified poles.

Provided \( A \) is the characteristic polynomial of the plant and \( X \) is that of the controller, then the characteristic polynomial of the closed-loop system, say \( C(s) \), which specifies the poles desired, is given by \( C = AX + BY \).

Thus the pole placement design is based on equation (1). However not all solution pairs \( X, Y \) are of interest: one must take the one in which \( Y \) has least degree. This leads to a proper controller whenever one exists.

2. REVIEW OF THEORY

It is well known [1] that \( K[s] \) is a principal ideal domain. Thus (1) is solvable if and only if any greatest common divisor of \( A \) and \( B \) divides \( C \). Writing \( D \) for a greatest
common divisor of $A$ and $B$ and denoting
\[ \tilde{A} = \frac{A}{D}, \quad \tilde{B} = \frac{B}{D}, \quad \tilde{C} = \frac{C}{D}, \]
one concludes that (1) has a solution if and only if $\tilde{C}$ is a polynomial. Therefore if $A$ and $B$ are coprime then (1) is solvable for any $C$.

Suppose that $X, Y$ is a particular solution pair of (1). Since the equation is linear, any and all solution pairs of (1) are given by
\[ X = \tilde{X} - \tilde{B}T, \quad Y = \tilde{Y} + \tilde{A}T, \]
where $T$ varies over $K[s]$. Thus the solution class of (1) is parametrized through $T$ in a simple manner.

It is well known [1] that $K[s]$ is a euclidean domain. Therefore if (1) is solvable and $B \neq 0$ there is a unique solution pair $X_{1\min}, Y_1$ of (1) such that either $X_{1\min} = 0$ or $\deg X_{1\min} < \deg \tilde{B}$. Further if (1) is solvable and $A \neq 0$ then there is a unique solution pair $X_2, Y_{2\min}$ of (1) such that either $Y_{2\min} = 0$ or $\deg Y_{2\min} < \deg \tilde{A}$. These two least-degree solution pairs coincide [4] whenever $\deg \tilde{A} + \deg \tilde{B} > \deg \tilde{C}$.

As a result, equation (1) with $A \neq 0$ and $B \neq 0$ can possess solution pairs $X, Y$ of arbitrarily high degree, limited only from below by $\deg X_{1\min}$ and $\deg Y_{2\min}$.

3. FIXED DEGREE SOLUTIONS

We shall study the class of solutions whose degrees are limited from above. We suppose that $A, B$ and $C$ in (1) are non-zero polynomials from $K[s]$ with $A$ and $B$ coprime. Hence (1) is solvable. Let
\[ p = \deg A, \quad q = \deg B, \quad r = \deg C. \]
If
\[ A = a_0 + a_1 s + \ldots + a_p s^p \]
then, for any integer $k \geq p$, we denote
\[ \text{vec}_k A = [a_0 \ a_1 \ldots a_p \ 0 \ldots 0]_{k-p}. \]

The existence result [5] is as follows. Let $m, n$ be non-negative integers and
\[ d = \max(m + p, \ n + q, \ r). \]
Then a solution pair $X, Y$ of (1) exists such that
\[ X = 0 \text{ or } \deg X \leq m, \quad Y = 0 \text{ or } \deg Y \leq n \quad (2) \]
if and only if $\text{vec}_d C$ is a $K$-linear combination of $\text{vec}_d A$, $\text{vec}_d s A$, $\ldots$, $\text{vec}_d s^m A$, $\text{vec}_d B$, $\ldots$, $\text{vec}_d s^n B$.

A special case of particular interest concerns the constant solutions of (1). Putting $m = n = 0$ we deduce [6] that a solution pair $X, Y$ of (1) exists in $K$ if and only if $\text{vec}_d C$ is a $K$-linear combination of $\text{vec}_d A$ and $\text{vec}_d B$. 

The set of solutions whose degrees are limited from above can be parametrized as follows [5]. Let \( m \geq q \) and \( n \geq p \). If \( n \geq r - q \) then the set of solutions \( X, Y \) of (1) that satisfy (2) is given as
\[
X = X_{1 \text{min}} - BT_1, \quad Y = Y_1 + AT_1,
\]
where \( T_1 \) varies over \( K[s] \) and
\[
\deg T_1 \leq \min(m - q, n - p);
\]
if \( m \geq r - p \) then the set of solutions \( X, Y \) of (1) that satisfy (2) is given as
\[
X = X_2 - BT_2, \quad Y = Y_2 \text{min} + AT_2,
\]
where \( T_2 \) varies over \( K[s] \) and
\[
\deg T_2 \leq \min(m - q, n - p).
\]
Indeed suppose that \( n \geq r - q \). Then (3) implies
\[
\deg X = q + \deg T_1 \leq m \quad \text{and} \quad \deg Y = \max(r - q, p + \deg T_1) \leq n
\]
so that \( \deg T_1 \leq m - q \) and \( \deg T_1 \leq n - p \). In case \( m \geq r - p \) then (4) implies
\[
\deg X = \max(r - p, q + \deg T_2) \leq m \quad \text{and} \quad \deg Y = p + \deg T_2 \leq n
\]
and again \( \deg T_2 \leq m - q \) and \( \deg T_2 \leq n - p \).

We note that at least one of the two conditions, \( m \geq r - p \) and \( n \geq r - q \), is always satisfied. Of course (3) can be used to parametrize the solution set (2) even if \( n < r - q \). Then, however, \( T_1 \) has a higher degree than shown and is not completely free in \( K[s] \). An analogous statement is true for (4) when \( m < r - p \). To illustrate, we parametrize the solution class of
\[
X + sY = s^2
\]
such that \( \deg X \leq 1 \) and \( \deg Y \leq 1 \). Using (3),
\[
X = -sT_1, \quad Y = s + T_1, \quad T_1 \text{ constant}
\]
while using (4),
\[
X = s^2 - T_2, \quad Y = T_2, \quad T_2 = s + r, \quad r \text{ constant}.
\]
4. EXAMPLES

Can the double integrator
\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = u, \quad y = x_1 \]
be converted into an harmonic oscillator using a \textit{proportional} output feedback?

The double integrator gives rise to the transfer function
\[ P(s) = \frac{1}{s^2} \]
and any harmonic oscillator has the characteristic polynomial
\[ C(s) = s^2 + \omega^2 \]
for some real constant \( \omega > 0 \). Thus the answer depends on the polynomial equation
\[ s^2 X + Y = s^2 + \omega^2 \]
having a constant solution pair \( X, Y \).

Since
\[ \begin{align*}
\text{vec}_2 A &= [0 \ 0 \ 1] \\
\text{vec}_2 B &= [1 \ 0 \ 0] \\
\text{vec}_2 C &= [\omega^2 \ 0 \ 1]
\end{align*} \]
the answer is an affirmative: the output feedback \( u = -\omega^2 y \) will do the job. The resulting system equations read
\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = u - \omega^2 x_1, \quad y = x_1. \]

On the other hand, the double integrator cannot be stabilized via proportional output feedback: the polynomial \( s^2 X + Y \) is not Hurwitz for any real numbers \( X \) and \( Y \).

As the second example, we consider the plant
\[ \dot{x}_1 = u - x, \quad y = x \]
and find all output feedback controllers that will alter its characteristic polynomial \( s + 1 \) to \( s^2 + 3s + 2 \).

These controllers possess the transfer functions
\[ Q(s) = \frac{X(s)}{Y(s)}, \]
where \( X, Y \) is the solution set of the equation
\[ (s + 1)X + Y = s^2 + 3s + 2 \]
such that \( \text{deg} X = 1 \) and \( \text{deg} Y \leq 1 \).
The condition \( m \geq r - p = 1 \) is verified. Therefore the solution set is given by

\[
X = s + 2 - T_2, \quad Y = (s + 1) T_2,
\]

where \( T_2 \) is any real polynomial of degree at most \( \min(m - q, n - p) = 0 \), hence any real constant.

A realization of the parametrized controller set is

\[
\begin{align*}
\dot{w} &= (T_2 - 2) w + (T_2 - 1) y \\
-u &= T_2 w + T_2 y.
\end{align*}
\]

The case \( T_2 = 0 \) leads to an unobservable realization while \( T_2 = 1 \) leads to an uncontrollable realization. A PI controller is obtained when \( T_2 = 2 \).

If desired, the parameter \( T_2 \) can be chosen so that a specific goal is achieved. For example, if the \( H_\infty \)-norm of the sensitivity function

\[
S(s) = \frac{s + 2}{s + 2 - T_2}
\]

is not to exceed 1, we should avoid the values \( 0 < T_2 < 4 \).

5. METHODS OF SOLUTION

Equation (1) can be solved in several ways [4]. One can distinguish parametric methods (where the polynomials are represented by their coefficients) and non-parametric ones (where the polynomials are represented by their functional values.) We shall describe three major parametric methods.

We suppose that \( A, B \) and \( C \) in (1) are non-zero real polynomials with \( A \) and \( B \) coprime. Hence (1) is solvable. For the sake of simplicity let

\[
\deg A = \deg B = N, \quad \deg C = 2N - 1.
\]

The Method of Indeterminate Coefficients [4] converts equation (1) into a system of \( 2N \) linear equations over the field of real numbers. Suppose we seek the least-degree solution pair \( X, Y \):

\[
\deg X \leq N - 1, \quad \deg Y \leq N - 1.
\]

The \( 2N \) coefficients of \( X, Y \) satisfy the system of equations

\[
\begin{bmatrix}
\text{vec}_{2N-1} X & \text{vec}_{2N-1} Y
\end{bmatrix}
\begin{bmatrix}
\text{vec}_{2N-1} A \\
\ldots \\
\text{vec}_{2N-1} s^{N-1} A \\
\text{vec}_{2N-1} B \\
\ldots
\end{bmatrix}
= \text{vec}_{2N-1} C.
\]
The system matrix is a Sylvester matrix and it has full rank since $A$ and $B$ are coprime.

The Method of Polynomial Reductions [3] reduces equation (1) to a polynomial equation that is much easier to solve. It consists of the substitutions

$$C' = C - A \left\lfloor \frac{deg C}{deg A} \right\rfloor C$$
$$C' = C - B \left\lfloor \frac{deg C}{deg B} \right\rfloor C$$
$$B' = B - A \left\lfloor \frac{deg B}{deg A} \right\rfloor A$$
$$A' = A - B \left\lfloor \frac{deg A}{deg B} \right\rfloor B$$

each reducing the degree of one of the polynomials $A, B, C$. The substitutions are repeated for the new polynomials $A', B', C'$ and will ultimately reduce all $A, B, C$ but one to zero. The resulting equation has a solution $X' = 0, Y' = 0$ and the solution pair $X, Y$ of (1) is obtained through the backward substitutions

$$X = X' + \left\lfloor \frac{deg C}{deg A} \right\rfloor C$$
$$Y = Y' + \left\lfloor \frac{deg C}{deg B} \right\rfloor C$$
$$X = X' - \left\lfloor \frac{deg B}{deg A} \right\rfloor A$$
$$Y = Y' - \left\lfloor \frac{deg A}{deg B} \right\rfloor B$$

The process involves the euclidean algorithm for $A, B$ and leads to the least-degree solution pair $X, Y$.

The Method of State-space Realization [2] combines matrix and polynomial operations. We write (1) as

$$X + \frac{B}{A}Y = \frac{C}{A}$$

and determine a reachable state-space realization $(F, G, H, J)$ of the rational function $B/A$. The $N$ coefficients of $Y$ satisfy the system of equations

$$\text{vec}_{N-1}Y \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{N-1} \end{bmatrix} = \text{vec}_{N-1}(C \mod A)$$

and the corresponding $X$ is recovered from (1), it is the least-degree solution pair. The system matrix is an observability matrix and it has full rank since $A$ and $B$ are coprime.

6. NUMERICAL EXPERIENCE

The method of indeterminate coefficients is straightforward and leads directly to a system of linear equations for the coefficients of the unknown polynomials. The
method of polynomial reductions solves the polynomial equation by polynomial means and is not suitable for pencil-and-paper calculations, for it requires a large number of logical operations. The method of state-space realization combines the two above: one unknown polynomial is obtained by solving a system of linear equations while the other results from polynomial manipulations.

The comparison of the methods with respect to the arithmetic complexity is quite clear [7]. The fastest is the method of polynomial reductions, where the operations count is proportional to $N^2$. For the other two methods the arithmetic complexity is proportional to $N^3$. The slowest method, however, is that of indeterminate coefficients because it leads to a larger system of linear equations than the method of state-space realization.

The comparison of the methods from the precision point of view [7] is not that simple, however. Provided the polynomials $A$ and $B$ have no (especially multiple) roots close to each other, the precision of all three methods is alike. The ill-conditioned data, however, make the method of polynomial reductions fail more often than that of indeterminate coefficients. The method of state-space realization shows no clear-cut tendency, it stays between the two preceding methods.

To conclude, polynomial reductions are fast but sensitive to data, indeterminate coefficients are robust but slow, and the method of state-space realization is universal but second best in each single aspect.

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