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Semantics Based on States of Affairs

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A small change in the construction of states of affairs given in [1] enables us to formulate all semantic concepts in terms of states of affairs and facts only.

1. INTRODUCTION

In [1] a variant of the theory of states of affairs was outlined. The disadvantage of this variant is that the states of affairs are assigned to synthetic formulas only. The semantic notions of satisfaction, truth etc. were defined in terms of states of affairs only for the case of synthetic formulas. We cannot obtain these concepts in general.

In the present article a new variant of the theory of states of affairs is given, which enables us to formulate the semantic concepts for arbitrary formulas. This variant may be, therefore, a basis for development of semantics.

Main difference consists in the fact that in the new variant we can eliminate only the unessential occurrences of $I - V$-terms but we cannot eliminate the occurrence of D^I. Therefore, for every formula we obtain a state of affairs, i.e. a pair, the first member of which consists at least of D^I.*

This difference provokes two essential changes of our conception. First, we need, in the case of logically true formulas, the state of affairs, the second member of which is a proper class (i.e. not a set). We apply, therefore, a strong variant of the set-theoretical ontology in which it is possible to construct ordered pairs with proper classes as members.

Furthermore, in contrast with [1], we cannot start with the concept of truth-value assignment. The starting point of our semantic constructions must be the concept of state of affairs and the concept of fact. By these concepts the concept of satisfaction

* For the construction of states of affairs see [1].

and other semantic concepts must be defined. Indeed, in this case, we must prove the adequacy of the concept of satisfaction defined in this manner for the customarily used notion of satisfaction.

Most definitions and theorems in the present article are the same as in [1]. We shall give them without proofs and comments. For typographical reasons we shall introduce some unessential changes in our notation.

2. THE LANGUAGE L, ITS METALANGUAGE AND ONTOLOGY OF L

The language L is an arbitrary language of applied first-order predicate calculus with identity. The logical constants are as follows:

$$\neg, \Rightarrow, \cdot, +, \Leftrightarrow, (\forall), (\exists), = .$$

The extralogical constants of L are following:

$a_1, a_2, \ldots, a_n$ (individual constants)

$P_1, P_2, \ldots, P_m$ (predicates; a predicate P_i is k_i-ary).

The variables of L are as follows:

$x_1, x_2, \ldots, x_p, \ldots$ (the number of variables is unlimited).

The concept of a formula of L, of a sentence of L and other syntactical concepts are defined as customary. The result of replacing of B by C in A is designed by $A(B/C)$.

The metalanguage ML of the language L contains following variables for arbitrary expressions of L:

$$A, B, C, D, A_1, B_1, C_1, D_1, A_2, \ldots$$

and following logical symbols:

$$\sim, \rightarrow, \&, \vee, \equiv, (\,), (E), = .$$

The ontology of L is the domain of all objects constructed by the principles of the set theory of Bernays-Morse (in the formulation given in [2]).

The ML-variables for objects of ontology are as follows

$$\alpha, \beta, \gamma, \delta, \alpha_1, \beta_1, \gamma_1, \delta_1, \alpha_2, \ldots$$

They have arbitrary classes as values. The membership relation is denoted as $\in$. A set is a class being a member of some class. When a class is not a set, it is a proper class. The universal class V is the class of all sets.*

The symbol $\{\ldots:\ldots\}$ is the abstraction operator (class of all sets..., that it holds...). The operations and relations

$$\cup,\ \cap,\ \backslash\ \text{(difference)},\ -\ \text{(complement)},\ \times,\ \subset,\ =$$

are defined for arbitrary classes. The operation $\{\ldots\}$ forms a class from sets. The symbol $\emptyset$ designs the empty set. The class of all unempty sets U is defined as follows:

$$U = V \backslash \{\emptyset\}\,.$$

We suppose that our ontology contains the natural numbers 0, 1, 2, 3, ...

The concept of ordered pair $\langle\alpha, \beta\rangle$ is defined in a usual way only for the case of sets α and β. We need, however, ordered pairs also in the case, when α and β are proper classes. We adopt a device cited in [2] and we define the ordered pair as follows:

$$\langle\alpha, \beta\rangle = \begin{cases} \{\{\alpha\}, \{\alpha, \beta\}\}\,, & \text{if } \alpha \text{ and } \beta \text{ are sets}\,, \\ (\alpha \times \{0\}) \cup (\beta \times \{1\})\,, & \text{if } \alpha \text{ or } \beta \text{ are proper classes}\,. \end{cases}$$

The second part of this definition satisfies the condition

$$\langle\alpha, \beta\rangle = \langle\gamma, \delta\rangle \equiv \alpha = \gamma \,\&\, \beta = \delta\,.$$

The variables for arbitrary formulas of set theory are as follows:

$$\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}, \mathscr{A}_1, \mathscr{B}_1, \mathscr{C}_1, \mathscr{D}_1, \mathscr{A}_2, \ldots\,.$$

The principle of abstraction:

(a) $(\alpha \text{ is a set} \,\&\, \mathscr{A}(\alpha)) \equiv \alpha \in \{\beta : \mathscr{A}(\alpha/\beta)\}$ ($\mathscr{A}$ does not contain β);

(b)
$$(\langle\alpha_1, \ldots, \alpha_i\rangle \text{ is a set} \,\&\, \mathscr{A}(\alpha_1, \ldots, \alpha_i) \equiv$$
$$\equiv \langle\alpha_1, \ldots, \alpha_i\rangle \in \{\langle\beta_1, \ldots, \beta_i\rangle : \mathscr{A}(\alpha_1/\beta_1, \ldots, \alpha_i/\beta_i)\}$$
($\mathscr{A}$ does not contain $\beta_1, \ldots, \beta_i$).

In contrast with [1], there are no truth-values in our ontology.

* When we assume in our ontology also the existence of individuals (Urelemente), some principles of the set theoretical basis must be changed, e.g. the following equivalence cannot hold:

$$\alpha \text{ is a set} \equiv \alpha \in V\,.$$

(see e.g. [3]). Our constructions are not dependent on these changes.

Definition 1. *The interpretation of L is a function I assigning* 1) *a domain of interpretation* D^I *to the language L and* 2) *exactly one denotatum* $I(A)$ *to every constant A in I such that*:

$$I(A) \in D^I, \text{ if } A \text{ is an individual constant ,}$$

$$I(A) \text{ is an } k\text{-ary relation on } D^I, \text{ if } A \text{ is a } k\text{-ary predicate .}$$

The variables for arbitrary interpretations are as follows:

$$I, J, K, I_1, J_1, K_1, I_2, \ldots .$$

Definition 2. *The valuation of variables in I is a function* V^I *assigning to every variable A exactly one value* $V^I(A)$ *such that* $V^I(A) \in D^I$.

The variables for arbitrary valuations in I are as follows:

$$V^I, U^I, W^I, V_1^I, U_1^I, W_1^I, V_2^I, \ldots ,$$

(or without index I, when the use in the given context is clear).

For the sake of simplicity of the text, we define a generalized concept of value for variables and constants in V and I.

Definition 3.

$$\text{If } A \text{ is a constant of } L, \text{ then } v_V^I(A) = I(A) ,$$

$$\text{if } A \text{ is a variable of } L, \text{ then } v_V^I(A) = V^I(A) .$$

We call the terms $v_V^I(A)$ $I-V$-terms. The sentences constructed from the formulas of set theory by replacing of variables by $I-V$-terms or by the term D^I will be called $I-V$-sentences. We shall use also the variables

$$\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}, \mathscr{A}_1, \mathscr{B}_1, \mathscr{C}_1, \mathscr{D}_1, \mathscr{A}_2, \ldots ,$$

for arbitrary $I-V$-expressions (i.e. $I-V$-sentences or $I-V$-terms).

Definition 4.

(1) *If A is a k-ary predicate and* $A_1, \ldots, A_k$ *are individual terms, then the* $I-V$*-translation of* $A(A_1, \ldots, A_k)$ *is the ML-sentence*

$$\langle v_V^I(A_1), \ldots, v_V^I(A_k) \rangle \in v_V^I(A) ;$$

(2) *if A and B are individual terms, then*

the $I-V$-translation of $A = B$ is the ML-sentence $v_V^I(A) = v_V^I(B)$;

(3) *if A is a formula and $\mathscr{A}$ is its $I-V$-translation, then*

the $I-V$-translation of $\neg A$ is the ML-sentence $\sim\mathscr{A}$;

(4) *if A and B are formulas and $\mathscr{A}$ and $\mathscr{B}$ are their $I-V$-translations, then*

the $I-V$-translation of $A . B$ is the ML-sentence $\mathscr{A} \,\&\, \mathscr{B}$;

the $I-V$-translation of $A + B$ is the ML-sentence $\mathscr{A} \vee \mathscr{B}$;

the $I-V$-translation of $A \Rightarrow B$ is the ML-sentence $\mathscr{A} \rightarrow \mathscr{B}$;

the $I-V$-translation of $A \Leftrightarrow B$ is the ML-sentence $\mathscr{A} \equiv \mathscr{B}$;

(5) *if A is a formula and $\mathscr{A}$ is its $I-V$-translation and x_i is a variable, then the $I-V$-translation of $(\forall x_i)\, A$ is the ML-sentence*

$$(\alpha_i)\,[\alpha_i \in D^I \rightarrow \mathscr{A}(v_V^I(x_i)/\alpha_i)]$$

the $I-V$-translation of $(\exists x_i)\, A$ is the ML-sentence

$$(E\alpha_i)\,[\alpha_i \in D^I \,\&\, \mathscr{A}(v_V^I(x_i)/\alpha_i)]\,.$$

Let us define the one-one correspondence between variables of L and variables of ML such that for every i the variable x_i corresponds to the variable α_i and vice versa. Then it is clear that to every formula A of L there is exactly one $I-V$-translation of A and vice versa. The set of $I-V$-translations of formulas of L is a proper subset of the set of $I-V$-sentences.

We bring a few theorems about $I-V$-sentences. The first of them is identical with Theorem 4 proved in [1]. We make only some changes in the notation.

Theorem 1. *If $\mathscr{A}$ is a $K-U$-sentence containing exactly the $K-U$-terms $v_U^K(A_1), \ldots, v_U^K(A_i)$ and possibly D^K, then*

$$(I)\,(V)\,[\mathscr{A}(K/I, U/V)] \equiv (\beta)\,(\alpha_1)\ldots(\alpha_i)\,[\omega(\beta, \alpha_1, \ldots, \alpha_i) \rightarrow$$
$$\rightarrow \mathscr{A}(D^K/\beta, v_U^K(A_1)/\alpha_1, \ldots, v_U^K(A_i)/\alpha_i)]\,.$$

By $\mathscr{A}(K/I, U/V)$ we mean the result of replacing of K by I and of U by V in $\mathscr{A}$. The condition $\omega(\beta, \alpha_1, \ldots, \alpha_i)$ is a abbreviation for

$$\beta \in U \,\&\, \omega_\beta^{\alpha_1} \,\&\, \ldots \,\&\, \omega_\beta^{\alpha_i}\,,$$

where

$$\omega_{\beta}^{\alpha_g} \text{ is } \alpha_g \in \beta, \text{ if } A_g \text{ is an individual term },$$

$$\text{is } \alpha_g \subset \underbrace{\beta \times \ldots \times \beta}_{k\text{-times}}, \text{ if } A_g \text{ is a } k\text{-ary predicate }.$$

Analogically, the condition $\omega(D^K, \alpha_1, \ldots, \alpha_i)$ is an abbreviation for $\omega_{D^K}^{\alpha_1} \& \ldots \& \omega_{D^K}^{\alpha_i}$, where

$$\omega_{D^K}^{\alpha_g} \text{ is } \alpha_g \in D^K, \text{ if } A_g \text{ is an individual term },$$

$$\text{is } \alpha_g \subset \underbrace{D^K \times \ldots \times D^K}_{k\text{-times}}, \text{ if } A_g \text{ is a } k\text{-ary predicate }.$$

In [1] these conditions are written without the abbreviation of the conditions limiting the quantificators. We shall use these abbreviations also in connection with abstraction-operators (e.g.)

$$\{\langle \beta, \alpha_1, \ldots, \alpha_i \rangle : \omega(\beta, \alpha_1, \ldots, \alpha_i) \& \mathscr{A}\}$$

mean

$$\{\langle \beta, \alpha_1, \ldots, \alpha_i \rangle : [(\beta \in U \& \omega_{\beta}^{\alpha_1} \& \ldots \& \omega_{\beta}^{\alpha_i}) \& \mathscr{A}]\} .$$

The following two theorems are evident consequences of Theorem 1.:

Theorem 2. *If $\mathscr{A}$ is a $K-U$-sentence containing exactly the $K-U$-terms $v_U^K(A_1), \ldots, v_U^K(A_i)$, then*

$$(I)(V)[\mathscr{A}(K/I, U/V)] \equiv (I)(\alpha_1)\ldots(\alpha_i)[\omega(D^I, \alpha_1, \ldots, \alpha_i) \rightarrow$$
$$\rightarrow (\mathscr{A}(v_U^K(A_1)/\alpha_1, \ldots, v_U^K(A_i)/\alpha_i)(K/I, U/V)]) .$$

Theorem 3. *If $\mathscr{A}$ is a $K-U$-sentence and $v_U^K(A_1), \ldots, v_U^K(A_i)$ are only some of* $K-U$-terms contained in it, then

$$(I)(V)[\mathscr{A}(K/I, U/V)] \equiv (I)(V)(\alpha_1)\ldots(\alpha_i)[\omega(D^I, \alpha_1, \ldots, \alpha_i) \rightarrow$$
$$\rightarrow (\mathscr{A}(v_U^K(A_1)/\alpha_1, \ldots, v_U^K(A_i)/\alpha_i)(K/I, U/V))] .$$

Note. On the right-hand side of equivalence in the Theorem 2. there must be the quantificator (I) because the formula in $[\ldots]$ contains D^I. But this formula does not contain V. On the right-hand side of the equivalence in the Theorem 3. there must be both (I) and (V), because $\mathscr{A}(K/I, U/V)$ contains still some $I-V$-terms (originally $K-U$-terms) not replaced by variables. Therefore, it contains I and V.

As in [1] we define three following auxiliary concepts:

Definition 5. *If A and B are formulas of L, then A and B are extensionally isomorph in I and V (abbreviated: $EIS_V^I(A, B)$), iff there are constants or variables free in A: $A_1, \ldots, A_i$ and there are constants or variables free in B: $B_1, \ldots, B_i$, such that $v_V^I(A_1) = v_V^I(B_1)$ and ... and $v_V^I(A_i) = v_V^I(B_i)$ and "A" = "$B(B_1/A_1, \ldots, \ldots, B_i/A_i)$".*

Definition 6. *If A and B are formulas of L, then A and B are +strongly equivalent in I and V (abbreviated: $+STREQ_V^I(A, B)$), iff there are formulas of L C and D such that: $EIS_V^I(A, C)$ and $EIS_V^I(B, D)$ and $(K)(U)((\mathscr{C} \equiv \mathscr{D})(I/K, V/U))$, where $\mathscr{C}$ and $\mathscr{D}$ are the $I-V$-translations of C and D.*

Note. In [1] was used $L\text{-}EQ(C, D)$ (C and D are L-equivalent) instead of the third member of the conjunction in the definiens of the present definition.

Definition 7. *If A is a formula of L and B is a constant or variable free in A, then A essentially contains B, iff $(EI)(EV)[\mathscr{A} \not\equiv_{\omega_{DI}^{\alpha}} (\alpha)\, \mathscr{A}(v_V^I(B)/\alpha)]$, where ω_{DI}^{α} is customary and $\mathscr{A}$ is the $I-V$-translation of A.*

The concept of strong equivalence ($STREQ$) is now defined as follows:

Definition 8. *If A and B are formulas of L, then $STREQ_V^I(A, B)$, iff the two following conditions are satisfied: 1) $+STREQ_V^I(A, B)$ and 2) for every C contained essentially in A there is a D essentially contained in B such that $v_V^I(C) = v_V^I(D)$ and vice versa (where C is a constant or a variable free in A and D is a constant or variable free in B).*

The concept $STREQ_V^I(A, B)$ expresses our intuitive ideas about the situation when A and B speak (in I and V) about the same thing (about the same state of affairs in I and V).

5. ABSTRACTION

The construction of states of affairs for given formula contains the application of three operation: abstraction, elimination and reduction. The operation of abstraction consists in application of certain form of the principle of abstraction defined for $I-V$-sentences.

We call the order of the $I-V$-terms in the following sequence

$$\text{(L)} \qquad v_V^I(P_1), \ldots, v_V^I(P_m), v_V^I(a_1), \ldots, v_V^I(a_n), v_V^I(x_1), \ldots, v_V^I(x_p), \ldots,$$

the lexicographical order of $I-V$-terms. Given an i-tuple of $I-V$-terms

$$v_V^I(A_1), \ldots, v_V^I(A_i),$$

the lexicographical permutation of the i-tuple mentioned is the i-tuple, in which each $I-V$-term on the left precedes in the lexicographical order (L) each member on the right. We shall designate this lexicographical permutation as follows:

$$v_V^I(A_{L_1}), \ldots, v_V^I(A_{L_i}).$$

The form of the abstraction principle for our purposes using the lexicographical permutation of $I-V$-terms is as follows:

Theorem 4. *If $\mathscr{A}$ is an $I-V$-sentence containing exactly the terms $v_V^I(A_1), \ldots, v_V^I(A_i)$* and possibly the term D^I, then

$$\mathscr{A} \equiv \langle D^I, v_V^I(A_{L_1}), \ldots, v_V^I(A_{L_i})\rangle \in \{\langle \beta, \alpha_{L_1}, \ldots, \alpha_{Li}\rangle : [\omega(\beta, \alpha_1, \ldots, \alpha_i) \to$$
$$\to \mathscr{A}(D^I/\beta, v_V^I(A_1)/\alpha_1, \ldots, v_V^I(A_i/\alpha_i)]\}$$

(*where the condition $\omega(\beta, \alpha_1, \ldots, \alpha_i)$ is the same as in the Note about Theorem* 1)

Definition 9. *If $\mathscr{A}$ is a $K-U$-sentence, then $ABS^L(\mathscr{A})$ is the result of application of the Theorem* 4 *on $\mathscr{A}$.*

Theorem 5. *$ABS^L(\mathscr{A}) \equiv \mathscr{A}$, in every I and V, where $\mathscr{A}$ is a $I-V$-sentence.*

Theorem 6. *If $\mathscr{A}$ and $\mathscr{B}$ are $K-U$-sentences containing the same $K-U$-terms $v_U^K(A_1), \ldots, v_U^K(A_i)$, then*

$$(\beta)(\alpha_1)\ldots(\alpha_i)\{\omega(\beta, \alpha_1, \ldots, \alpha_i) \to [\mathscr{A}(D^K/\beta, v_U^K(A_1)/\alpha_1, \ldots, v_U^K(A_i)/\alpha_i) \equiv$$
$$\equiv \mathscr{B}(D^K/\beta, v_U^K(A_1)/\alpha_1, \ldots, v_U^K(A_i)/\alpha_i)]\} \equiv$$
$$\equiv [\{\langle \beta, \alpha_1, \ldots, \alpha_i\rangle : \omega(\beta, \alpha_1, \ldots, \alpha_i) \,\&\, \mathscr{A}(D^K/\beta, v_U^K(A_1)/\alpha_1, \ldots, v_U^K(A_i)/\alpha_i)\} =$$
$$= \{\langle \beta, \alpha_1, \ldots, \alpha_i\rangle : \omega(\beta, \alpha_1, \ldots, \alpha_i) \,\&\, \mathscr{B}(D^K/\beta, v_U^K(A_1)/\alpha_1, \ldots, v_U^K(A_i)/\alpha_i)\}].$$

(This theorem is identical with Theorem 11 in [1].)

Theorem 7. *If* $\mathscr{A}$ *and* $\mathscr{B}$ *are* $K-U$*-sentences containing the same terms* $v_U^K(A_1), \ldots, v_U^K(A_i)$, *then*

$$(\beta)(\alpha_1)\ldots(\alpha_i)\{\omega(\beta, \alpha_1, \ldots, \alpha_i) \rightarrow [\mathscr{A}(D^K/\beta, v_U^K(A_1)/\alpha_1, \ldots, v_U^K(A_i)/\alpha_i) \rightarrow$$
$$\rightarrow \mathscr{B}(D^K/\beta, v_U^K(A_1)/\alpha_1, \ldots, v_U^K(A_i)/\alpha_i)]\} \equiv$$
$$\equiv [\{\langle\beta, \alpha_1, \ldots, \alpha_i\rangle : \omega(\beta, \alpha_1, \ldots, \alpha_i) \,\&\, \mathscr{A}(D^K/\beta, v_U^K(A_1)/\alpha_1, \ldots, v_U^K(A_i)/\alpha_i)\} \subset$$
$$\subset \{\langle\beta, \alpha_1, \ldots, \alpha_i\rangle : \omega(\beta, \alpha_1, \ldots, \alpha_i) \,\&\, \mathscr{B}(D^K/\beta, v_U^K(A_1)/\alpha_1, \ldots, v_U^K(A_i)/\alpha_i)\}] .$$

(This theorem follows from the principle of abstraction and the definition of inclusion.)

Theorem 8. *If* $\mathscr{A}$ *and* $\mathscr{B}$ *are* $K-U$*-sentences containing the same terms* $v_U^K(A_1), \ldots, v_U^K(A_i)$, *then*

$$(I)(V)\ \ (\mathscr{A} \equiv \mathscr{B})(K/I, U/V)] \equiv [\{\langle\beta, \alpha_1, \ldots, \alpha_i\rangle : \omega(\beta, \alpha_1, \ldots, \alpha_i) \,\&$$
$$\&\, \mathscr{A}(D^K/\beta, v_U^K(A_1)/\alpha_1, \ldots, v_U^K(A_i)/\alpha_i)\} =$$
$$= \{\langle\beta, \alpha_1, \ldots, \alpha_i\rangle : \omega(\beta, \alpha_1, \ldots, \alpha_i) \,\&\, \mathscr{B}(D^K/\beta, v_U^K(A_1)/\alpha_1, \ldots, v_U^K(A_i)/\alpha_i)\}] .$$

(This theorem corresponds to Theorem 12 in [1].)

6. REDUCTION

The operation of $I-V$-reduction is the same as in [1] (the following theorems are proved in [1]).

Definition 10. *If* $\mathscr{A}$ *and* $\mathscr{B}$ *are* $I-V$*-terms, then* $\mathscr{A}$ *is the* $I-V$*-reduced term of* $\mathscr{B}$, *iff* $\mathscr{A}$ *is the lexicographically first member of the set of* $I-V$*-terms having the same denotatum as* $\mathscr{B}$.

Definition 11. *If* $\mathscr{A}$ *is an* $I-V$*-sentence containing exactly* $I-V$*-terms* $v_V^I(A_1), \ldots, v_V^I(A_i)$ *and* $v_V^I(B_1), \ldots, v_V^I(B_i)$ *are their* $I-V$*-reduced terms, then the* $I-V$*-reduced form of* $\mathscr{A}$ *(abbreviated:* $R_V^I(\mathscr{A})$*) is the sentence*

$$\mathscr{A}(v_V^I(A_1)/v_V^I(B_1), \ldots, v_V^I(A_i)/v_V^I(B_i)) .$$

Theorem 9. *For every I and V:* $R_V^I(\mathscr{A}) \equiv \mathscr{A}$, *where* $\mathscr{A}$ *is an* $I-V$*-sentence.*

Definition 12. *I and V define an analogical reduction as K and U, iff for every terms A and B it holds:* $v_V^I(A)$ *is* $I-V$*-reduced term of* $v_V^I(B)$, *iff* $v_U^K(A)$ *is* $K-U$*-reduced term of* $v_U^K(B)$.

The abbreviation for the concept just defined is as follows: $R_V^I = R_V^I$.
We may express the same idea by a ML-condition as follows:

Theorem 10. $R_V^I = R_U^K$, *iff for every terms A and B it holds following*:

$$v_V^I(A) = v_V^I(B) \equiv v_U^K(A) = v_U^K(B).$$

Theorem 11. *If $\mathscr{A}$ and $\mathscr{B}$ are $I-V$-sentences, then*

$$R_V^I(\mathscr{A} \equiv \mathscr{B}) \equiv [R_V^I(\mathscr{A}) \equiv R_V^I(\mathscr{B})].$$

Theorem 12. *If $\mathscr{A}$ is a $K-U$-sentence, then*

$$(I)(V)[(R_U^K(\mathscr{A}))(K/I, U/V)] \equiv (I)(V)[(R_V^I = R_U^K) \to \mathscr{A}(K/I, U/V)].$$

Theorem 13. *If A and B are formulas of L and $\mathscr{A}$ and $\mathscr{B}$ are their $K-U$-translations, then*

$$+STREQ_U^K(A, B) \equiv (I)(V)[(R_V^I = R_U^K) \to (\mathscr{A} \equiv \mathscr{B})(K/I, U/V)].$$

7. ELIMINATION

The main difference from [1] consists in the fact that we define the elimination only as $ELIM_1$ in [1]; we do not consider the elimination of D^I.

Definition 13. *If $\mathscr{A}$ is a $K-U$-sentence and $\mathscr{B}$ is a $K-U$-term, then $\mathscr{A}$ contains essentially $\mathscr{B}$, iff* $(EI)(EV)[\mathscr{A} \not\equiv (\alpha)(\omega(D^K, \alpha) \to \mathscr{A}(\mathscr{B}/\alpha)(K/I, U/V))]$.

Definition 14. *If $\mathscr{A}$ is a $K-U$-sentence and $\mathscr{B}$ is a $K-U$-term, then $\mathscr{A}$ contains unessentially $\mathscr{B}$, iff $\mathscr{A}$ contains $\mathscr{B}$ but not essentially.*

Definition 15. *If $\mathscr{A}$ is a $K-U$-sentence containing unessentially exactly the $K-U$-terms $v_U^K(A_1), \ldots, v_U^K(A_i)$, then the eliminated form of $\mathscr{A}$ (abbreviated: $ELIM(\mathscr{A})$) is*

$$(\alpha_1)\ldots(\alpha_i)[\omega(D^K, \alpha_1, \ldots, \alpha_i) \to \mathscr{A}(v_U^K(A_1)/\alpha_1, \ldots, v_U^K(A_i)/\alpha_i)].$$

Theorem 14. *For every I and V*: $ELIM(\mathscr{A}) \equiv \mathscr{A}$, *where $\mathscr{A}$* is an *$I-V$-sentence.*

Under given I and V we can construct for every formula A of L a state of affairs assigned to this formula A in the interpretation I and valuation V (abbreviated: $\mathscr{S}_V^I(A)$) in the unique manner. Let $\mathscr{A}$ is the $I-V$-translation of A; first we transform $\mathscr{A}$ in the eliminated form $ELIM(\mathscr{A})$, secondly in the $I-V$-reduced form $R_V^I(ELIM(\mathscr{A}))$ and finally in the abstraction-form $ABS^L(R_V^I(ELIM(\mathscr{A})))$. This abstraction-form – let we designate it as $\alpha \in \beta$ – is uniquely determined and, therefore, the members α and β are uniquely determined, too. The state of affairs $\mathscr{S}_V^I(A)$ is the ordered pair $\langle \alpha, \beta \rangle$. In contrast with [1] the present construction assigns a state of affairs to an arbitrary formula of L (not to a synthetic formula only).

Definition 16. *If A is an L-formula and $\mathscr{A}$ is its $I-V$-translation, then $\mathscr{S}_V^I(A) = \langle \alpha, \beta \rangle$, if "$\alpha \in \beta$" is identical with "$ABS^L(R_V^I(ELIM(\mathscr{A})))$".*

The existence of states of affairs for non-synthetic formulas is now guaranteed by change of the procedure of elimination: we cannot eliminate the term D^I and, therefore, the abstraction is possible in every case.

As in [1] we can see the form of a state of affairs more detailed in the following theorem:

Theorem 15. *If A is an formula of L and $\mathscr{A}$ is its $I-V$-translation, then*

$$\mathscr{S}_V^I(A) = \langle \langle D^I, v_V^I(A_{L_1}), \ldots, v_V^I(A_{L_i}) \rangle , \\ \{ \langle \beta, \alpha_{L_1}, \ldots, \alpha_{L_i} \rangle : [\omega(\beta, \alpha_1, \ldots, \alpha_i) \, \& \\ \& \, R_V^I(ELIM(\mathscr{A})) \, (D^I/\beta, v_V^I(A_1)/\alpha_1, \ldots, v_V^I(A_i)/\alpha_i)]\} \rangle ,$$

where $v_V^I(A_1), \ldots, v_V^I(A_i)$ are all term resting in $R_V^I(ELIM(\mathscr{A}))$.

The theorem of adequacy holds for changed states of affairs without restriction to synthetic formulas only.

Theorem 16. *If A and B are arbitrary formulas of L, then*

$$STREQ_V^I(A, B) \equiv \mathscr{S}_V^I(A) = \mathscr{S}_V^I(B) .$$

The proof of this theorem is identical with the proof given in [1] except for the argument that for synthetic formulas there exists always an $(i + 1)$-tuple of the objects abstracted from $R_V^I(ELIM(\mathscr{A}))$ (where $\mathscr{A}$ is the $I-V$-translation of A). This argument is changed in the following way: in the new construction of states of affairs we can never eliminate the term D^I and, therefore, the $(i + 1)$-tuple mentioned above will exist in every case (for arbitrary formulas).

Now, we can define the fundamental concepts of the extensional semantics in terms of states of affairs. First, we must distinguish between "facts" and states of affairs which are not "facts". We introduce the predicate "... is a fact" (abbreviated: "$FACT(\dots)$") as follows:

Definition 17.

$FACT(\langle\alpha, \beta\rangle) \equiv (EI)(EV)(EA)(A$ is a formula & $\langle\alpha, \beta\rangle = \mathscr{S}_V^I(A)$ & $\alpha \in \beta)$;
(a pair $\langle\alpha, \beta\rangle$ is a fact, iff $\langle\alpha, \beta\rangle$ is the state of affairs assigned to some formula A in some interpretation I and in some valuation V and it holds that $\alpha \in \beta$).

From this definition it follows

Theorem 17. *If A is a formula and $\mathscr{S}_V^I(A) = \langle\alpha, \beta\rangle$, then $FACT(\mathscr{S}_V^I(A)) \equiv \alpha \in \beta$.*

It holds, furthermore, the following theorem:

Theorem 18. *If A is a formula and $\mathscr{A}$ is its $I-V$-translation, then*

$$FACT(\mathscr{S}_V^I(A)) \equiv ABS^L(R_V^I(ELIM(\mathscr{A}))) \equiv \mathscr{A}\,.$$

Proof. Let $\mathscr{S}_V^I(A) = \langle\alpha, \beta\rangle$. Then the sentence $ABS^L(R_V^I(ELIM(\mathscr{A})))$ is identical with "$\alpha \in \beta$". Because $FACT(\mathscr{S}_V^I(A)) \equiv \alpha \in \beta$ (by Theorem 17) it holds that $FACT(\mathscr{S}_V^I(A))$ is equivalent with $ABS^L(R_V^I(ELIM(\mathscr{A})))$ and also with $\mathscr{A}$ (Theorems 5, 9 and 14).

Now, we introduce a relation defined on formulas, interpretations and valuations "A is satisfied by V in I" (abbreviated: $SAT^I(A, V)$).

Definition 18. *If A is a formula, then $SAT^I(A, V) \equiv FACT(\mathscr{S}_V^I(A))$.*

In [1] the concept of satisfaction was defined in the usual manner on the ground of the function assigning truth-values to formulas. But in the present development of semantics we do not dispose of the concept of truth-value assignment. We define the satisfaction of a formula A in I and V by saying that the state of affairs assigned to A in I and V is a fact. This definition is a new one and we must first prove its adequacy, i.e. we must prove that the new concept SAT has (extensionally) the same properties as the customary concept of satisfaction.

First, we must prove an auxiliary theorem.

Theorem 19. *If A is a formula and $\mathscr{A}$ is its $I-V$-translation, then $(\alpha_i)(\alpha_i \in D^I \rightarrow \rightarrow \mathscr{A}(v_V^I(x_i)/\alpha_i) \equiv \mathscr{A}^U$ for every valuation U^I differing from V at most in the value for x_i; where $\mathscr{A}^U$ is an $I-U$-translation of A (therefore: "$\mathscr{A}^U$" = "$\mathscr{A}(V/U)$").*

Proof. The expression on the left-hand side of the equivalence means that $\mathscr{A}(v_V^I(x_i)/\alpha_i)$ holds for every member α_i of D^I. All valuations U have as values for x_i exactly all possible members of D^I (on the other places they have the same value as V). Therefore, the statement that $\mathscr{A}^U$ holds under all valuations U^I means exactly the same fact as the statement that $\mathscr{A}(v_V^I(x_i)/\alpha_i)$ holds for every member α_i of D^I.

Now, we can prove the theorem about the adequacy of the concept SAT.

Theorem 20.

(1) *If A is a k-ary predicate and $A_1, \ldots, A_i$ are individual terms, then*

$$SAT^I(A(A_1, \ldots, A_i), V) \equiv \langle v_V^I(A_1), \ldots, v_V^I(A_i)\rangle \in v_V^I(A)\ ;$$

(2) *if A and B are individual terms, then $SAT^I(A = B, V) \equiv v_V^I(A) = v_V^I(B)$;*

(3) *if A is a formula, then $SAT^I(\neg A, V) \equiv \sim SAT^I(A, V)$;*

(4) *if A and B are formulas, then $SAT^I(A\,.\,B, V) \equiv SAT^I(A, V)\,\&\,SAT^I(B, V)$;*

(5) *if A and B are formulas, then $SAT^I(A + B, V) \equiv SAT^I(A, V) \vee SAT^I(B, V)$;*

(6) *if A and B are formulas, then $SAT^I(A \Rightarrow B, V) \equiv SAT^I(A, V) \rightarrow SAT^I(B, V)$;*

(7) *if A and B are formulas, then $SAT^I(A \Leftrightarrow B, V) \equiv SAT^I(A, V) \equiv SAT^I(B, V)$;*

(8a) *if A is a formula containing a free variable x_i, then*
$SAT^I((\forall x_i)\, A, V) \equiv SAT^I(A, U)$ for every valuation U^I differing from V at most in the value for x_i;

(8b) *if A does not contain x_i, then $SAT^I((\forall x_i)\, A, V) \equiv SAT^I(A, V)$;*

(9a) *if A is a formula containing a free variable x_i, then*
$SAT^I((\exists x_i)\, A, V) \equiv SAT^I(A, U)$ for some valuation U^I differing from V at most in the value for x_i;

(9b) *if A does not contain x_i, then $SAT^I((\exists x_i)\, A, V) \equiv SAT^I(A, V)$.*

Proof. We shall prove the cases (1)' – (4), (8a) and (8b) only. The other cases are dependent on the cases mentioned.

(1) $SAT^I(A(A_1, \ldots, A_i), V) \equiv FACT(\mathscr{S}_V^I(A(A_1, \ldots, A_i)))$. The right-hand side of this equivalence is by Theorem 18 equivalent with the $I-V$-translation $\langle v_V^I(A_1), \ldots, v_V^I(A_i)\rangle \in v_V^I(A)$.

(2) $SAT^I(A = B, V) \equiv FACT(\mathscr{S}_V^I(A = B))$. The right-hand side is by Theorem 18 equivalent with the $I-V$-translation $v_V^I(A) = v_V^I(B)$.

(3) $SAT^I(\neg A, V) \equiv FACT(\mathscr{S}^I_V(\neg A))$. Let $\mathscr{A}$ be the $I-V$-translation of A. The $I-V$-translation of $\neg A$ is then $\sim\mathscr{A}$ and, therefore, the right-hand side of the equivalence is by Theorem 18 equivalent with $\sim\mathscr{A}$. This sentence is (by Theorem 18 again) equivalent with $\sim FACT(\mathscr{S}^I_V(A))$ and also with $\sim SAT(A, V)$ (by Definition 18).

(4) $SAT^I(A \,.\, B, V) \equiv FACT(\mathscr{S}^I_V(A \,.\, B))$. Let again $\mathscr{A}$ and $\mathscr{B}$ be the $I-V$-translations of A and B. The $I-V$-translation of $A \,.\, B$ is then $\mathscr{A} \,\&\, \mathscr{B}$ and, therefore, the right-hand side of the equivalence is by Theorem 18 equivalent with $\mathscr{A} \,\&\, \mathscr{B}$ and, by Theorem 18 again, is equivalent with $FACT(\mathscr{S}^I_V(A)) \,\&\, FACT(\mathscr{S}^I_V(B))$. By Definition 18 we have, therefore, $SAT^I(A, V) \,\&\, SAT^I(B, V)$.

(8a) $SAT^I((\forall x_i)\, A, V) \equiv FACT(\mathscr{S}^I_V((\forall x_i)\, A))$. The right-hand side is by Theorem 18 equivalent with the $I-V$-translation

$$(\alpha_i)\,(\alpha_i \in D^{I'} \rightarrow \mathscr{A}(v^I_V(x_i)/\alpha_i))\,.$$

By auxiliary Theorem 19 this expression is equivalent with the statement that $\mathscr{A}^U$ holds for every valuation U^I differing from V at most in the value for x_i. This statement means (by Theorem 18) that $FACT(\mathscr{S}^I_U(A))$ for every U^I mentioned and by Definition 18 that $SAT^I(A, U)$ for every U^I mentioned.

(8b) In the case that A does not contain x_i as the free variable the formula $(\forall x_i)\, A$ is logically equivalent with A and, therefore, $SAT^I((\forall x_i)\, A, V) \equiv SAT^I(A, V)$.

The proof of the cases (5), (6), (7) and (9), (9b) is evident by the fact that the formulas having the form $A + B$, $A \Rightarrow B$, $A \Leftrightarrow B$, $(\exists x_i)\, A$ are equivalent with some formulas containing the terms $\neg$, $\cdot$, $(\forall)$ only.

The concept SAT defined in Definition 18 is, therefore, adequate to the customary concept of satisfaction and it may be the foundation for development of semantics.

Definition 19. *If A and B are formulas, then*

A is true in I $\equiv (V) FACT(\mathscr{S}^I_V(A))$ $\qquad (VER^I(A))$;

A is false in I $\equiv (V) \sim FACT(\mathscr{S}^I_V(A))$ $\qquad (FALS^I(A))$;

A is logically true $\equiv (I)\,(V)\, FACT(\mathscr{S}^I_V(A))$ $\qquad (L\text{-}VER(A))$;

A is logically false $\equiv (I)\,(V) \sim FACT(\mathscr{S}^I_V(A))$ $\qquad (L\text{-}FALS(A))$;

A and B are equivalent in I $\equiv (V)\,(FACT(\mathscr{S}^I_V(A)) \equiv FACT(\mathscr{S}^I_V(B)))$ $\qquad (EQ^I(A, B))$;

A and B are logically equivalent $\equiv (I)\,(V)\,(FACT(\mathscr{S}^I_V(A)) \equiv FACT(\mathscr{S}^I_V(B)))$

$(L\text{-}EQ(A, B))$.

In the preceding article [1] there was no possibility to define the concepts of logical truth and of logical falsehood in terms of states of affairs. The logically true and logically false formulas have assigned no states of affairs. In the present construction we do not eliminate the term D^I for the domain of I and, therefore, the operation of abstraction is always applicable. We obtain a state of affairs in every case. Thus, Definition 19 is possible for arbitrary formulas.

Let us look in detail at the states of affairs assigned to analytic formulas, i.e. to logically true and logically false formulas. First we prove an auxiliary theorem about the eliminated form of analytic formulas.

Theorem 21. *If $L\text{-}VER(A)$ or $L\text{-}FALS(A)$ and $\mathscr{A}$ is the $I-V$-translation of A in an arbitrary I and V, then $ELIM(\mathscr{A})$ contains no $I-V$-terms, but it contains D^I.*

Proof. If $L\text{-}VER(A)$ or $L\text{-}FALS(A)$, then $(I)(V)\,FACT(\mathscr{S}^I_V(A))$ or $(I)(V) \sim \sim FACT(\mathscr{S}^I_V(A))$. This means by Theorem 18: $(I)(V)\,\mathscr{A}$ or $(I)(V) \sim \mathscr{A}$. From Theorem 2 we can see that the sentence $\mathscr{A}$ and the sentence $\sim\mathscr{A}$ contains no $I-V$-term essentially. Therefore, all terms must be eliminated by $ELIM$ and the forms $ELIM(\mathscr{A})$ and $ELIM(\sim\mathscr{A})$ contain no $I-V$-terms. But they must contain the term D^I because: 1) $\mathscr{A}$ or $\sim\mathscr{A}$ contains D^I or 2) it contains at least an $I-V$-term. If it contains D^I, then $ELIM(\mathscr{A})$ or $ELIM(\sim\mathscr{A})$ contains D^I too. If $\mathscr{A}$ or $\sim\mathscr{A}$ does not contain D^I, then D^I appears in $ELIM(\mathscr{A})$ or $ELIM(\sim\mathscr{A})$ by elimination of some $I-V$-terms (see the definition of elimination).

Now, we prove two theorems about the form of states of affairs assigned to analytic formulas.

Theorem 22. *If A is a formula, then $L\text{-}VER(A) \equiv (I)(V)\,[\mathscr{S}^I_V(A) = \langle D^I, U\rangle]$.*

Proof. (I) Let (1) $L\text{-}VER(A)$, i.e. by Definition 19 (2) $(I)(V)\,FACT(\mathscr{S}^I_V(A))$. Furthermore, let $\mathscr{A}$ be the $I-V$-translation of A. From (1) it follows (by Theorem 21) that $ELIM(\mathscr{A})$ and, therefore, also $R^I_V(ELIM(\mathscr{A}))$ contains only D^I and contains no $I-V$-term. This means that the abstraction-form of the sentence mentioned will contain on the left-hand side the term D^I only. The form of the state of affairs will be as follows:

$$(3) \qquad \langle D^I, \{\beta : \beta \in U \;\&\; R^I_V(ELIM(\mathscr{A}))\,(D^I/\beta)\}\rangle\,.$$

Now, we must ascertain what class is on the right-hand side of (3). We can distribute the abstraction-operator between the members of conjuction:

$$(4) \qquad \{\beta : \beta \in U\} \cap \{\beta : R^I_V(ELIM(\mathscr{A}))\,(D^I/\beta)\}\,.$$

The left-hand side of the intersection is equal with U; the content of the right-hand side can be ascertained starting from (2). The assumption (2) means by Theorem 18 that $(I)(V)\mathscr{A}$ and, therefore, by Theorems 9 and 14 that

$$(5) \qquad (I)(V)\, R_V^I(ELIM(\mathscr{A}))\,.$$

In accordance with the above ascertainment the sentence (5) contains only D^I and it is by Theorem 1 equivalent with

$$(6) \qquad (\beta)\,[\beta \in U \rightarrow R_V^I(ELIM(\mathscr{A}))\,(D^I/\beta)]\,.$$

Furthermore, by Theorem 7, it follows from (6):

$$(7) \qquad \{\beta : \beta \in U\} \subset \{\beta : R_V^I(ELIM(\mathscr{A}))\,(D^I/\beta)\}\,.$$

Now, the left-hand side of (7) is U, the right-hand side is identical with the second member of the intersection (4). The inclusion (7) follows from the assumption (1) and the question what is the class on the right-hand side of (7) can be solved by the assertainment what classes α satisfy the inclusion $U \subset \alpha$. Evidently, there are only two classes α satisfying the inclusion mentioned: the class U and the universal class V. Therefore, the right-hand side of (4) can be identical either with U or with V. In the first case (4) is $U \cap U$, in the second case $U \cap V$. We can see that it follows from the assumption that the class (4) is in every case identical with U. From the fact just mentioned and from (3) we can conclude that the state of affairs of a logically true formula must be identical with $\langle D^I, U\rangle$ in every I and V.

(II) It holds also the converse. If $(I)(V)[\mathscr{S}_V^I(A) = \langle D^I, U\rangle]$, then evidently $(I)(V)\,FACT(\mathscr{S}_V^I(A))$, because $D^I \in U$ in every I and V. This means that $L\text{-}VER(A)$.

Theorem 23. *If A is a formula, then* $L\text{-}FALS(A) \equiv (I)(V)[\mathscr{S}_V^I(A) = \langle D^I, \emptyset\rangle]$.

Proof. (I) We suppose (1) $L\text{-}FALS(A)$, i.e. (2) $(I)(V) \sim FACT(\mathscr{S}_V^I(A))$. Furthermore, let $\mathscr{A}$ be the $I-V$-translation of A. Analogically as in proof of Theorem 22 we can see that $\mathscr{S}_V^I(A)$ is

$$(3) \qquad \langle D^I, \{\beta : \beta \in U \;\&\; R_V^I(ELIM(\mathscr{A}))\,(D^I/\beta)\}\rangle\,.$$

We must ascertain what class is the second member of (3). We transform again the expression for this member as follows:

$$(4) \qquad U \cap \{\beta : R_V^I(ELIM(\mathscr{A}))\,(D^I/\beta)\}\,.$$

It follows from the assumption by Theorem 18 that it holds

$$(5) \qquad (I)(V) \sim R_V^I(ELIM(\mathscr{A}))\,.$$

Analogically as above we have

$$(6) \qquad (\beta)\,[\beta \in U \rightarrow \sim R_V^I(ELIM(\mathscr{A}))\,(D^I/\beta)]$$

and equivalently (by transposition):

$$(7) \qquad (\beta)\,[R_V^I(ELIM(\mathscr{A}))\,(D^I/\beta) \rightarrow \sim \beta \in U]\,.$$

Furthermore, the following inclusion follows from (7) by Theorem 7:

$$(8) \qquad \{\beta : R_V^I(ELIM(\mathscr{A}))\,(D^I/\beta)\} \subset \{\beta : \sim\beta \in U\}\,.$$

The class on the right-hand side is evidently $\{\emptyset\}$. What class can be α in the inclusion $\alpha \subset \{\emptyset\}$? Evidently α can be either $\emptyset$ or $\{\emptyset\}$. Therefore, if (4) is either $U \cap \emptyset$ or $U \cap \{\emptyset\}$, the result is $\emptyset$. Therefore, the first element of $\mathscr{S}_V^I(A)$ is D^I and the second one is $\emptyset$ in every I and V.

(II) When $(I)\,(V)\,[\mathscr{S}_V^I(A) = \langle D^I, \emptyset\rangle]$, then $(I)\,(V) \sim FACT(\mathscr{S}_V^I(A))$, because $D^I \notin \emptyset$ in every I and V. This means that $L\text{-}FALS(A)$.

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