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A supplement to Gottwald's note on fuzzy cardinals


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We supplement the review of fuzzy cardinality definitions placed in [3]. To be exact, we present approaches in which cardinality of a finite fuzzy subset is expressed by a fuzzy natural number and indicate the most appropriate one.

S. Gottwald placed in [3] a comparative review of approaches to the problem how to define fuzzy cardinality, i.e. how to count elements of a universe which are in its fuzzy subset. In accordance with the concepts presented in [3], cardinality of a fuzzy subset was defined either as a non-negative real number or as a family of usual cardinals. In this note we shall present and compare such approaches in which cardinality of finite fuzzy subset is expressed by means of a fuzzy number. To this end, we must introduce some notation and terminology.

Throughout this note, by a fuzzy subset $A$ of some fixed universal set $U$ we shall mean a function $A : U \to I$, where $I := [0, 1]$ with $=$ standing for “equals by the definition”. Membership grade of an element $x \in U$ in $A$ will be denoted by $A(x)$. The classical subset $\{x : A(x) > 0\}$ will be called support of $A$ and denoted $\text{supp}(A)$. If support of a fuzzy subset is finite, then that subset is called finite, too. Throughout the paper we shall assume that $A$ is finite and $\text{card}(\text{supp}(A)) = n$, where $\text{card}(M)$ denotes the usual cardinality of a classical subset $M$ of $U$. The subset $A_i := \{x : A(x) \geq i\}$, where $i \in I_0$ and $I_0 := (0, 1]$, is called $i$-level set of $A$. The sequence $a_0 \geq a_1 \geq a_2 \geq \ldots \geq a_n > a_{n+1} = a_{n+2} = \ldots$ is defined in the following way: $a_0 := 1$, $a_i (1 \leq i \leq n)$ denotes the $i$th element in descending sequence consisting of positive membership grades in $A$, $a_i := 0$ for $i > n$.

Let $N := \{0, 1, 2, \ldots\}$. If $F : N \to I$ (i.e. $U := N$), the $F$ will be called fuzzy natural number (in short, fn-number). $F$ is said to be convex iff $F(j) \geq \min(F(i), F(j+1))$ for all $i, j \in N$. Each fn-number $F$ has a decreasing sequence of positive membership grades $a_i (1 \leq i \leq n)$.

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$F(k)$ for each triplet $i \leq j \leq k$ (cf. [4]). Let $\oplus$ denote addition of fn-numbers. Then the fn-number $F \oplus G$ is defined by membership grades

$$(F \oplus G)(k) := \sup_{i+j=k} \min (F(i), G(j)).$$

As a chronologically first fuzzy approach to cardinality of finite fuzzy subsets, we shall consider the fn-number $FG\text{Count}^\circ$ (see [1, 7]) with membership grades

$$FG\text{Count}^\circ(k) := \begin{cases} \max \{ t \in I_0 : \text{card} (A_t) = k \} , \\ 0 \text{ if card} (A_t) < k \text{ for each } t . \end{cases}$$

The values $FG\text{Count}^\circ(k)$ may be considered degrees to which cardinality of $A$ equals $k$. One can easily notice (see [1]) that $FG\text{Count}^\circ(k)$

(a) is always normalized, i.e. there exists a natural number $h$ such that $FG\text{Count}^\circ(h) = 1$,

(b) is strictly decreasing on its support,

(c) is a non-convex fn-number,

(d) does not fulfill the additivity property

$$FG\text{Count}^\circ \oplus FG\text{Count}^\circ = FG\text{Count}^\circ_{nB} \oplus FG\text{Count}^\circ_{uB},$$

where $A \cap B$ and $A \cup B$ denote (resp.) intersection and union of $A$ and $B$, i.e.

$$(A \cap B)(x) := \min (A(x), B(x)), (A \cup B)(x) := \max (A(x), B(x)).$$

In order to avoid the lack of convexity, an important modification of the definition of $FG\text{Count}^\circ$ was proposed in [2] and [8]. As a consequence, we get then a new fn-number defining fuzzy cardinality, namely the $FG\text{Count}_A$ where

$$FG\text{Count}_A(k) := \begin{cases} \max \{ t \in I_0 : \text{card} (A_t) \geq k \} , \\ 0 \text{ if card} (A_t) < k \text{ for each } t . \end{cases}$$

Let $T$ be a finite fn-number such that $T(0) = g_0$, $T(1) = g_1$, ..., $T(s) = g_s$, and $T(r) = 0$ for $r = s + 1, s + 2, ...$. In such a case we shall use the following "vectorial" notation $T = (g_0, g_1, ..., g_s)$.

It is easy to prove (see e.g. [2], [6], [8]) that the following propositions are valid:

(a) $FG\text{Count}_A(k) = \max_{j \leq k} FG\text{Count}^\circ(j)$,

(b) $FG\text{Count}_A = (a_0, a_1, ..., a_s)$. Hence $FG\text{Count}_A$ is convex.

(c) If $A \subseteq B$, then $FG\text{Count}_A \subseteq FG\text{Count}_B$ (monotonicity).

Remark. $Y \subseteq Z := (Y(x) \leq Z(x) \text{ for all } x \in U)$.

(d) $FG\text{Count}_A \oplus FG\text{Count}_B = FG\text{Count}_{A \cap B} \oplus FG\text{Count}_{A \cup B}$ (additivity). 

Let $D$ denote a classical $n$-element subset of $U$. Then, contrary to expectation, we get $FG\text{Count}_D = (1, 1, ..., 1)$ with support of $FG\text{Count}_D$ consisting of $n + 1$ elements. This result is sensible provided that $FG\text{Count}_A(k)$ defines degree to which $A$ has at least rather than exactly $k$ elements. Thus $FG\text{Count}_A$ as definition of fuzzy cardinality, is unsatisfactory. Namely, for classical subsets it does not collapse to
usual cardinal number. That is why a new definition of fuzzy cardinality was introduced in [2]. To be exact, the new definition is again a simple modification of the previous one.

Let $\mathcal{F}_k(A)$ denote the family of $k$-element classical subsets of $\mathcal{U}$ containing $A$. Then fuzzy cardinality of $A$ will be defined by the finite fn-number $\text{Crd}_A(k)$ with membership grades

$$\text{Crd}_A(k) := \max \left\{ \min_{x \in A} A(x) \right\},$$

where $A$ is empty, what implies $\mathcal{F}_0(A) = \{\emptyset\}$, we additionally put $\min \{A(x) := 1\}$.

One can consider $\text{Crd}_A(k)$ to be degree to which cardinality of $A$ equals $k$. It is easy to verify that (cf. [2], [6])

(a) $\text{Crd}_A = (0, 0, \ldots, 0, 1, a_{m+1}, a_{m+2}, \ldots, a_m)$, where $m := \text{card}(A)$ and the constant sequence composed of zeros is $m$-element one. Thus $\text{Crd}_A$ is always convex.

(b) $\text{Crd}_A = (0, \ldots, 0, 1)$ with the figure one placed at the $(n + 1)$th position and $D$ as previously.

(c) $\text{Crd}_A \oplus \text{Crd}_B = \text{Crd}_{A \cup B} \oplus \text{Crd}_B$.

(d) $\text{Crd}_A = \text{FGCount}_A$ iff $\text{card}(A) = 0$.

Unfortunately, the monotonicity does not hold for $\text{Crd}$-cardinality. But it is quite obvious that property (b) excludes, in principle, monotonicity. On the other hand, property (b) is, from the practical as well set-theoretical points of view, more important than monotonicity.

This is well-known that the theory of fuzzy subsets is closely connected with the Łukasiewicz many-valued logic (see e.g. [5]). Indeed, it suffices to interpret each membership grade $A(x)$ as representing the truth-value of the statement "$x$ is in $A". Therefore, the next approach is based on that logic.

Let $\mathcal{P}_k(A)$ denote the family of all the $k$-element classical subsets of $\text{supp}(A)$. Moreover, let $p \rightarrow q := \min(1, 1 - p + q)$ (Łukasiewicz implication operator) and $\Leftrightarrow q := \min(p \rightarrow q, q \rightarrow p)$ for $p, q \in I$. Then $\text{deg}(R, S) := \inf_{x \in U} (R(x) \Leftrightarrow S(x))$ for arbitrary fuzzy subsets $R$ and $S$ of $U$. One can consider $\text{deg}(R, S)$ to be degree to which $R$ equals $S$. Let us define finite fn-number $\text{Cd}_A(k)$ by means of membership grades

$$\text{Cd}_A(k) := \max \left\{ \text{deg}(A, Y) \right\} : Y \in \mathcal{P}_k(A),$$

$$\left\{ \begin{array}{ll}
0 & \text{if } \mathcal{P}_k(A) \text{ is empty.}
\end{array} \right.$$
(c) \(Cd_A = (1 - a_1, 1 - a_2, \ldots, 1 - a_p, a_{p+1}, \ldots, a_n)\), where \(p := \min \{i : a_i + a_{i+1} \leq 1\}\). Hence \(Cd_A\) is always convex.

(d) At most one cardinal number is "favoured", i.e., there exists at most one natural number \(k_f\) such that \(Cd_A(k_f) > 0.5\).

(e) \(FGCount^A = 2Cd_{0,5A}\), where membership grades in \(0.5A\) and \(2Cd_{0,5A}\) are defined as follows: \((0.5A)(x) := 0.5A(x)\) and \((2Cd_{0,5A})(k) := \min (1, 2Cd_{0,5A}(k))\).

(f) \(Cd_A \oplus Cd_B = Cd_{A \cup B} \oplus Cd_{A \cap B}\).

(g) Let \(A'\) denote the complement of \(A\), i.e., \(A'(x) := 1 - A(x)\). If \(U\) is finite and \(\text{card}(U) = m\), then \(Cd_A(j) = Cd_B(m - j)\) for \(j = 0, 1, \ldots, m\).

One can easily give counterexamples that both the important properties (d) and (g) do not hold for \(FGCount^A\) and \(Cr_{m, A}\). Obviously (g) is a counterpart of the elementary law \(\text{card}(D^c) = m - \text{card}(D)\), where \(D\) denotes now a classical subset of \(m\)-element universe.

To summarize the discussion, it seems to be more suitable to define cardinality of a finite fuzzy subset as a fuzzy natural rather than positive real number (or a family consisting of usual cardinals). Then the fn-number \(Cd_A\) is, from the set-theoretical point of view, defined in a most natural way and fulfills many natural postulates (see e.g. properties (b), (f), (g)) except the monotonicity (what is, however, explicable).

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REFERENCES


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