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A Note on Fuzzy Cardinals

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We compare different notions of fuzzy cardinals and discuss which is the most appropriate one.

In last years, a variety of papers on fuzzy sets and other fuzzy topics was concerned with set-algebraic operations for and properties of fuzzy sets. However, only few remarks are devoted to fuzzy cardinals.

In classical set theory the cardinality of a set is a measure of its size or "power". In the fuzzy case one has to differentiate: there are measures of fuzziness and measures of power.

Here measures of fuzziness are not our main concern. The interested reader may consult e.g. [1], [2], [5], [9].

Fuzzy cardinals as measures of power of fuzzy sets are considered e.g. in [2], [3], and [6]. To describe and compare these definitions needs some notation.

A fuzzy set $A$ over some universe of discourse $X$ is a function $A : X \rightarrow [0, 1]$. Instead of $A(x)$ for $x \in X$ we write also $x \in A$ for this membership value of $x$ in $A$.

The universe of discourse $X$ shall be fixed throughout the paper. By $\mathcal{F}(X)$ we denote the class of all fuzzy sets over $X$; for every $A \in \mathcal{F}(X)$, the support $|A|$ of $A$ is the classical set

$$|A| = \{x \in X | (x \in A) \neq 0\}.$$ 

As a first, but very rough measure of power for fuzzy sets one can consider for each $A \in \mathcal{F}(X)$

$$\text{card}_a A = \overline{|A|},$$

with $\overline{M}$ for the classical cardinality of the classical set $M$.

For fuzzy sets $A$ with finite support $|A|$ one has in the book [6] of A. Kaufmann as further cardinalities for fuzzy sets
\[
\text{card}_1 A = \sum_{x \in A} A(x) = \sum_{x \in A} (x \in A), \\
\text{card}_2 A = \sum_{x \in A} A^2(x) = \sum_{x \in A} (x \in A)^2.
\]

A. De Luca and S. Termini [2] consider \text{card}_1 A also for fuzzy sets \(A\) with denumerable support, in which case \(\sum_{x \in A} A(x)\) can be a divergent series in the sense of analysis; but in case of convergence it is absolutely convergent.

To explain also the essential points of the definition of fuzzy cardinals in the authors paper [3], we introduce for every \(A \in \mathcal{F}(X)\) and every \(0 \leq i \in [0, 1]\) the level sets
\[
A^i = \{x \in X \mid (x \in A) = i\},
\]
which themselves are classical sets. Furthermore, put \(W^+ = (0, 1]\). Obviously, every fuzzy set \(A\) can be characterized by the family \((A^i)_{i \in W^+}\) of its level sets.

Now, [3] leads to the definition
\[
\text{card}_A = \text{card}_{W^+}(A^i),
\]
which is independent of the cardinality of \(|A|\). Hence, \(\text{card}_A\) is a family of usual cardinals of usual sets.

It is easy to see that, given \(\text{card}_A\), one can get any one of \(\text{card}_A\) for \(k = 0, 1, 2\). Put always \(a_i = A^i\). Then clearly
\[
\text{card}_0 A = \sum_{i \in W^+} a_i
\]
with summation understood as usual addition of cardinals. In case of a finite support \(|A|\) there is a finite subset \(I = \{i_1, \ldots, i_k\} \subseteq W^+\) such that: \(a_i \neq 0\) iff \(i \in I\). Furthermore, with the finite cardinals as the natural numbers, in this case each of \(a_i\) is a natural number. Hence now
\[
\text{card}_1 A = \sum_{i \in I} i \cdot a_i, \\
\text{card}_2 A = \sum_{i \in I} i^2 \cdot a_i
\]
for Kaufmann’s [6] notions of fuzzy cardinals. Because of \(a_i = 0\) if \(i \in W^+ \setminus I\), we write by abuse of language
\[
\text{card}_j A = \sum_{i \in W^+} i^j \cdot a_i
\]
for \(j = 1, 2\). To do the same thing with denumerable supports as de Luca/Termini [2], we have to add \(\infty\) as a “real”, which can be done e.g. as sketched in [4] (giving \(\infty\) already as an “integer”). Now, there exists a countable subset \(I = \{i_1, i_2, i_3, \ldots\} \subseteq W^+\) such that \(a_i = 0\) for \(i \in W^+ \setminus I\), and [2] leads to
\[ \text{card}_1 A = \sum_{i=1}^{n} l(i, a_i) \]

(a\text{, always a natural number or } \infty). \text{ Again by abuse of language we can write:}

\[ \text{card}_A = \sum_{i \in \mathbb{W}} l(i, a_i) \cdot \]

In the same way it is possible to understand the entropy \( d(A) \) of a fuzzy set \( A \) (cf. [2]), and also other measures of fuzziness (cf. [5]). In general, the structure of such definitions is

\[ f(A) = O(\text{card}_A) \]

\( A \) any fuzzy set, \( O \) some operator.

Hence, to choose \( \text{card}_A \) as the fuzzy cardinality of a fuzzy set \( A \in \mathcal{F}(X) \) seems to be the most promising variant. The essential idea behind that definition is also independent of the choice of the set \([0, 1]\) as set of generalized membership grades — it does work equally well also in the case of L-fuzzy sets (cf. e.g. [8]). Furthermore, almost the same idea applied to the set \( \mathbb{W} = \{0, 1/2, 1\} \) as set of membership grades was used by D. Klaua [7] to give a set-theoretical construction of interval numbers.

As a further advantage, from the set-theoretical point of view adopted in [3], \( \text{card}_A \) is the result of a fuzzification of the usual definition of cardinals in any one of the standard systems of set theory.

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REFERENCES


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