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ON DISCRETE CONTROL PROBLEMS HAVING A MINMAX TYPE OBJECTIVE FUNCTIONAL*

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Based on recent results for static minmax problems with constraints, a set of necessary conditions is presented for a class of discrete control problems with an objective functional of the minmax type. Two cases are considered which have either a finite or an infinite set of possible objective functionals. In both cases, usual constraints on control and/or state may be present.

1. INTRODUCTION

Recently, several results appeared in the area of mathematical programming which took into the account the minmax type objective function and also various constraints, especially those of equality and inequality type [1]–[4]. On the other hand, the present formulation of necessary optimality conditions for these so-called static minmax problems also enables one to include such constraints as are given implicitly by their "conical approximation" [5]–[7]. However, such general formulation will not be pursued in this contribution. The corresponding type of mathematical programming problems was studied earlier in [8] in fairly general setting, but the given conditions were not worked out in such detail as in [1]–[4].

If now the problem is to study discrete optimal control systems with a minmax type objective functional one can easily envision that the above mentioned results make it possible to deal with this type of optimization problem in a straightforward way. Namely, any discrete control problem is, after all, a mathematical programming one, as often used in [5]–[7] when studying classical discrete control problems. And this is also the case of a minmax type objective functional.

For convenience, the basic results for static minmax problems are summarized in the next section. These results are then applied to a discrete control problem with

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a minmax type objective functional. For the sake of simplicity only the so-called explicit case [7] of state and state-dependent control constraints, given as a system of equalities and inequalities, is studied in detail. More involved case having the constraining sets of a general structure will be described elsewhere. To make a comparison with the case of a classical objective functional simpler, the notation of [7] is preserved when convenient. If not otherwise stated all vectors are considered as column-vectors, however, gradients of various functions are treated always as row-vectors to be able to use a more efficient matrix notation. Subscripts will be occasionally used to indicate the respective partial derivative.

2. STATIC MINMAX PROBLEMS

The aim is to minimize the function

$$F(z) = \sup_{y \in Y} \Phi(z, y) \tag{1}$$

subject to the constraints

$$h^i(z) = 0 \,, \quad i = 1, \ldots, p \,, \quad g^i(z) \leqq 0 \,, \quad i = 1, \ldots, q \,, \tag{2}$$

where $\Phi : E^n \times Y \to E^1$, h^i and g^i are the real functions on E^n, and Y is a compact metric space, in general, representing an index set. All functions are assumed to be continuously differentiable. Then one can alternatively replace "sup" by "max" in (1).

The inclusion of equality type constraints into our formulation of a static minmax problem is important having in mind later application to discrete control problems. In fact, the system dynamics are almost exclusively given as a set of equations which, in turn, must be treated as constraints in the resulting mathematical programming problem. One has the following result [3]–[4].

Theorem 1. Let $\hat{z}$ be a solution to the static minmax problem (1)–(2). Then there exist multipliers $\chi_1, \ldots, \chi_p$, $\sigma_1, \ldots, \sigma_q$, with $\sigma_i \leqq 0$, $i = 1, \ldots, q$, and vectors $y^1, \ldots, y^r \in Y(\hat{z}) = \{y \in Y \mid \Phi(\hat{z}, y) = \sup_{w \in Y} \Phi(\hat{z}, w)\}$, $1 \leqq r \leqq n + 1$, together with multipliers $\mu_1, \ldots, \mu_r$, with $\mu_i \leqq 0$, $i = 1, \ldots, r$, such that

$$\sum_{i=1}^{r} \mu_i \, \Phi_z(\hat{z}, y^i) + \sum_{i=1}^{p} \chi_i \, h^i_z(\hat{z}) + \sum_{i=1}^{q} \sigma_i \, g^i_z(\hat{z}) = 0 \,, \tag{3}$$

$$\sigma_i \, g^i(\hat{z}) = 0 \,, \quad i = 1, \ldots, q \,. \tag{4}$$

Furthermore, if p' is the number of nonzero χ_i, q' the number of nonzero σ_i, and r' the number of nonzero μ_i, then

$$1 \leqq p' + q' + r' \leqq n + 1 \,. \tag{5}$$

For practical reasons, it is sometimes convenient to specify this theorem for the case of a finite index set Y. To do this assume that

(6) $$F(z) = \max_{1 \leqq i \leqq r} \Phi^i(z) .$$

Similar to the inclusion of a finite number of inequality constraints in mathematical programming (complementary slackness condition), one obtains from Theorem 1 the following corollary.

Corollary 1. Let $\hat{z}$ be a solution to the static minmax problem (2) and (6). Then there exist multipliers $\chi_1, \ldots, \chi_p, \sigma_1, \ldots, \sigma_q$, with $\sigma_i \leqq 0$, $i = 1, \ldots, q$, and $\mu_1, \ldots, \mu_r$, with $\mu_i \leqq 0$. $i = 1, \ldots, r$, such that not all of them are zero and

(7) $$\sum_{i=1}^{r} \mu_i \Phi_z^i(\hat{z}) + \sum_{i=1}^{p} \chi_i h_z^i(\hat{z}) + \sum_{i=1}^{q} \sigma_i g_z^i(\hat{z}) = 0 ,$$

(8) $$\sigma_i g^i(\hat{z}) = 0 , \quad i = 1, \ldots, q ,$$

(9) $$\mu_i(\Phi^i(\hat{z}) - F(\hat{z})) = 0 , \quad i = 1, \ldots, r .$$

3. DISCRETE MINMAX CONTROL PROBLEM

Let a discrete dynamical system be described by the equations

(10) $$x_{k+1} = f^k(x_k, u_k) , \quad k = 0, 1, \ldots, K - 1 ,$$

where $x_k \in E^n$ is the state, $u_k \in E^n$ is the control, and $f^k : E^n \times E^m \to E^n$. The aim is to find a control sequence $\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{K-1}$ and a corresponding trajectory $\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_K$, determined by (10), satisfying the constraints

(11) $$x_k \in A_k \subset E^n , \qquad k = 0, 1, \ldots, K ,$$

(12) $$u_k \in U_k(x_k) \subset E^m , \quad k = 0, 1, \ldots, K - 1 ,$$

and minimizing the objective functional

(13) $$J = \sup_{y \in Y} J(y) = \sup_{y \in Y} \sum_{k=0}^{K-1} h^k(x_k, u_k, y) ,$$

where $h^k : E^n \times E^m \times Y \to E^1$. Y is a compact metric space, and the constraints (11)–(12) are given in an explicit way as

(14) $$A_k = \{x \in E^n \mid S^k(x) = 0,\ s^k(x) \leqq 0\} , \quad k = 0, 1, \ldots, K ,$$

(15) $$U_k(x) = \{(x, u) \in E^n \times E^m \mid Q^k(x, u) = 0,\ q^k(x, u) \leqq 0\} ,$$
$$k = 0, 1, \ldots, K - 1 .$$

Here $S^k : E^n \to E^\varrho$, $s^k : E^n \to E^\pi$, $Q^k : E^n \times E^m \to E^\gamma$ and $q^k : E^n \times E^m \to E^\varkappa$. The inequality sign for vectors is to be taken componentwise.

It will be assumed that all of the above functions are continuously differentiable. In fact, there are no particular difficulties to handle a general case of constraints (11)–(12) when they are described in an implicit way by conical approximations [5]–[7]. However, then a more lengthy exposition would be necessary to give a satisfactory treatment.

To avoid a trivial satisfaction of further stated results let us assume that for any $x \in E^n$, the gradients of the corresponding active constraints in (14) are linearly independent. Similarly, for any $(x, u) \in E^n \times E^m$ let also the partial gradients of the corresponding active constraints in (15), with respect to u, be linearly independent. Some consequences in this respect can be found in the mentioned references [5]–[7].

4. NECESSARY OPTIMALITY CONDITIONS

It is not difficult to realize that the formulated discrete control problem with a minmax type objective functional (10)–(15) is of the form required in Theorem 1. Namely, if one introduces a vector $z = (x_0, x_1, \ldots, x_K, u_0, u_1, \ldots, u_{K-1})$, one can see that a static minmax problem in the variable z results. Due to a special structure of constraints (10)–(12) it is possible, similar to a classical objective functional, to express the respective necessary conditions in a more familiar form. As the pertinent calculations are fairly straightforward following [7], they are omitted here. Let us only formulate the final result.

Theorem 2. Consider a discrete optimal control problem (10)–(15). If $\hat{u}_0, \hat{u}_1, \ldots$ $\ldots, \hat{u}_{K-1}$ is an optimal control sequence and $\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_K$ a corresponding trajectory, then there exist vectors $y^1, \ldots, y^r \in \hat{Y} = \{y \in Y \mid J(y) = \sup_{w \in Y} J(w)$ along the optimal solution$\}$, $1 \leqq r \leqq K(n+1) + Km + 1$, together with a vector multiplier $\mu = (\mu_1, \ldots, \mu_r) \in E^r$, with $\mu_i \leqq 0$, $i = 1, \ldots, r$, and row-vector multipliers

$$\lambda_k \in E^n, \quad \psi_k \in E^\varrho, \quad \nu_k \in E^\pi, \quad k = 0, 1, \ldots, K, \tag{16}$$

$$\zeta_k \in E^\gamma, \quad \xi_k \in E^\varkappa, \quad k = 0, 1, \ldots, K-1,$$

such that (all expressions are to be evaluated along the optimal solution)

(i) if $\mu = 0$, then at least one of the vectors λ_k, ψ_k is nonzero;

(ii) the costate row-vectors λ_k satisfy the adjoint equation

$$\lambda_k = H_x^{k+1} + \zeta_k Q_x^k + \xi_k q_x^k + \psi_k S_x^k + \nu_k s_x^k, \quad k = 0, 1, \ldots, K-1,$$

where

$$H^{k+1}(x, u) = \sum_{i=1}^{r} \mu_i h^k(x, u, y^i) + \lambda_{k+1} f^k(x, u), \quad k = 0, 1, \ldots, K-1,$$

and $\lambda_0 = 0$, $\lambda_K = \psi_K S_x^K + \nu_K s_x^K$;

(iii) the optimal control sequence $\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{K-1}$ satisfies the relations

$$H_u^{k+1} + \zeta_k Q_u^k + \xi_k q_u^k = 0, \quad k = 0, 1, \ldots, K-1;$$

(iv) $v_k \leqq 0$, $v_k s^k = 0$, $k = 0, 1, \ldots, K$;

(v) $\xi_k \leqq 0$, $\xi_k q^k = 0$, $k = 0, 1, \ldots, K - 1$.

If the maximum principle form of condition (iii) is required, the additional assumption of the so-called directional convexity has to be imposed, see [5] and [7]. One can now see that the possible number of indices from $\hat{Y}$ can be rather high, being limited only by the dimension of the corresponding mathematical programming problem. However, in practical cases this drawback is often eliminated by the fact that $\hat{Y}$ contains only several elements.

Now let Y be a finite set. Then owing to Corollary 1 the following result is easily established.

Corollary 2. If $\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{K-1}$ is an optimal control sequence and $\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_K$ a corresponding trajectory in (10)–(15), with $Y = \{1, 2, \ldots, r\}$, then there exists a vector multiplier $\mu = (\mu_1, \ldots, \mu_r) \in E^r$ and row-vector multipliers (16) such that the conditions (i)–(v) of Theorem 2 are satisfied for

$$H^{k+1}(x, u) = \sum_{i=1}^{r} \mu_i h^k(x, u, i) + \lambda_{k+1} f^k(x, u), \quad k = 0, 1, \ldots, K - 1,$$

and, moreover,

$$\mu_i \leqq 0, \quad \mu_i(\hat{J} - J(i)) = 0, \quad i = 1, \ldots, r,$$

where $\hat{J}$ denotes the corresponding optimal value of (13).

5. CONCLUSIONS

It was shown that the existing results for static minmax problems can be applied when dealing with discrete control problems having a max-type objective functional. For this reason it is necessary to treat in the first row static minmax problems with equality side constraints. Then the system dynamics, described by a set of equations, are easily included in the respective static minmax problem. For the sake of simplicity only the case of explicit constraints was studied more in detail. However, the same approach can be used when dealing with a more general type of constraints along the lines of [7].

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