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The GUHA Method and the Three-valued Logic

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The theory of the GUHA method of automatic hypotheses determination is modified by generalizing the definition of a model (= experimental material). The generalization consists in allowing absence of information for some objects and properties. The three-valued logic is used as a means for treatment of such models. Appropriate modifications of algorithms for hypotheses determination follow immediately from the developed theory.

This paper is a continuation of [1], [2], [3]. (The reader can use the paper [4], written in English, instead of [1], [2]; a short summary of parts of [3] used here will be given below.) As stated in [3], the main principle of the GUHA method is to obtain automatically all the interesting hypotheses verifiable on the basis of some experimental material. It is necessary for every particular realization of this general task to define mathematically three notions:

(a) experimental material,
(b) hypotheses,
(c) verification.

We present here a generalization of the notion of experimental material. In contradistinction to [1]–[4], where it was supposed that, for each object and each property, we know whether or not the object possesses the property, we shall now allow that for some objects and some properties we have no information. This can happen in practice for several reasons, e.g. some patients could not be examined, etc. Hence we have some "empty fields" in the experimental material or — better — we have a sign "unknown" (say, $\times$).*

* The content of this paper was referred in the seminar on applications of mathematical logic at the Mathematico-physical faculty of the Charles University Prague in October 1969 and February 1970. We thank Professor H. B. Curry, who told us during his visit at Prague in September 1970 that the system of three-valued logic formulated below appears in [5].

The referee, Professor O. Zich, has pointed out the necessity of a methodological discussion of the problem of vagueness in connection with (the present version of) the GUHA method. We agree completely, but we do not include any such discussion into the present paper.

Models considered in [1]–[4] will now be called two-valued models. We further define

1. A structure $\mathcal{M} = \langle M, P_1, \ldots, P_n \rangle$ is a *three-valued model* if M is a non-empty finite set and P_i $(i = 1, \ldots, n)$ are functions mapping M into the three-element set $\{0, 1, \times\}$. Elements of M are called *objects*, the n-tuple $K(a) = \langle P_1(a), \ldots, P_n(a) \rangle$ is called the *card* of a. The *canonical three-valued model* of the type n is the model $\mathcal{M}$ whose objects are all the n-tuples of elements 0, 1, $\times$ and such that $P_i(\langle u_1, \ldots, u_n \rangle) = u_i$.

2. *Three-valued truth-functions* associated to the logical connectives &, $\vee$, $\rightarrow$, $\neg$ are defined by the following tables:

& $u \, . \, v$

$u \backslash v$	0	1	$\times$
0	0	0	0
1	0	1	$\times$
$\times$	0	$\times$	$\times$

$\vee$ $u(+)\,v$

$u \backslash v$	0	1	$\times$
0	0	1	$\times$
1	1	1	1
$\times$	$\times$	1	$\times$

$\rightarrow$ $u(\rightarrow)\,v$

$u \backslash v$	0	1	$\times$
0	1	1	1
1	0	1	$\times$
$\times$	$\times$	1	$\times$

$\neg$

u	0	1	$\times$
$\bar{u}$	1	0	$\times$

These tables correspond to the intuitive understanding of logical connectives if 1 is understood as the value "known that the object has the property", 0 as "known that the object does not have the property" and $\times$ as "unknown whether the object has the property or not". For example, one knows that the object satisfies a conjunction iff one knows that it satisfies both the members of that conjunction, one knows that the object does not satisfy the conjunction iff one knows that at least one member is not satisfied, and in other cases one does not know whether or not the object satisfies the conjunction.

3. Given a model $\mathscr{M}$, we denote the properties $P_1, \ldots, P_n$ by the propositional variables $p_1, \ldots, p_n$ and for every formula $\Phi(p_1, \ldots, p_n)$, we define the *three-valued associated function* $F_\Phi^\mathscr{M}$ of Φ and $\mathscr{M}$ by the obvious definition (see [1] p. 35 or [4] p. 299–300). Further we define the *canonical three-valued associated function* F_Φ of Φ as the three-valued function associated to Φ and to the canonical three-valued model.

4. An object a is said to *satisfy* Φ if $F_\Phi^\mathscr{M}(a) = 1$ and is said to *decide* Φ if either $F_\Phi^\mathscr{M}(a) = 1$ or $F_\Phi^\mathscr{M}(a) = 0$. (Obviously, a satisfies $\neg\Phi$ iff $F_\Phi^\mathscr{M}(a) = 0$ and a does not decide Φ iff $F_\Phi^\mathscr{M}(a) = \times$.)

The aim of this paper is to give theoretical foundations of automatical proceeding of three-valued models in accordance with the main principle of the GUHA method. The present theory is motivated by the following idea: even if we do not know whether the object has a property or not, either the object has or does not have the property. In other words our three-valued model is particular information on a two-valued model which is possessed by anybody able to decide for all the objects in the model which properties they have. Let us call the latter model "the heavenly model"; our purpose is to find most possible hypotheses verified by the "heavenly model", using only our "earthly model".

* * *

5. A *two-valued card* is an n-tuple of zeros and ones; a *three-valued card* is an n-tuple of zeros, ones and crosses. A two-valued card $\langle u_1, \ldots, u_n \rangle$ is said to be a *two-valued completion* (2-*v.c.*) of a three-valued card $\langle v_1, \ldots, v_n \rangle$ if, for each $i = 1, \ldots, n$, $v_i = 1$ implies $u_i = 1$ and $v_i = 0$ implies $u_i = 0$. Similarly a two-valued model $\langle \bar{M}, \bar{P}_1, \ldots, \bar{P}_n \rangle$ is said to be a 2-*v.c.* of a three-valued model $\langle M, P_1, \ldots, P_n \rangle$ if, for each $a \in M$ and each $i = 1, \ldots, n$, $P_i(a) = 1$ implies $\bar{P}_i(a) = 1$ and $P_i(a) = 0$ implies $\bar{P}_i(a) = 0$.

Since the fact that an object satisfies (decides) Φ depends only on its card, we shall say that a *card satisfies* (*decides*) Φ instead of saying that an object with this card does.

6. Lemma. *If a card u satisfies a formula Φ then every* 2-*v.c. of u satisfies Φ.*

Proof. (By induction.) If $\langle u_1, \ldots, u_n \rangle$ satisfies p_i then $u_i = 1$ and consequently $v_i = 1$ for every 2-v.c. $\langle v_1, \ldots, v_n \rangle$ of $\langle u_1, \ldots, u_n \rangle$. Similarly, if a card satisfieds $\neg p_i$ then every of its 2-v.c.'s. Suppose that the following holds for Φ_1, Φ_2: if a card satisfies Φ_j then every of its 2-v.c.'s does. Then the same holds for formulas $\neg\Phi_j$, $(\Phi_1 \& \Phi_2)$. (The negation is obvious. If u satisfies $\Phi \& \Phi_2$ then u satisfies Φ_1 and satisfies Φ_2, hence every 2-v.c. of u satisfies Φ_1 and satisfies Φ_2, which means that it satisfies $\Phi_1 \& \Phi_2$. If u satisfies $\neg(\Phi_1 \& \Phi_2)$ then u satisfies $\neg\Phi_1$ or satisfies $\neg\Phi_2$ hence every 2-v.c. of u satisfies $\neg(\Phi_1 \& \Phi_2)$.) Similarly other connectives.

7. Lemma. *For every formula Φ, there is a card not deciding Φ.*

Proof. One verifies easily by induction that the card $\langle \times, \times, \ldots, \times \rangle$ decides no card.

8. Theorem. *Φ is a tautology of the (classic, two-valued) propositional calculus if and only if the three-valued canonical function associated to Φ has never value* 0.

Proof. If Φ is not a tautology then there is a two-valued card satisfying $\neg\Phi$. Conversely, if there is a three-valued card satisfying $\neg\Phi$ then each of its 2-v.c.'s satisfies $\neg\Phi$ and therefore Φ is not a tautology.

9. *Remark.* Formulas equivalent in classic propositional calculus can have different three-valued canonical functions; e.g. formulas $p \to q$ and $\neg p \vee (p \,\&\, q)$ are equivalent but the card $\langle \times, 1 \rangle$ satisfies $p \to q$ and does not decide $\neg p \vee (p \,\&\, q)$.

10. Formulas Φ and Ψ are said to be (*three-valued-*) *semantically equivalent* if they have the same canonical three-valued associated function. (Denotation: $\Phi \Leftrightarrow_3 \Psi$).

11. Lemma.

(1) $$\neg(p \,\&\, q) \Leftrightarrow_3 (\neg p \vee \neg q),$$

(2) $$\neg(p \vee q) \Leftrightarrow_3 (\neg p \,\&\, \neg q),$$

(3) $$(p \to q) \Leftrightarrow_3 (\neg p \vee q) \Leftrightarrow_3 \neg(p \,\&\, \neg q),$$

(4) $$p \vee p \Leftrightarrow_3 p,$$

(5) $$p \,\&\, (q \vee r) \Leftrightarrow_3 (p \,\&\, q) \vee (p \,\&\, r),$$

(6) $$p \vee (q \,\&\, r) \Leftrightarrow_3 (p \vee q) \,\&\, (p \vee r).$$

Proof by truth-tables.

12. Lemma. *If $\Phi_1 \Leftrightarrow_3 \Phi_2$ then*

$$(\Phi_1 \,\&\, \Psi) \Leftrightarrow_3 (\Phi_2 \,\&\, \Psi), \quad (\Phi_1 \vee \Psi) \Leftrightarrow_3 (\Phi_2 \vee \Psi), \quad \neg\Phi_1 \Leftrightarrow_3 \neg\Phi_2 .$$

Proof. Obvious.

13. (1) A *letter* is a propositional variable or a negated propositional variable. (2) A *fundamental disjunction* of the length n is a disjunction of n distinct letters. (Every elementary disjunction is a fundamental disjunction, but also e.g. $p \vee \neg p \vee q$ is a fundamental disjunction.)

14. Theorem. *Every formula is semantically equivalent to a conjunction of some fundamental disjunctions.*

Proof. The theorem follows by Lemmas 11 and 12.

15. (1) A formula is said to be *in fundamental form* if it is a conjunction of some fundamental disjunctions. (A fundamental disjunction is considered as a one-element conjunction of fundamental disjunction.)

(2) Let $\Phi = \bigwedge_{i=1}^{k} D_i$ be a formula in fundamental form and let u be a card. Φ is said to be singular w.r.t. u if there is an i such that D_i a non-elementary disjunction (i.e. some propositional variable has two occurrences in D_i) and u does not satisfy D_i. Otherwise Φ is regular w.r.t. u.

16. Theorem. *Let Φ be a formula in fundamental form and let u be a card.*

(1) *If Φ is singular w.r.t. u then u does not satisfy Φ, whether each 2-v.c. of u satisfies Φ or not.*

(2) *If Φ is regular w.r.t. u then u satisfies Φ iff each 2-v.c. satisfies Φ.*

Proof. (1) Since u does not satisfy D_i, u does not satisfy Φ. At the same time, if u is $\langle \times, 0\rangle$ and Φ is $(p \vee \neg p \vee q)$ then each 2-v.c. satisfies Φ, but if Φ is $(p \vee \neg p \vee q) \,\&\, (p \vee q)$ then $\langle 0, 0\rangle$ is a 2-v.c. of u and does not satisfy Φ.

(2) The implication $\Rightarrow$ follows by Lemma 6. Conversely suppose that u does not satisfy $\Phi = \bigwedge_{i=1}^{n} D_i$. Then there is a D_i such that u does not satisfy D_i. Since Φ is regular, D_i must be an elementary disjunction. Let D_i be $\bigvee_{k=1}^{k_0} \varepsilon_{i_k} p_{i_k}$ $(i_1 < \ldots < i_{k_0}$, $\varepsilon_{i_k} = 0$ or $1)$. Put $v_i = u_i$ if $u_i = 0$ or 1, $v_{i_k} = \bar{\varepsilon}_{i_k}$ if $u_{i_k} = \times$ and finally $v_i = 1$ for i distinct from all i_k and such that $u_i = \times$. Then v is a 2-v.c. of u and v satisfies $\neg D_i$. Consequently, v satisfies $\neg \Phi$, which completes the proof.

17. Corollary. *If Φ is in fundamental form, then Φ is regular w.r.t. all the cards iff Φ is in normal (conjunctive-disjunctive) form, i.e. iff it is a conjunction of elementary disjunctions. For such a formula and for an arbitrary card u we have: u satisfies Φ if and only if each 2-v.c. of u satisfies Φ.*

* * *

18. In this section, let $\mathcal{M}$ be a fixed three-valued model. A formula Φ is said to be *true* in $\mathcal{M}$ if the function associated to Φ and $\mathcal{M}$ equals identically 1. (Note that no formula is true in the canonical model.)

19. An elementary disjunction D (built up from some of the variables $p_1, \ldots, p_n$) is a *prime disjunction* of $\mathcal{M}$ if

(1) D is true in $\mathcal{M}$ and

(2) no elementary disjunction obtained by omitting some components in D is true in $\mathcal{M}$.

20. The *two-valued model corresponding to $\mathscr{M}$* is the submodel of the two-valued canonical model whose field consists of all two-valued cards which are 2-v.c.'s of some cards of objects in $\mathscr{M}$.

21. Lemma. *Let D be an elementary disjunction. The following are equivalent:*

(1) *D is true in $\mathscr{M}$,*
(2) *D is true in every 2-v.c. of $\mathscr{M}$,*
(3) *D is true in the two-valued model corresponding to $\mathscr{M}$.*

Proof. (1) implies (2) by Lemma 6. (2) implies (3) since every card in the two-valued model corresponding to $\mathscr{M}$ occurs in some 2-v.c. of $\mathscr{M}$. (3) implies (1) by Corollary 17.

22. Corollary. *Let D be an elementary disjunction. The following are equivalent:*

(1) *D is a prime disjunction of $\mathscr{M}$,*
(2) *D is a prime disjunction of the two-valued model corresponding to $\mathscr{M}$.*

23. Theorem. (1) *Every conjunction of some prime disjunctions of a three-valued model $\mathscr{M}$ is true in $\mathscr{M}$.*

(2) *If a formula Φ is true in $\mathscr{M}$ then it is logically equivalent to a conjunction of some prime disjunctions of $\mathscr{M}$.*

Proof. (1) Is obvious by the definition.

(2) If Φ is true in $\mathscr{M}$ then Φ is true in the two-valued model $\overline{\overline{\mathscr{M}}}$ corresponding to $\mathscr{M}$ and, by [1], Φ is logically equivalent to a conjunction of some prime disjunctions of $\overline{\overline{\mathscr{M}}}$. The theorem follows by Corollary 22.

24. Theorem. *A formula Φ is true in every 2-v.c. of a three-valued model $\mathscr{M}$ iff it is a logical consequence of the prime disjunctions of $\mathscr{M}$.*

Proof. If Φ is true in every 2-v.c. of $\mathscr{M}$ then it is true in the two-valued model corresponding to $\mathscr{M}$ and therefore it is a logical consequence of the prime disjunctions of $\mathscr{M}$ by Theorem 13. Conversely, if Φ is a consequence of the prime disjunctions of $\mathscr{M}$ then Φ is true in the two-valued model corresponding to $\mathscr{M}$ and hence in every 2-v.c. of $\mathscr{M}$.

25. *Remark.* This theorem enables us to "determine on the basis of the earthly model most possible hypotheses (of the form "Φ is true") verified by the heavently model" since the "heavenly model" is one of the 2-v.c.'s of the "earthly model".

26. We shall now consider three-valued models from another point of view. Let us introduce a new unary propositional connective !; the formula $!\Phi$ is to be read "*known that Φ*". Formulas containing the connective ! are called *generalized formulas*, formulas not containing ! are called *Boolean formulas*. The truth function for ! is defined by the following table:

u	0	1	×
$!u$	0	1	0

The function associated to Φ and $\mathscr{M}$ is defined for generalized formulas in the same way as for Boolean formulas with the following supplement

$$F^{\mathscr{M}}_{!\Phi}(a) = !(F^{\mathscr{M}}_{\Phi}(a)) .$$

The definition of the semantical equivalence and Lemmas 11, 12 generalize for generalized formulas.

27. Φ is a *two-valued formula* if the canonical function associated to Φ has never the value $\times$. (Note that no Boolean formula is two-valued.)

28. Lemma.

(1) $$!(\Phi \mathbin{\&} \Psi) \Leftrightarrow_3 (!\Phi \mathbin{\&} !\Psi) .$$

(2) $$!(\Phi \vee \Psi) \Leftrightarrow_3 (!\Phi \vee !\Psi) .$$

(3) *If Φ is two-valued then $!\Phi \Leftrightarrow_3 \Phi$.*

Proof. Obvious.

29. Theorem. *Every generalized formula is semantically equivalent to a formula built up from the formulas p_i, $!p_i$, $!(\neg p_i)$ using connectives &, $\vee$, $\neg$.*

This can be proved by induction on formulas. Since the theorem will not be used in this paper, the proof is left to the reader.

30. Theorem. *Φ is true in $\mathscr{M}$ if and only if $!\Phi$ is true in $\mathscr{M}$.*

Proof. Obvious.

31. Corollary. *Let D be an elementary disjunction $\bigvee_i \varepsilon_i p_i$. D is true in $\mathscr{M}$ iff $\bigvee_i !(\varepsilon_i p_i)$ is true in $\mathscr{M}$. In other words D is true in $\mathscr{M}$ iff it is true in the 2-v.c. of $\mathscr{M}$ which results from $\mathscr{M}$ by changing the function associated with p_i and $\mathscr{M}$*

(1) *to the function associated with $!p_i$ if $\varepsilon_i = 1$ and*

(2) *to the function associated with $\neg !(\neg p_i)$ if $\varepsilon_i = 0$.*

(Crosses in columns corresponding to variables not occurring in D may be completed arbitrarily.) Evidently, this 2-v.c. is the most unfavorable one w.r.t. the validity of D.

32. *Remarks.* A programmer of a computer programme determining to a given three-valued model (input) all its prime disjunctions (beginning from one-element ones) will have to decide how to code three-valued models in the computer's memory. One has at least three possibilities:

(1) To code every card in one cell, two bits being reserved for each of symbols 0, 1, ×. In this way we restrict the number of properties in the model to one half of the number of bits in one cell.

(2) To code every card in two cells similarly as elementary disjunction are coded. In this way we restrict the number of objects in model.

(3) When models with few crosses are proceeded one could construct from $\mathcal{M}$ the two-valued model corresponding to $\mathcal{M}$. The resulting model can be proceeded by an old programme described in [1].

Programmes using possibilities (1), (2) above should respect some facts economizing the computer's work analogous to the facts described in [1]. Furthermore, the computer should respect the following fact: if an object occurs whose card consists solely of crosses then the model has no prime disjunctions. If a property occurs whose associated function consists solely of crosses then the model has no prime disjunction containing this property.

* * *

33. Now we generalize the theory described in [2] for the three-valued logic. It is possible to define the notion of an almost true (Boolean) formula in the three-valued model in a similar way as in [2]. Let a member p be given, $0 < p \leqq 1$.

34. Φ is *p-true* (*almost true*) in $\mathcal{M}$, if at least $100p$ percent objects satisfy Φ.

35. An elementary disjunction D is called an *almost prime disjunction* of $\mathcal{M}$ if (1) D is almost true in $\mathcal{M}$ and (2) no elementary disjunction obtained by omitting some letters in D is almost true in $\mathcal{M}$.

36. Theorem. *If Φ is almost true in $\mathcal{M}$ then it is logically implied by a conjunction of some prime and almost prime disjunctions of $\mathcal{M}$.*

Proof. Fully analogous to the proof of Theorem 1 in [2].

37. Further we want to define the notion of a relatively almost true implication in a three-valued model $\mathcal{M}$.

38. Let $K \to D$ be an implication such that K is an elementary conjunction, D is an elementary disjunction and no variable occurs both in K and in D. Let m_{11}, $m_{1\times}$, m_{10}, $m_{\times 1}$, $m_{\times\times}$, $m_{\times 0}$, m_{01}, $m_{0\times}$, m_{00} be defined as follows: m_{11} is the number of cards in $\mathcal{M}$ satisfying K and D, $m_{1\times}$ is the number of cards satisfying K and not deciding D, m_{10} is the number of cards satisfying K and $\neg D$ etc. An implication

$K \to D$ is said to be *3-v.-relatively almost true*, if

$$\frac{m_{11}}{m_{11} + m_{10} + m_{1\times} + m_{\times\times} + m_{\times 0}} \geqq p$$

(if $p = 1$ we say that the implication $K \to D$ is *3-v.-relatively true*).

39. Theorem. *Every 3-v.-relatively almost true implication is logically implied by a conjunction of some prime and almost prime disjunctions.*

Proof. This can be proved in the same way as the Theorem 2 in [2] using the fact that a 3-v.-relatively almost true implication is almost true.

40. Lemma. *$K \to D$ is 3-v.-relatively almost true in $\mathcal{M}$ iff $K \to D$ is relatively almost true in each 2-v.c. of $\mathcal{M}$.*

Proof. 1. Suppose that $K \to D$ is 3-v.-relatively almost true in $\mathcal{M}$. Let $\overline{\mathcal{M}}$ be a 2-v.c. of $\mathcal{M}$, let $\overline{m}_{10}$, $\overline{m}_{11}$, $\overline{m}_{01}$, $\overline{m}_{00}$ be the numbers defined w.r.t. $\overline{\mathcal{M}}$ and $K \to D$ similarly as in $\mathcal{M}$. We have

$$\overline{m}_{11} = m_{11} + \varepsilon \quad \text{where} \quad \varepsilon \leqq m_{\times 1} + m_{\times\times} + m_{1\times}$$

and

$$\overline{m}_{10} \leqq m_{10} + m_{\times\times} + m_{\times 0} + m_{1\times}$$

which implies

$$\frac{\overline{m}_{11}}{\overline{m}_{11} + \overline{m}_{10}} \geqq \frac{m_{11} + \varepsilon}{m_{10} + m_{\times\times} + m_{\times 0} + m_{1\times} + m_{11} + \varepsilon} \geqq$$

$$\geqq \frac{m_{11}}{m_{11} + m_{10} + m_{\times\times} + m_{\times 0} + m_{1\times}} \geqq p\,.$$

2. If $K \to D$ is relatively almost true in every 2-v.c. of $\mathcal{M}$ then it must be relatively almost true in the 2-v.c. $\overline{\mathcal{M}}$ which results by completing each card satisfying K and not deciding D or deciding neither K nor D or not deciding K and satisfying $\neg D$ to a card which satisfies K and $\neg D$ and each card, not deciding K satisfying D to a card which satisfies $\neg K$ and D. (That is possible by Corollary 17.) If $\overline{m}_{11}$, $\overline{m}_{10}$ etc. are frequencies of $\overline{\mathcal{M}}$ then we have

$$p \leqq \frac{\overline{m}_{11}}{\overline{m}_{11} + \overline{m}_{10}} \leqq \frac{m_{11}}{m_{11} + m_{10} + m_{1\times} + m_{\times\times} + m_{\times 0}}\,.$$

41. Let $K \to D$ be an implication (K is an elementary conjunction and D is an elementary disjunction, no variable occurs both in K and in D) logically equivalent to some prime or almost prime disjunction (say A) of $\mathcal{M}$. The antecedent K is said to be *good* (with respect to A) if the following holds:

(a) there are at least s objects in $\mathcal{M}$ satisfying K (s is a fixed number less than the number of all cards in the model),
(b) if A is almost true then $K \rightarrow D$ is 3-v.-relatively almost true.

We say that a conjunction K_1 is a *part* of K_2 iff every letter of K_1 is a letter of K_2.

42. Theorem. *If K is a good antecedent with respect to A then every elementary conjunction which is a part of K is also a good antecedent w.r.t. A.*

Proof. By Theorem 3 of [2] and Lemma 40.

43. To summarize, the task of the computer proceeding three-valued models using the present theory can be formulated in the same words as it was formulated for two-valued models, namely, to generate all the elementary disjunctions (or the elementary disjunctions belonging to some probe) and print the prime and almost prime ones. Secondly, to each prime and almost prime disjunction find all its maximal good antecedents.

* * *

Finally, we want to generalize the theory developed in [3] for three-valued models. We recall some definitions.

44. Let $\mathcal{M}$ be a two-valued model with m objects and n properties denoted by $p_1, \ldots, p_{n-1}, q$. The property q is called the *preferred property*, p's are *symptoms*. Let k be the frequency of q and let $0 < k < m$. For every elementary conjunction K built up from some of the symptoms, let r be the frequency of K and let a be the frequency of $K \,\&\, q$. The numbers b, c, d, s, l are defined by the following "frequency table":

	q	$\neg q$	
K	a	b	r
$\neg K$	c	d	s
	k	l	m

(For example b is the frequency of $K \,\&\, \neg q$; $b + d = l$.)
Put

$$\sigma(a, r, k, m) = \frac{r!\, s!\, k!\, l!}{m!\, a!\, b!\, c!\, d!}$$

and

$$\Delta(a, r, k, m) = \sum_{i=a}^{\min(r,k)} \sigma(i, r, k, m).$$

K is said to be *associated* with q iff

(1) $$\frac{a}{r} > \frac{k}{m}$$

and

(2) $$\Delta(a, r, k, m) \leqq \sigma_0 ,$$

where σ_0 is a given small number (e.g. $\sigma_0 = 0{\cdot}05$).

(This definition is based on the so-called (one-sided) exact Fisher's test.)

In [3] an algorithm is described, which, given a two-valued model, generates all the elementary conjunctions K and prints those associated with q and prime in $\mathcal{M}$ (in the sense that K is not equivalent in $\mathcal{M}$ to any of its proper subconjunctions).

45. Now let $\mathcal{M} = \langle M, P_1, \ldots, P_{n-1}, Q\rangle$ be a three-valued model and let $p_1, \ldots$ $\ldots, p_{n-1}, q$ be variables denoting the corresponding properties. We suppose that each object in the model decides the preferred property Q. Given an elementary conjunction K built up from some of the symptoms, let the numbers of objects satisfying both K and q, satisfying $K \,\&\, \neg q$ not deciding K and satisfying q etc. respectively be given by the following table:

	q	$\neg q$	
K	a	b	r
$K\times$	i	j	u
$\neg K$	c	d	s
	k	l	m

(In p articular u is the number of objects not deciding K.)

46. Theorem. *K is associated with q in every 2-v.c. of $\mathcal{M}$ if and only if*

(1) $$\frac{a}{r+j} > \frac{k}{m},$$

(2) $$\Delta(a, r + j, k, m) \leqq \sigma_0 .$$

Proof. If K is associated with q in every 2-v.c. of $\mathcal{M}$ then also in the 2-v.c. where the frequencies are as follows:

	q	$\neg q$	
K	a	$b + j$	$r + j$
$\neg K$	$c + i$	d	$s + i$
	k	l	m

We denote such a 2-v.c. by $\mathcal{M}_{\times}(K)$. This implies (1) and (2) by the definition. Conversely, suppose that (1) and (2) hold. The following was proved in [3] (and is easy to prove):

If

(3) $$\frac{a}{r+1} > \frac{k}{m}$$

then

$$\Delta(a, r, k, m) < \Delta(a, r+1, k, m);$$

if

(4) $$\frac{a}{r} > \frac{k}{m}$$

then

$$\Delta(a, r, k, m) > \Delta(a+1, r+1, k, m).$$

Let $i = i_1 + i_2$ and $j = j_1 + j_2$, where i_1, i_2, j_1, j_2 are nonnegative integers. Consider an arbitrary 2-v.c. $\mathcal{M}'$ of $\mathcal{M}$ with the following frequencies:

	q	$\neg q$	
K	$a + i_1$	$b + j_1$	$r + i_1 + j_1$
$\neg K$	$c + i_2$	$d + j_2$	$s + i_2 + j_2$
	k	l	m

Then we have

$$\frac{a + i_1}{r + i_1 + j_1} \geqq \frac{a}{r + j_1} \geqq \frac{a}{r + j}$$

and

$$\Delta(a + i_1, r + i_1 + j_1, k, m) \leqq \Delta(a, r + j_1, k, m) \leqq$$
$$\leqq \Delta(a, r + j, k, m).$$

This implies that K is associated with q and $\mathcal{M}'$ by (1) and (2), which completes the proof.

47. The preceding Theorem enables us to define: K *is associated with* q in the three-valued model $\mathcal{M}$ if (1) and (2) hold. We see that it is the màtter of a slight modification to change the algorithm described in [3] such that it proceeds three-valued models and, for every such model, it finds succesively its prime conjunctions associated with the preferred property.

48. Proceeding the model $\mathcal{M}$ we could also omit every object having some crosses in its card (or omit in every moment all objects not deciding the conjunction proceeded) and then use the old algorithm. The following theorem compares this method with the method consisting in applying Theorem 46. Given K, we denote by $\mathcal{M}_0(K)$ the submodel of $\mathcal{M}$ which results by omitting all the objects not deciding K.

49. Theorem. *If K is associated with q in $\mathcal{M}$ then K is associated with q in $\mathcal{M}_0(K)$ (on the same level of significance σ_0).*

Proof. The following tables describe frequencies concerning K and q and the models $\mathcal{M}$, $\mathcal{M}_\times(K)$, $\mathcal{M}_0(K)$ respectively (where $\mathcal{M}_\times(K)$ is defined as in 46):

$\mathcal{M}$

	q	$\neg q$	
K	a	b	r
$K_\times$	i	j	u
$\neg K$	c	d	s
	k	l	m

$\mathcal{M}_\times(K)$

	q	$\neg q$	
K	a	β	ϱ
$\neg K$	γ	d	τ
	k	l	m

where $\beta = b + j$, $\gamma = c + i$, $\varrho = r + j$, $\tau = s + i$,

$\mathcal{M}_0(K)$

	q	$\neg q$	
K	a	b	r
$\neg K$	c	d	s
	$\varkappa$	λ	μ

where $\varkappa = k - i$, $\lambda = l - j$, $\mu = m - u$.

Evidently it suffices to prove that if K is associated with q in $\mathcal{M}_\times(K)$ then also in a submodel of $\mathcal{M}_\times(K)$ where we have the following frequencies

	q	$\neg q$	
K	a	$\beta - 1$	$\varrho - 1$
$\neg K$	γ	d	τ
	k	$l - 1$	$m - 1$

or

	q	$\neg q$	
K	a	β	ϱ
$\neg K$	$\gamma - 1$	d	$\tau - 1$
	$k - 1$	l	$m - 1$

respectively. Both cases are analogous and we consider only the first one. Evidently,

$a/\varrho > k/m$ implies $a/(\varrho - 1) > k/(m - 1)$. Further let $0 \leqq p \leqq \min(r, k) - a$. We prove the inequality

$$(*) \qquad \sigma(a + p, \varrho, k, m) > \sigma(a + p, \varrho - 1, k, m - 1) .$$

This is equivalent to

$$\frac{\varrho!\, \tau!\, k!\, l!}{m!\,(a + p)!\,(b - p)!\,(\gamma - p)!\,(d + p)!} >$$
$$> \frac{(\varrho - 1)!\, \tau!\,(l - 1)!\, k!}{(m - 1)!\,(a + p)!\,(\beta - 1 - p)!\,(\gamma - p)!\,(d + p)!} .$$

By equivalent transformations we obtain:

$$\frac{\varrho l}{m(\beta - p)} > 1 ,$$
$$\varrho(m - k) > m(\varrho - a - p) ,$$
$$m(a + p) > \varrho k ,$$
$$\frac{a + p}{\varrho} > \frac{k}{m} ,$$

the last inequality follows from $a/\varrho > k/m$. From (*) we obtain

$$\Delta(a, \varrho, k, m) = \sum_{i=a}^{\min(\varrho, k)} \sigma(i, \varrho, k, m) > \sum_{i=a}^{\min(\varrho - 1, k)} \sigma(i, \varrho - 1, k, m - 1) =$$
$$= \Delta(a, \varrho - 1, k, m - 1) .$$

In fact, if $k < \varrho$ then both sums have the same number of members, every member of the first sum being greater than the corresponding member of the second sum, and if $\varrho \leqq k$ then the first sum has, moreover, an additional member.

50. It is easy to show that the converse theorem does not hold. For example, put $a = d = i = j = 5$, $b = c = 0$. A short calculation shows that K is associated with q in $\mathscr{M}_0(K)$ on the level 1%, whereas K is not associated with q in $\mathscr{M}$.

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VÝTAH

Metoda GUHA a trojhodnotová logika

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Cílem metody GUHA je generovat automaticky všechny zajímavé hypotézy verifikované na základě daného experimentálního materiálu. Při každé konkrétní realizaci tohoto cíle je nutno přesně definovat tyto pojmy: experimentální materiál, hypotéza, verifikace. V této práci je (v porovnání s předchozími) zobecněn pojem experimentálního materiálu. Pracuje se s trojhodnotovými modely, tj. se strukturami typu $\mathcal{M} = \langle M, P_1, \ldots, P_n \rangle$ kde M je neprázdná konečná množina a P_i jsou funkce zobrazující M do tříprvkové množiny $\{0, 1, \times\}$. Hodnoty $P_i(a) = 1, 0, \times$ odpovídají případům „vím, že a má P_i," „vím, že a nemá P_i" a „nevím, zda a má P_i". Cílem práce je dát teoretický základ pro automatické zpracování trojhodnotových modelů vzhledem k principu metody GUHA a v analogii k existujícím realizacím pro dvojhodnotové modely. Z podané teorie (opírající se o trojhodnotovou logiku Kleeneho) je zřejmé, jakým způsobem je třeba modifikovat existující realizace při vyšetřovaném zobecnění pojmu experimentálního materiálu.

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