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A Graphical Way to Solve the Boolean Matrix Equations $AX = B$ and $XA = B$

U. J. Nieminen

A graphical way to find all the solutions of the Boolean matrix equations $AX = B$ and $XA = B$ is proposed and an example is given.

1. INTRODUCTION AND BASIC CONCEPTS

As shown by Ledley in [2, pp. 448-484] and in [3, 479-494], the determination of the solutions for the Boolean matrix equations $AX = B$ and $XA = B$ has important applications to switching theory and logical problems. A way to find all the solutions is given in the books cited above. Recently, Rudeanu [4] has derived a complete solution to the equations $AX = B$ and $XA = B$ in parametric form. In this paper we apply a well known graph-theoretic representation of a Boolean matrix to find a graphical way to determine the complete solution to the equations $AX = B$ and $XA = B$. We assume that the reader is familiar with the basic concepts in graph theory.

By a Boolean matrix $Q = [q_{ij}]$ we shall mean in this paper a $(0, 1)$-matrix. The join of two Boolean $n \times m$ matrices $A$ and $B$ is the matrix $[a_{ij} \lor b_{ij}]$, and the product of the matrices $C$ and $D$ of orders $n \times p$ and $p \times m$, respectively, is an $n \times m$ matrix $CD = [c_{ij}d_{ij}]$. Further, $A^T$ is the transpose of $A$ and $A'$ the complement of $A$, i.e. $A^T = [a_{ji}]$ and $A' = [a'_{ij}]$. $A \geq B$ if and only if $a_{ij} \geq b_{ij}$ for any index pair $ij$.

It is well known that with every $m \times n$ Boolean matrix $Q$ one can naturally associate a bipartite graph $G_b(Q)$ as follows (see e.g. Hedetniemi [1]): The set of vertices $V(G_b(Q))$ of $G_b(Q)$ consists of two disjoint subsets $\{u_i | i = 1, \ldots, m\}$ and $\{v_j | j = 1, \ldots, n\}$ which correspond to the rows and columns of $Q$, respectively. An edge $(u_i, v_j)$ joining $u_i$ and $v_j$ belongs to the edge set $E(G_b(Q))$ only if $q_{ij} = 1$.
Conversely, every bipartite graph $G_b$ can be translated into a Boolean matrix according to the rules above. In the following we shall concentrate on the equation $AX = B$. As known, the solution of $XA = B$ is analogous to that of $AX = B$.

2. THE BOOLEAN MATRIX EQUATION $AX = B$

Consider the product of two Boolean matrices $A$ and $B$, and let the vertex sets of the bipartite graphs $G(A)$ and $G(B)$ be $V(G(A)) = \{u_{Ai} \mid i = 1, \ldots, m\} \cup \{v_{As} \mid s = 1, \ldots, k\}$ and $V(G(B)) = \{u_{Bj} \mid j = 1, \ldots, n\} \cup \{v_{Bj} \mid j = 1, \ldots, n\}$. Let us draw the bipartite graphs $G(A)$ and $G(B)$ such that the vertices in the sets $\{v_{As}\}$ and $\{u_{Bj}\}$ are common, and denote the graph thus obtained by $G(AB)$. Then, according to the formula $AB = \sum_{s,j} u_{Ai} v_{Bj}$, in the bipartite graph $G(AB)$ a vertex $u_{Ai}$ is connected by an edge to a vertex $v_{Bj}$ if and only if there is a path of length two from $u_{Ai}$ to $v_{Bj}$ in the graph $G(A)$ $G(B)$. As an illustration, see the graphs of Fig. 1. This graphical form of the product of two Boolean matrices can be applied to the determination of a complete solution to $AX = B$.

As shown in the literature, the equation $AX = B$ has a solution if and only if the matrix $(A'B')'$ is a solution to $AX = B$, i.e. $A(A'B')' = B$. Moreover, the solutions of $AX = B$ form a join semilattice, denoted by $L_{ij}(X)$, where $(A'B')'$ is the greatest element. Hence, if $AQ = B$, $Q \cup (A'B')' = (A'B')'$. Thus, in order to obtain the complete set of solutions, one needs to determine the greatest element and the minimum elements of the semilattice $L_{ij}(X)$, if such exist. First we consider a direct way to determine the graph $G_b((A'B')')$, and the matrix $(A'B')'$ as well, and then we show an obvious way to find all the solutions of $AX = B$.

Assume that the equation $AX = B$ has a solution. Now clearly a bipartite graph $G_b(X_b)$ corresponds to the greatest solution of $AX = B$, if in the graph $G(A) \ G_b(X_b)$ every vertex $u_{Ai}$, corresponding to $u_{Bi}$ in $G_b(B)$, is connected by a path of length
two to every vertex \(v_{X_0-j}\), corresponding to \(v_{Bj}\) in \(G_b(B)\), for which \((u_{Bi}, v_{Bj}) \notin E(G_b(B))\), i.e. \((u_{Bi}, v_{Bj}) \in E(G_b(B'))\). Thus the following simple rule can be obtained to find the graph \(G_b(X'_0)\):

**Rule 1.** Connect in \(G_b(X'_0)\) the vertices \(u_{A1}, ..., u_{Ar}\) corresponding to the vertices \(v_{X_0} \in V(G_b(A))\), to all the vertices \(v_{X_0-j}\) for which \((u_{Bi}, v_{Bj}) \in E(G_b(B'))\).

It should be noted that the matrix \(X_0\) determined by the rule above does not give any indication of the non-consistency of the equation \(AX = B\).

As an illuminating example, consider the following consistent Boolean matrix equation

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
\]

The graphs \(G_b(A)\) and \(G_b(B)\) are given in Fig. 2, and the graph \(G_b(X'_0)\) can be seen in the graph \(G_b(A) \cup G_b(X'_0)\) determined by Rule 1. Hence,

\[
X_0 = \begin{bmatrix}
1 & 0 \\
1 & 1 \\
1 & 0
\end{bmatrix}.
\]

**Fig. 2.**

Consider now a way to find all the solutions of \(AX = B\). We construct a solution matrix base, denoted by \(Z_1, Z_2, ..., Z_t\), where every \(Z_w, w = 1, ..., t\), is a Boolean matrix of the order of \(X\) and corresponds to an edge, say \((u_{Bi}, v_{Bj})\), of \(G_b(B)\) such that \(G_b(Z_w)\) contains any edge which gives in \(G_b(A) \cup G_b(Z_w)\) a path of length two from \(u_{A1}, ..., u_{Ar}\) \(\in V(G_b(A))\) and no edge such that there would be a path of length two in \(G_b(A) \cup G_b(Z_w)\) determining an edge of \(G_b(B')\). Since the matrix product is distributive with respect to the join operation and \(AZ_w \leq B\), \(A(Z_1 \cup Z_2 \cup ... \cup Z_t) = B\) according to the definition of the matrices \(Z_w\) if \(Z_w > 0\) for any \(w, w = 1, ..., t\). Furthermore, as every \(G_b(Z_w)\) contains all the edges giving in \(G_b(A) \cup G_b(Z_w)\) the edge of \(G_b(B)\) which determines \(G_b(Z_w)\), \(Z_1 \cup ... \cup Z_t = (A^TB') = X_0\), the greatest element of the solution join semilattice \(L(X)\). According to the definition of \(Z_w\), the matrix equation \(AX = B\) is consistent if and only if \(Z_w > 0\), i.e. \(E(G_b(Z_w)) \neq \emptyset\), for any \(w, w = 1, ..., t\).
A matrix $Q$ is a solution of $AX = B$, if $Q \cap Z_w > 0$ for every $w$, and $Q_0$ is a minimum element of $L_u(X)$ if and only if the equation $Q_{00} \cap Z_w > 0$ does not hold for any matrix $Q_{00} < Q_0$, $w = 1, \ldots, t$.

For the determination of a matrix $Z_u$ corresponding to an edge $(u_{Z_1}, v_{Z_1}) \in E(G_b(B))$ we obtain the following simple rule:

**Rule 2.** Connect in $G_b(Z_u)$ the vertices of $\Gamma_{u_{Ai}} = \{v_{Ai}, \ldots, v_{Ai}\} = \{y_{Z_{Ai}}, \ldots, y_{Z_{Ai}}\}$, $u_{Ai} \in V(G_b(A))$, to $v_{x_{Z_{Ai}}}$ and remove then the edges which belong to $G_b(X')$.

Consider as an example the matrix equation in (1). Fig. 3 shows the determinations of the basis matrices $Z_1$, $Z_2$, and $Z_3$ corresponding to the edges $(u_{Z_1}, v_{Z_1})$, $(u_{Z_2}, v_{Z_2})$, and $(u_{Z_3}, v_{Z_3})$, respectively. The dotted lines in Fig. 3 mean the edges of $G_b(X')$. Since $Z_1, Z_2, Z_3 > 0$, the equation in (1) is consistent.

As one can readily check, the minimum elements of $L_u(X)$ are $X_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ and

$$Z_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$Z_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$Z_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$
The other solutions to \(AX = B\), which are between \(X_1\) and \(X_0\) in \(L_u(X)\), are \(X_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}\) and \(X_4 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}\).

There is another way to construct a solution matrix base. After determining the matrices \(Z_1, \ldots, Z_n\) defined above, we substitute the matrix \(Z_w\), \(w = 1, \ldots, t\), by a set \(\{Y_{lw}, Y_{2w}, \ldots, Y_{nw}\}\) of matrices, where \(Y_{lw} \cup \ldots \cup Y_{nw} = Z_w\). Every solution to \(AX = B\) is obtained by forming all possible joins \((U^,)^=\) of the matrices in the sets \(\{Y_{lw}, Y_{2w}, \ldots, Y_{nw}\}\) such that \((U^,)^=\) \(\cap Z_w > 0\) for any value of \(w\).

In the example considered before,

\[
\begin{align*}
Y_{11} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \\
Y_{12} &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \\
Y_{21} &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \\
Y_{22} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } Y_{31} &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}.
\end{align*}
\]

Thus \(X_1 = Y_{12} \cup Y_{22} \cup Y_{31} = Y_{12} \cup Y_{22} \cup Y_{31}, \quad X_2 = Y_{11} \cup Y_{21} \cup Y_{31}, \quad X_3 = Y_{11} \cup Y_{21} \cup Y_{31} = Y_{12} \cup Y_{22} \cup Y_{31}, \quad X_4 = Y_{12} \cup Y_{21} \cup Y_{31} = Y_{12} \cup Y_{22} \cup Y_{31} \cup Y_{31} \cup Y_{31} = Y_{11} \cup Y_{12} \cup Y_{21} \cup Y_{31} = Y_{12} \cup Y_{22} \cup Y_{31} = Y_{11} \cup Y_{21} \cup Y_{31} \cup Y_{31}.\)

In the case of the equation \(XA = B\), Rule 1 and Rule 2 can be expressed as follows:

**Rule 1.** Connect in \(G_b(X')\) the vertices \(v_{e_{Ai}} = \{u_{Ai1}, \ldots, u_{Ait}\} = \{v_{Xo,1'}, \ldots, v_{Xo,ir}\}, \quad v_{e_{At}} \in V(G_b(A)),\) to all the vertices \(u_{Xo,j}\) for which \((u_{Bj}, u_{Bi}) \in E(G_b(B))).\)

**Rule 2.** Connect in \(G_b(Z_w)\) the vertices of \(v_{e_{Aj}} = \{u_{Aj1}, \ldots, u_{Ajr}\} = \{v_{Zw,1'}, \ldots, v_{Zw,kr}\}, v_{e_{Aj}} \in V(G_b(A)),\) to \(u_{Zwi}\) and remove then the edges which belong to \(G_b(X')\).

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