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A Graphical Way to Solve the Boolean Matrix Equations $AX = B$ and $XA = B$

U. J. NIEMINEN

A graphical way to find all the solutions of the Boolean matrix equations $AX = B$ and $XA = B$ is proposed and an example is given.

1. INTRODUCTION AND BASIC CONCEPTS

As shown by Ledley in [2, pp. 448–484] and in [3, 479–494], the determination of the solutions for the Boolean matrix equations $AX = B$ and $XA = B$ has important applications to switching theory and logical problems. A way to find all the solutions is given in the books cited above. Recently, Rudeanu [4] has derived a complete solution to the equations $AX = B$ and $XA = B$ in parametric form. In this paper we apply a well known graphtheoretic representation of a Boolean matrix to find a graphical way to determine the complete solution to the equations $AX = B$ and $XA = B$. We assume that the reader is familiar with the basic concepts in graph theory.

By a Boolean matrix $Q = [q_{ij}]$ we shall mean in this paper a (0, 1)-matrix. The join of two Boolean $n \times m$ matrices A and B is the matrix $[a_{ij} \cup b_{ij}]$, and the product of the matrices C and D of orders $n \times p$ and $p \times m$, respectively, is an $n \times m$ matrix $CD = [\bigcup_s c_{is} d_{sj}]$. Further, A^{T} is the transpose of A and A' the complement of A, i.e. $A^{\mathrm{T}} = [a_{ji}]$ and $A' = [a'_{ij}]$. $A \geqq B$ if and only if $a_{ij} \geqq b_{ij}$ for any index pair ij.

It is well known that with every $m \times n$ Boolean matrix Q one can naturally associate a bipartite graph $G_b(Q)$ as follows (see e.g. Hedetniemi [1]): The set of vertices $V(G_b(Q))$ of $G_b(Q)$ consists of two disjoint subsets $\{u_i \mid i = 1, \ldots, m\}$ and $\{v_j \mid j = 1, \ldots, n\}$ which correspond to the rows and columns of Q, respectively. An edge (u_i, v_j) joining u_i and v_j, belongs to the edge set $E(G_b(Q))$ only if $q_{ij} = 1$

 in Q. Conversely, every bipartite graph G_b can be translated into a Boolean matrix according to the rules above.

In the following we shall concentrate on the equation $AX = B$. As known, the solution of $XA = B$ is analogous to that of $AX = B$.

2. THE BOOLEAN MATRIX EQUATION $AX = B$

Consider the product of two Boolean matrices A and B, and let the vertex sets of the bipartite graphs $G_b(A)$ and $G_b(B)$ be $V(G_b(A)) = \{u_{Ai} \mid i = 1, \ldots, \ldots, m\} \cup \cup \{v_{As} \mid s = 1, \ldots, k\}$ and $V(G_b(B)) = \{u_{Bs} \mid s = 1, \ldots, k\} \cup \{v_{Bj} \mid j = 1, \ldots, n\}$. Let us draw the bipartite graphs $G_b(A)$ and $G_b(B)$ such that the vertices in the sets $\{v_{As}\}$ and $\{u_{Bs}\}$ are common, and denote the graph thus obtained by $G_b(A)\, G_b(B)$. Then, according to the formula $AB = [\bigcup_s a_{is}b_{sj}]$, in the bipartite graph $G_b(AB)$ a vertex u_{ABi} is connected by an edge to a vertex v_{ABj} if and only if there is a path of length two from u_{Ai} to v_{Bj} in the graph $G_b(A)\, B_b(B)$. As an illustration, see the graphs of Fig. 1. This graphical form of the product of two Boolean matrices can be applied to the determination of a complete solution to $AX = B$.

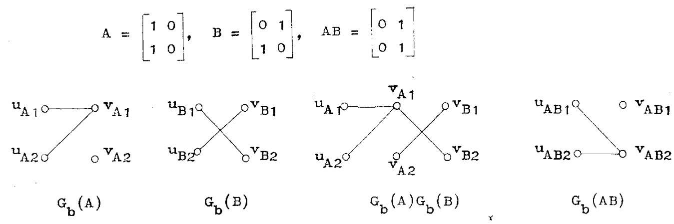

Fig. 1.

As shown in the literature, the equation $AX = B$ has a solution if and only if the matrix $(A^TB')'$ is a solution to $AX = B$, i.e. $A(A^TB')' = B$. Moreover, the solutions of $AX = B$ form a join semilattice, denoted by $L_\cup(X)$, where $(A^TB')'$ is the greatest element. Hence, if $AQ = B$, $Q \cup (A^TB')' = (A^TB')'$. Thus, in order to obtain the complete set of solutions, one needs to determine the greatest element and the minimum elements of the semilattice $L_\cup(X)$, if such exist. First we consider a direct way to determine the graph $G_b((A^TB')')$, and the matrix $(A^TB')'$ as well, and then we show an obvious way to find all the solutions of $AX = B$.

Assume that the equation $AX = B$ has a solution. Now clearly a bipartite graph $G_b(X_0)$ corresponds to the greatest solution of $AX = B$, if in the graph $G_b(A)\, G_b(X_0')$ every vertex u_{Ai}, corresponding to u_{Bi} in $G_b(B)$, is connected by a path of length two to every vertex $v_{X_0'j}$, corresponding to v_{Bj} in $G_b(B)$, for which $(u_{Bi}, v_{Bj}) \notin E(G_b(B))$, i.e. $(u_{Bi}, v_{Bj}) \in E(G_b(B'))$. Thus the following simple rule can be obtained to find the graph $G_b(X_0')$:

Rule 1. *Connect in $G_b(X_0')$ the vertices $\Gamma u_{Ai} = \{v_{Ai_1}, \ldots, v_{Ai_r}\} = \{u_{X_0'i_1}, \ldots, u_{X_0'i_r}\}$, $u_{A\,i} \in V(G_b(A))$, to all the vertices $v_{X_0'j}$ for which $(u_{Bi}, v_{Bj}) \in E(G_b(B'))$.*

It should be noted that the matrix X_0 determined by the rule above does not give any indication of the non-consistency of the equation $AX = B$.

As an illuminating example, consider the following consistent Boolean matrix equation

$$(1) \qquad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

The graphs $G_b(A)$ and $G_b(B)$ are given in Fig. 2, and the graph $G_b(X_0')$ can be seen in the graph $G_b(A)\, G_b(X_0')$ determined by Rule 1. Hence,

$$X_0 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

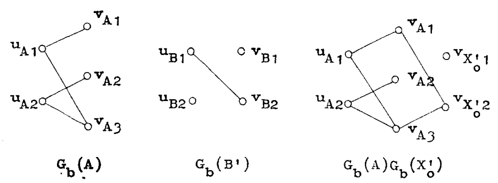

Fig. 2.

Consider now a way to find all the solutions of $AX = B$. We construct a solution matrix base, denoted by $Z_1, Z_2, \ldots, Z_t$, where every Z_w, $w = 1, \ldots, t$, is a Boolean matrix of the order of X and corresponds to an edge, say (u_{Bi}, v_{Bj}), of $G_b(B)$ such that $G_b(Z_w)$ contains any edge which gives in $G_b(A)\, G_b(Z_w)$ a path of length two from u_{Ai} to v_{Z_wj} $(=v_{Bj})$ and no edges such that there would be a path of length two in $G_b(A)\, G_b(Z_w)$ determining an edge of $G_b(B')$. Since the matrix product is distributive with respect to the join operation and $AZ_w \leqq B$, $A(Z_1 \cup Z_2 \cup \ldots \cup Z_t) = B$ according to the definition of the matrices Z_w, if $Z_w > 0$ for any w, $w = 1, \ldots, t$. Furthermore, as every $G_b(Z_w)$ contains all the edges giving in $G_b(A)\, G_b(Z_w)$ the edge of $G_b(B)$ which determines $G_b(Z_w)$, $Z_1 \cup \ldots \cup Z_t = (A^T B')' = X_0$, the greatest element of the solution join semilattice $L_\cup(X)$. According to the definition of Z_w, the matrix equation $AX = B$ is consistent if and only if $Z_w > 0$, i.e. $E(G_b(Z_w)) \neq \emptyset$, for any w, $w = 1, \ldots, t$.

A matrix Q is a solution of $AX = B$, if $Q \cap Z_w > 0$ for every w, and Q_0 is a minimum element of $L_\cup(X)$ if and only if the equation $Q_{00} \cap Z_w > 0$ does not hold for any matrix $Q_{00} < Q_0$, $w = 1, \dots, t$.

For the determination of a matrix Z_w corresponding to an edge $(u_{Bi}, v_{Bj}) \in E(G_b(B))$ we obtain the following simple rule:

Rule 2. *Connect in* $G_b(Z_w)$ *the vertices of* $\Gamma u_{Ai} = \{v_{Ai_1}, \dots, v_{Ai_r}\} = \{u_{Z_w i_1}, \dots, u_{Z_w i_r}\}$, $u_{Ai} \in V(G_b(A))$, *to* $v_{Z_w j}$ *and remove then the edges which belong to* $G_b(X_0')$.

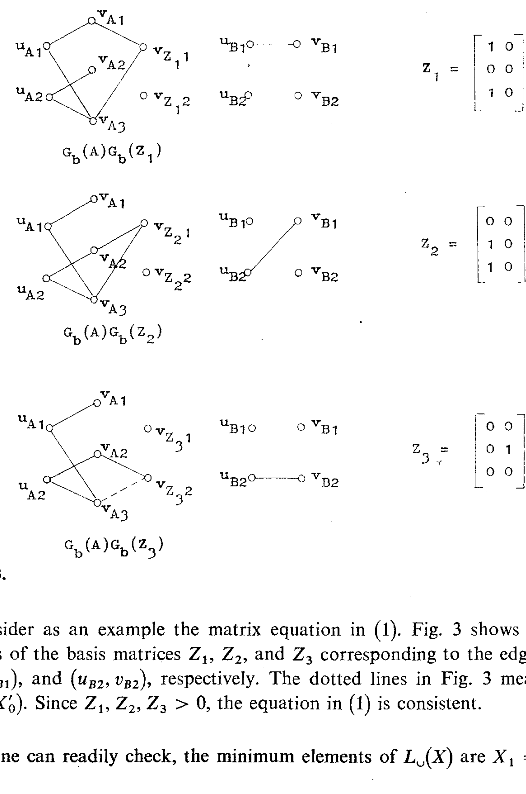

Fig. 3.

Consider as an example the matrix equation in (1). Fig. 3 shows the determinations of the basis matrices Z_1, Z_2, and Z_3 corresponding to the edges (u_{B1}, v_{B1}), (u_{B2}, v_{B1}), and (u_{B2}, v_{B2}), respectively. The dotted lines in Fig. 3 mean the edges of $G_b(X_0')$. Since $Z_1, Z_2, Z_3 > 0$, the equation in (1) is consistent.

As one can readily check, the minimum elements of $L_\cup(X)$ are $X_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ and

$X_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$. $Z_1 \cup Z_2 \cup Z_3 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} = X_0 = (A^T B')'$. The other solutions to $AX =$ $= B$, which are between X_1 and X_0 in $L_\cup(X)$, are $X_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $X_4 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$.

There is an other way to construct a solution matrix base. After determining the matrices $Z_1, \ldots, Z_t$ defined above, we substitute the matrix Z_w, $w = 1, \ldots, t$, by a set $\{Y_{1w}, Y_{2w}, \ldots, Y_{s_w w}\}$ of matrices, where $Y_{1w} \cup \ldots \cup Y_{s_w w} = Z_w$, $Y_{kw} > 0$ and Y_{kw} contains a single one for any k, $k = 1, \ldots, s_w$. Every solution to $AX = B$ is obtained by forming all possible joins $(\bigcup Y)$ of the matrices in the sets $\{Y_{1w}, \ldots, Y_{s_w w}\}$ such that $(\bigcup Y) \cap Z_w > 0$ for any value of w.

In the example considered before,

$$Y_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad Y_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad Y_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Y_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad Y_{31} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Thus $X_1 = Y_{12} \cup Y_{22} \cup Y_{31} = Y_{12} \cup Y_{31} = Y_{22} \cup Y_{31}$, $X_2 = Y_{11} \cup Y_{21} \cup Y_{31}$, $X_3 = Y_{11} \cup Y_{22} \cup Y_{31} = Y_{11} \cup Y_{12} \cup Y_{22} \cup Y_{31}$, $X_4 = Y_{12} \cup Y_{21} \cup Y_{31} = Y_{12} \cup$ $\cup Y_{21} \cup Y_{22} \cup Y_{31}$, and $X_0 = Y_{11} \cup Y_{12} \cup Y_{21} \cup Y_{31} = Y_{11} \cup Y_{21} \cup Y_{22} \cup Y_{31} =$ $= Y_{11} \cup Y_{12} \cup Y_{21} \cup Y_{22} \cup Y_{31}$.

In the case of the equation $XA = B$, Rule 1 and Rule 2 can be expressed as follows:

Rule 1′. *Connect in $G_b(X_0')$ the vertices $\Gamma v_{Ai} = \{u_{Ai_1}, \ldots, u_{Ai_r}\} = \{v_{X_0' i_1}, \ldots, v_{X_0' i_r}\}$, $v_{Ai} \in V(G_b(A))$, to all the vertices $u_{X_0' j}$ for which $(u_{Bj}, u_{Bi}) \in E(G_b(B'))$.*

Rule 2′. *Connect in $G_b(Z_w)$ the vertices of $\Gamma v_{Aj} = \{u_{Aj_1}, \ldots, u_{Aj_r}\} = \{v_{Z_w j_1}, \ldots$ $\ldots, v_{Z_w j_r}\}$, $v_{Aj} \in V(G_b(A))$, to $u_{Z_w i}$ and remove then the edges which belong to $G_b(X_0')$.*

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