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WEAK AVERAGING
OF STOCHASTIC EVOLUTION EQUATIONS

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Summary. A theorem on continuous dependence of solutions to stochastic evolution equations on coefficients is established, covering the classical averaging procedure for stochastic parabolic equations with rapidly oscillating both the drift and the diffusion term.

Keywords: stochastic evolution equations, weak convergence, averaging methods

AMS classification: 60H15

1. INTRODUCTION AND MAIN RESULTS

This paper is devoted to integral continuity type results on continuous dependence of solutions of stochastic evolution equations on coefficients, in which, roughly speaking, the convergence of coefficients is defined via pointwise convergence of their indefinite integrals. Such theorems are known to be fairly general and, in particular, they provide justification for the averaging procedures for equations with rapidly oscillating coefficients. As a motivation let us consider a stochastic differential equation

\[ dx_t = a(t, x_t) \, dt + b(t, x_t) \, dw(t), \quad x_0 = x_0 \]

in \( \mathbb{R}^d \), \( w(t) \) being a standard Wiener process in \( \mathbb{R}^d \) defined on a probability space \( (\Omega, \mathcal{F}, P) \), and \( a, b \), being, roughly speaking, lipschitzian in the space variable uniformly in \( \varepsilon \geq 0 \). It was proved in [12] and [4] that if

\[ \lim_\varepsilon \rightarrow 0^+ \int_0^t a_\varepsilon(s, x) \, ds = \int_0^t a_0(s, x) \, ds, \quad x \in \mathbb{R}^d, \ t \geq 0, \]

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and

\[
\lim_{t \to 0^+} \int_0^t \| b_t(s, x) - b_0(s, x) \|^2 \, ds = 0, \quad x \in \mathbb{R}^d, \ t \geq 0,
\]

then \( x_t(t) \to x_0(t) \) in \( L^2(\Omega) \). Obviously, the assumption (1.3) excludes rapidly oscillating diffusion coefficients, but, as shown in the quoted papers, is essentially indispensable for the \( L^2 \)-convergence. On the other hand, as shown by R. Z. Khas’minskii, under (1.2) and

\[
\lim_{t \to 0^+} \int_0^t b_t(s, x) b^*_t(s, x) \, ds = \int_0^t b_0(s, x) b^*_0(s, x) \, ds, \quad x \in \mathbb{R}^d, \ t \geq 0,
\]

the finite-dimensional distributions of the process \( x_t \) tend to those of \( x_0 \) in law (see [5], [6]; cf. [11] for additional information).

The results on \( L^2 \)-convergence were extended to semilinear stochastic evolution equations in [10], [7], [8]. In the present paper we establish a result on the convergence in law under hypotheses similar to (1.2) and (1.4), thus covering, in particular, the case of stochastic parabolic equations with both the drift and the diffusion coefficient rapidly oscillating.

To state our results let us introduce some notation. Let \( U, H \) be real separable Hilbert spaces with norms \( |\cdot|_U, |\cdot|_H \), respectively, let \((\cdot, \cdot)\) denote the inner product in \( H \). (If there is no danger of confusion we will omit the subscript \( H \).) Let \( L(U, H) \) be the space of all bounded linear mappings from \( U \) to \( H \), whose norm will be denoted by \( |\cdot|_{L(U, H)} \). If \( A \in L(H) \) then \( A^* \) denotes the adjoint operator. Further, \( |A|_{\mathcal{K}} \) stands for the nuclear norm of \( A \in L(H) \), provided \( A \) is a nuclear operator, that is

\[
|A|_{\mathcal{K}} = \sup \left\{ \sum_i |(Ae_i, f_i)| ; \{e_i\}, \{f_i\} \text{ orthonormal bases of } H \right\}.
\]

The space of all \( H \)-valued continuous functions on \([0, T]\) will be denoted by \( C([0, T], H) \). As usual, if \((\mu_\alpha)_{\alpha \in \Gamma} \) is a net of Borel probabilities on a separable metric space \( M, \Gamma \) a right directed ordered set, we say that \( \mu_\alpha \rightharpoonup \mu \) weakly in \( M \) provided

\[
\int_M f \, d\mu_\alpha \to \int_M f \, d\mu
\]

for any bounded continuous function \( f : M \to \mathbb{R} \). If \( \xi_\alpha : (\mathcal{F}_\alpha, \mathbb{P}_\alpha) \to M \) are \( M \)-valued random variables, then \( \xi_\alpha \rightharpoonup \xi_0 \) weakly in \( M \) means that \( \xi_\alpha(\mathbb{P}_\alpha) \rightharpoonup \xi_0(\mathbb{P}_0) \) weakly in \( M \), where the probability measures \( \xi_\alpha(\mathbb{P}_\alpha) \) are defined by

\[
\xi_\alpha(\mathbb{P}_\alpha)(C) = \mathbb{P}_\alpha(\xi_\alpha \in C)
\]
for any Borel set $C \subset M$.

We will consider stochastic evolution equations

$$
\begin{align*}
\mathrm{d}x_{\varepsilon}(t) &= (Ax_{\varepsilon}(t) + a_{\varepsilon}(t, x_{\varepsilon}(t))) \mathrm{d}t + b_{\varepsilon}(t, x_{\varepsilon}(t)) \mathrm{d}w_{\varepsilon}(t), \\
x_{\varepsilon}(0) &= \eta_{\varepsilon}
\end{align*}
$$

for $\varepsilon \in [0, \varepsilon_0)$, assuming

(A1) $A : \text{Dom}(A) \to H$ is an infinitesimal generator of an analytic $C_0$-semigroup $(e^{At})$ on $H$.

(A2) The mappings $a_{\varepsilon} : \mathbb{R}^+ \times H \to H$, $b_{\varepsilon} : \mathbb{R}^+ \times H \to L(U, H)$ are measurable
and there exists a constant $L$ such that

$$
|a_{\varepsilon}(t, x)|_H + |b_{\varepsilon}(t, x)|_{L(U, H)} \leq L(1 + |x|_H)
$$

for all $t \geq 0$, $x \in H$, and $\varepsilon \in [0, \varepsilon_0)$.

(A3) The mappings $a_{\varepsilon}$, $b_{\varepsilon}$ are Lipschitz continuous uniformly in $\varepsilon$, i.e.

$$
|a_{\varepsilon}(t, x) - a_{\varepsilon}(t, y)|_H + |b_{\varepsilon}(t, x) - b_{\varepsilon}(t, y)|_{L(U, H)} \leq L|x - y|_H
$$

for all $t \geq 0$, $x, y \in H$, and $\varepsilon \in [0, \varepsilon_0)$.

(A4) $w_{\varepsilon}(t)$ are $U$-valued Wiener processes with nuclear covariance operators $W_{\varepsilon}$ such that

$$
\sup_{\varepsilon \in [0, \varepsilon_0)} \text{tr } W_{\varepsilon} < \infty,
$$

$w_{\varepsilon}$ being defined on a filtered probability space $(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t \geq 0}, \mathcal{P}_\varepsilon)$ and

$(\mathcal{F}'_t)$-adapted.

(A5) $\eta_{\varepsilon}$ are $H$-valued $\mathcal{F}'_0$-measurable random variables, $\varepsilon \in [0, \varepsilon_0)$.

The solutions of (1.5) will be understood in the mild sense, i.e. as the solutions of the integral equations

$$
x_{\varepsilon}(t) = e^{At}\eta_{\varepsilon} + \int_0^t e^{A(t-s)}a_{\varepsilon}(s, x_{\varepsilon}(s)) \mathrm{d}s + \int_0^t e^{A(t-s)}b_{\varepsilon}(s, x_{\varepsilon}(s)) \mathrm{d}w_{\varepsilon}(s).
$$

By Theorem 7.4 in [2] or Theorem 1.4 of [9] there exists a unique mild solution $x_{\varepsilon} \equiv x_{\varepsilon}(\cdot; \eta_{\varepsilon})$ of (1.5) and $x_{\varepsilon} \in C([0, T], H)$ almost surely.

**Theorem 1.1.** Let the assumptions (A1)-(A5) be fulfilled. Let $T > 0$ be arbitrary but fixed and suppose that

$$
\lim_{\varepsilon \to 0} \int_0^T \int_0^t e^{A(t-s)}a_{\varepsilon}(s, x) \mathrm{d}s = \int_0^T e^{A(t-s)}a_0(s, x) \mathrm{d}s \quad \text{in } H
$$
for any $x \in H$, $t \in [0, T]$; and

$$
\lim_{\varepsilon \to 0^+} \left| \int_0^t e^{\varepsilon(t-s)} U_t(s, x)e^{\varepsilon^2(t-s)} \, ds \right|_{L^\infty} = 0
$$

for all $x \in H, t \in [0, T]$, where

$$
U_t(t, x) = b_t(t, x)W_t\beta_0^t(t, x) - b_0(t, x)W_0\beta_0^t(t, x).
$$

If $\eta_\varepsilon \to \eta_0$ weakly in $H$ as $\varepsilon \to 0^+$ then $x_\varepsilon(\cdot, \eta_\varepsilon) \to x_0(\cdot, \eta_0)$ weakly in $C([0, T], H)$ as $\varepsilon \to 0^+$.

Due to the analyticity of the semigroup $(e^{\varepsilon t})$ the rather complicated assumptions (1.6), (1.7) may be replaced by more restrictive but verifiable hypotheses.

**Proposition 1.2.** Assume (A1) and (A2). If

$$
\lim_{\varepsilon \to 0^+} \int_0^t a_\varepsilon(s, x) \, ds = \int_0^t a_0(s, x) \, ds \quad \text{in } H
$$

for any $t \in [0, T], x \in H$ then (1.6) is satisfied. Analogously, if

$$
\lim_{\varepsilon \to 0^+} \left| \int_0^t b_\varepsilon(s, x)W_t\beta_0^t(s, x) \, ds - \int_0^t b_0(s, x)W_0\beta_0^t(s, x) \, ds \right|_{L^\infty} = 0
$$

for all $t \in [0, T], x \in H$ then (1.7) is fulfilled.

It will be shown in Example 2.9 that the assumptions (1.6), (1.7) are weaker than (1.8), (1.9).

As we have already mentioned, Theorem 1.1 contains the classical averaging procedure as a particular case, as can be shown in a standard way (see e.g. [10], Theorem 4). Because of its importance we state this corollary as a separate theorem. Let us consider equations

$$
\begin{align*}
\dot{x}_\varepsilon(t) & = \varepsilon(Ax_\varepsilon(t) + a(t, x_\varepsilon(t))) \, dt + \varepsilon^{1/2} b(t, x_\varepsilon(t)) \, dw(t), \\
x_\varepsilon(0) & = \eta_\varepsilon
\end{align*}
$$

for $\varepsilon \in (0, \varepsilon_0)$, where

(B2) the mappings $a: \mathbb{R}_+ \times H \to H$, $b: \mathbb{R}_+ \times H \to L(U, H)$ are measurable and there exists a constant $L$ such that

$$
\begin{align*}
|a(t, x)|_H + |b(t, x)|_{L(U, H)} & \leq L(1 + |x|_H), \\
|a(t, x) - a(t, y)|_H + |b(t, x) - b(t, y)|_{L(U, H)} & \leq L|x - y|_H
\end{align*}
$$
for all \( x, y \in H, t \geq 0 \);

(B3) \( w(t) \) is a \( U \)-valued \( (\mathcal{F}_t) \)-adapted Wiener process with a nuclear covariance operator \( W \), defined on a filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P) \);

(B4) \( \eta_\varepsilon : \Omega \rightarrow H, \varepsilon \in (0, \varepsilon_0) \), are \( \mathcal{F}_0 \)-measurable random variables.

**Theorem 1.3.** Assume (A1), (B2), (B3) and (B4). Suppose further that there exist Lipschitz continuous functions \( \tilde{a} : H \rightarrow H \) and \( \tilde{b} : H \rightarrow \mathcal{L}(U, H) \) such that

\[
\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T a(s, x) \, ds = \tilde{a}(x) \quad \text{in } H
\]

for any \( x \in H \), and

\[
\lim_{T \rightarrow \infty} \left| \frac{1}{T} \int_0^T b(s, x) W b^*(s, x) \, ds - \tilde{b}(x) W b^*(x) \right|_{L^\infty} = 0
\]

for all \( x \in H \). Set

\[
\bar{x}_\varepsilon(t, \eta_\varepsilon) = x_\varepsilon \left( \frac{t}{\varepsilon}, \eta_\varepsilon \right), \quad t \geq 0, \varepsilon \in (0, \varepsilon_0),
\]

\( x_\varepsilon(\cdot, \eta_\varepsilon) \) being the mild solutions to (1.10). Let \( x_0(\cdot, \eta_0) \) be the mild solution of the problem

\[
(1.11) \quad \left\{ \begin{array}{ll}
\mathrm{d}x_0(t) = (Ax_0(t) + \tilde{a}(x_0(t))) \, dt + \tilde{b}(x_0(t)) \, dw(t), \\
x_0(0) = \eta_0.
\end{array} \right.
\]

If \( \eta_\varepsilon \rightarrow \eta_0 \) weakly in \( H \) as \( \varepsilon \rightarrow 0^+ \) then \( \bar{x}_\varepsilon(\cdot, \eta_\varepsilon) \rightarrow x_0(\cdot, \eta_0) \) weakly in \( C([0, T], H) \) as \( \varepsilon \rightarrow 0^+ \) for any fixed \( T > 0 \).

2. PROOFS

To start with, let us recall a few well-known results.

**Proposition 2.1.** Let \( \mu_n, n \geq 0 \), be centered Gaussian measures on a separable Hilbert space \( Y \) with covariance operators \( \Gamma_n \). Then \( \mu_n \rightarrow \mu_0 \) weakly in \( Y \) if and only if \( \| \Gamma_n - \Gamma_0 \|_{L^1} \rightarrow 0, n \rightarrow \infty \).

For the proof, see e.g. [1]. We will need the following criterion for weak convergence of measures. Let \( (M, d) \) be a separable metric space, denote by \( BL(M) \) the space of all bounded Lipschitz functions on \( M \), that is

\[
BL(M) = \left\{ f : M \rightarrow \mathbb{R} ; \| f \|_{BL} \equiv \sup_{x \in M} |f(x)| + \sup_{x \neq y \in M} \frac{|f(x) - f(y)|}{d(x, y)} < \infty \right\}.
\]

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Let $\mu_n, \mu$ be Borel probabilities on $M$, then $\mu_n \rightarrow \mu$ weakly in $M$ if and only if

$$\lim_{n \to \infty} \int_M f \, d\mu_n = \int_M f \, d\mu$$

for any $f \in BL(M)$, see e.g. [3], Theorem 11.3.3. Moreover, by the same theorem, the metrics $\beta$, defined for probability measures on $M$ by

$$\beta(\mu, \nu) = \sup \left\{ \left| \int_M f \, d(\mu - \nu) \right| : \|f\|_{BL} \leq 1 \right\},$$

metrizes the weak convergence.

The next proposition looks almost obvious, nonetheless, it will be very useful in the sequel.

Let $U, V, H$ be real separable Hilbert spaces, let $w$ be a $U$-valued $(\mathcal{F}_t)$-adapted Wiener process with a nuclear covariance operator $W$, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, let $0 \leq s < t$.

**Proposition 2.2.** Let $\alpha : H \rightarrow V$ be a Lipschitz continuous mapping, let $\sigma : [s, t] \times H \rightarrow L(U, V)$ be a measurable mapping such that

$$|\sigma(r, x)|_{L(U, V)} \leq M(1 + |x|_H), \quad |\sigma(r, x) - \sigma(r, y)|_{L(U, V)} \leq M|x - y|_H$$

for a constant $M$ and any $r \in [s, t]$, $x, y \in H$. Let $g \in BL(V)$, define

$$\psi(y) = \mathbb{E} \left( \alpha(y) + \int_s^t \sigma(r, y) \, dw(r) \right), \quad y \in H.$$

Let $u : \Omega \rightarrow H$ be a $\mathcal{F}_s$-measurable random variable with $\mathbb{E} |u|_H^2 < \infty$. Then

$$\mathbb{E} \left( g \left( \alpha(u) + \int_s^t \sigma(r, u) \, dw(r) \right) \right) = \psi(u) \quad \mathbb{P}\text{-almost surely.} \quad (2.1)$$

**Proof.** To simplify notation, we will treat the case $\alpha = 0$; it can be seen easily that this leads to no loss of generality.

Take an arbitrary $\gamma > 0$, let $\{z_i : i \in \mathbb{N}\}$ be a dense subset of $H$ and define a Borel partition $\{A(i), \ i \in \mathbb{N}\}$ of $H$ by

$$A(1) = \{\xi \in H ; |\xi - z_i| < \gamma\}, \quad A(i + 1) = \{\xi \in H ; |\xi - z_i| < \gamma\} \setminus \bigcup_{j < i} A(j).$$
We may assume that \( z_j \in A(j) \) for any \( j \in \mathbb{N} \). Define

\[
n_{x}(\omega) = \sum_{i=1}^{\infty} \chi_{A(i)}(u(\omega)) z_i, \quad \omega \in \Omega,
\]

\[
\psi_{z}(\xi) = \sum_{i=1}^{\infty} \chi_{A(i)}(\xi) \mathbb{E}g \left( \int_{s}^{t} \sigma(r, z_i) \, dw(r) \right), \quad \xi \in H,
\]

where \( \chi_{A(i)} \) stands for the indicator function of the set \( A(i) \). Obviously,

\[
E|u - u_{\gamma}|_H^2 < \gamma^2
\]

and

\[
g \left( \int_{s}^{t} \sigma(r, u_{\gamma}) \, dw(r) \right) = \sum_{i=1}^{\infty} \chi_{A(i)}(u_{\gamma}) g \left( \int_{s}^{t} \sigma(r, z_i) \, dw(r) \right) \quad \text{P-a.s.}
\]

Therefore

\[
E \left[ g \left( \int_{s}^{t} \sigma(r, u_{\gamma}) \, dw(r) \right) \right] = \sum_{i=1}^{\infty} \chi_{A(i)}(u_{\gamma}) \mathbb{E} \left[ g \left( \int_{s}^{t} \sigma(r, z_i) \, dw(r) \right) \right] = \psi_{z}(u_{\gamma}).
\]

Furthermore,

\[
E \left[ g \left( \int_{s}^{t} \sigma(r, u_{\gamma}) \, dw(r) \right) - g \left( \int_{s}^{t} \sigma(r, u) \, dw(r) \right) \right]^2 \leq \|g\|_{L_{2}(\mathcal{F})}^{2} \mathbb{E} \left[ \int_{s}^{t} \sigma(r, u_{\gamma}) - \sigma(r, u) \right] \, dw(r)_{V}^2
\]

\[
\leq \|g\|_{L_{2}(\mathcal{F})}^{2} \text{tr}(W) \int_{s}^{t} \mathbb{E} |\sigma(r, u_{\gamma}) - \sigma(r, u)|_{L_{2}(V, V)}^2 \, dr
\]

\[
\leq \|g\|_{L_{2}(\mathcal{F})}^{2} M \text{tr}(W) \int_{s}^{t} \mathbb{E} |u - u_{\gamma}|_{H}^2 \, dr
\]

\[
\leq \|g\|_{L_{2}(\mathcal{F})}^{2} M^2 \text{tr}(W)(t - s)\gamma^2,
\]

hence

\[
E \left[ g \left( \int_{s}^{t} \sigma(r, u_{\gamma}) \, dw(r) \right) \right] \xrightarrow{\gamma \to 0^+} E \left[ g \left( \int_{s}^{t} \sigma(r, u) \, dw(r) \right) \right] \quad \text{in } L_{2}(\Omega).
\]
Analogously, take $\xi \in H$, then there exists $\xi_0 \in \mathbb{N}$ such that $\xi \in A(\xi_0)$, so

$$\left| \psi_n(\xi) - \psi(\xi) \right|^2 = \left| \mathbb{E}_\gamma \left( \int_0^t \sigma(r, z_{\xi_0}) \, dw(r) \right) - \mathbb{E}_\gamma \left( \int_0^t \sigma(r, \xi) \, dw(r) \right) \right|^2 \leq \|\sigma\|^2_{L^2} M^2 \text{tr}(W(t - s)) \gamma^2.$$ 

This yields

$$\psi_n(\xi) \xrightarrow{n \to \infty} \psi(\xi) \quad \text{for all } \xi \in H$$

and (2.1) follows. \hfill \Box

We are prepared to return to the equation (1.5). From now on, the hypotheses of Theorem 1.1 are assumed to be fulfilled.

**Lemma 2.3.** For every $T > 0$ and $p \in [2, \infty)$ there exists a constant $K$ such that for any $s \in [0, e_0)$ we have

$$\mathbb{E}|x_n(t)|_p^p \leq K(1 + |\eta_n|_p^p), \quad 0 \leq t \leq T,$$

provided the right-hand side is finite.

Since the estimate in (A2) is uniform in $\varepsilon$, Lemma 2.3 follows virtually from [2], Theorem 7.4, or [9], Theorem 1.4, as tracing the proofs of these theorems one can observe easily that the constant $K$ can be obtained independent of $\varepsilon \in [0, e_0)$.

Given $N \in \mathbb{N}$, let us define

$$g^N(x) = \begin{cases} 
\frac{x}{N} & \text{if } |x| \leq N, \\
\frac{|x|}{|x|} & \text{otherwise,}
\end{cases}$$

and set $\eta^N = g^N(\eta)$. By assumption, $\eta_n \rightarrow \eta_0$ weakly in $H$, so the continuity of $g^N$ yields that $\eta^N \rightarrow \eta_0^N$ weakly in $H$ for any $N \in \mathbb{N}$.

Let us fix $T > 0$ and an arbitrary sequence $\varepsilon_n \in (0, e_0)$, $\varepsilon_n \rightarrow 0$. For brevity we set

$$a_{\varepsilon_n} = a_n, \quad b_{\varepsilon_n} = b_n, \quad w_{\varepsilon_n} = w_n, \quad p_{\varepsilon_n} = p_n$$

and so forth. Further, set

$$x_n = x_{\varepsilon_n} (\cdot, \eta_n), \quad x^N_n = x_{\varepsilon_n} (\cdot, \eta^N_n), \quad x^N = x_0 (\cdot, \eta^N).$$

We denote the integral with respect to the measure $P_\varepsilon$ simply by $\mathbb{E}$ with omitted subscript as it leads to no ambiguity.
Proposition 2.4. For any $N \in \mathbb{N}$ and $\gamma > 0$ there exists $\varrho > 0$ such that

$$\sup_{n \geq 0} E \max_{0 \leq s, t \leq T} |x_n^N(t) - x_n^N(s)|^2 \leq \gamma.$$ 

Proof. Let us denote by $K_i$ generic constants independent of $n \geq 0$. Choose $p > 2$ and $\lambda \in (0, \frac{1}{2} - \frac{1}{p})$. By Remark following Theorem 1.1 in [9] we have

$$E \sup_{0 \leq s, t \leq T} \left| \int_0^t e^{\lambda(t-s)} b_n(r, x_n^N(r)) dw_n(r) - \int_0^s e^{\lambda(t-s)} b_n(r, x_n^N(r)) dw_n(r) \right|^2 \leq K_1 \zeta^{2\lambda},$$

for any $\zeta > 0$. Indeed,

$$E \sup_{0 \leq s, t \leq T} \left| \int_0^t e^{\lambda(t-s)} b_n(r, x_n^N(r)) dw_n(r) - \int_0^s e^{\lambda(t-s)} b_n(r, x_n^N(r)) dw_n(r) \right|^2 \leq K_3 C^2 \lambda^2$$

and (2.2) follows by (A4) and Lemma 2.3. By an analogous procedure we obtain

$$E \sup_{0 \leq s, t \leq T} \left| \int_0^t e^{\lambda(t-s)} a_n(r, x_n^N(r)) dr - \int_0^s e^{\lambda(t-s)} a_n(r, x_n^N(r)) dr \right|^2 \leq K_3 \lambda^{2\lambda}. $$

(2.3)
First, we prove that there exists $\theta_1 > 0$ such that

\begin{equation}
\sup_{n\geq 0} \sup_{0 < t < 2e_1} |x_{n}^N(t) - \eta_n^N|^2 \leq K_4\gamma
\end{equation}

(with a constant $K_4$ independent of $\theta_1$, of course). In fact, by the definition of a mild solution

$$x_{n}^N(t) - \eta_n^N = \left[e^{At} - I\right]\eta_n^N + \int_0^t e^{A(t-s)}a_n(s, x_{n}^N(s)) \, ds$$

$$+ \int_0^t e^{A(t-s)}b_n(s, x_{n}^N(s)) \, dw_n(s)$$

\[\equiv J_1 + J_2 + J_3.\]

By (2.2) and (2.3) we have

\begin{equation}
\sup_{n\geq 0} \sup_{0 < t < \zeta} \left( |J_2|^2 + |J_3|^2 \right) \leq K_5\zeta^2
\end{equation}

for any $\zeta > 0$. Since the random variables $\eta_n^N$ weakly converge, they are tight by the Prokhorov theorem, hence there exists a compact set $C \subseteq H$ such that

$$\sup_{n\geq 0} P_n\{\eta_n^N \notin C\} \leq \gamma.$$ 

As is well known, $e^{At}x \rightarrow x$ as $t \searrow 0$ uniformly in $x \in C$, hence we can find $\theta_1 \in (0, \gamma^{1/2})$ such that

$$\sup_{0 < t < 2e_1} \sup_{x \in C} |e^{At}x - x| < \sqrt{\gamma}.$$ 

Then

\begin{align*}
E \sup_{0 < t < 2e_1} |e^{At}\eta_n^N - \eta_n^N|^2 &\leq \gamma + E_X \sup_{0 < t < 2e_1} |e^{At}\eta_n^N - \eta_n^N|^2 \\
&\leq \gamma + K_6 E_X \sup_{0 < t < \zeta} |\eta_n^N|^2 \\
&\leq \gamma + K_7 E_X \sup_{0 < t < \zeta} |\eta_n^N|^2 < \gamma + K_8 N^2 P_n\{\eta_n^N \notin C\} \leq K_7\gamma.
\end{align*}

This estimate together with (2.5) yields (2.4).

The semigroup $(e^{At})$ is analytic by (A1), hence the $L(H)$-valued function $t \mapsto e^{At}$ is uniformly continuous on $[\theta_1, T]$ (cf. e.g. [2], Theorem A.7). So we can find $\theta_2 \in (0, \gamma^{1/2})$ such that

\begin{equation}
\sup_{\theta_1 < t < T} \sup_{|s| \leq \theta_2} |e^{At} - e^{As}| \leq \sqrt{\gamma}.
\end{equation}

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Take \( s, t \in [e_2, T] \), \( t > s \), then

\[
x_n^N(t) - x_n^N(s) = (e^{At} - e^{As})\eta_n^N + \left\{ \int_0^t e^{A(t-r)}a_n(r, x_n^N(r))\,dr - \int_0^s e^{A(s-r)}a_n(r, x_n^N(r))\,dr \right\} + \left\{ \int_0^t e^{A(t-r)}b_n(r, x_n^N(r))\,dw_n(r) - \int_0^s e^{A(s-r)}b_n(r, x_n^N(r))\,dw_n(r) \right\}
\]

\[= Q_1 + Q_2 + Q_3.\]

Obviously,

\[
\sup_{\eta \geq 0} \sup_{[t, s] \subseteq [e_2, T]} \left\{ |Q_2|^2 + |Q_3|^2 \right\} \leq K_{s, e_2}^2 \leq K_{s, \gamma}
\]

by (2.2) and (2.3). Finally,

\[
\sup_{\eta \geq 0} \sup_{[t, s] \subseteq [e_2, T]} \left\{ |e^{At} - e^{As}| \eta_n^N \right\} \leq N^2 \gamma
\]

by (2.6), so we have obtained

\[
(2.7) \quad \sup_{\eta \geq 0} \sup_{[t, s] \subseteq [e_2, T]} \left\{ |x_n^N(t) - x_n^N(s)|^2 \right\} \leq K_{\gamma}.
\]

Combining the estimates (2.7) and (2.4) we complete the proof of Proposition 2.4.

\[\Box\]

**Corollary 2.5.** For any \( N \in \mathbb{N} \) and \( \gamma > 0 \) there exists a partition \( \{0 = t_0 < \ldots < t_k = T\} \) of the interval \([0, T]\) such that

\[
(2.8) \quad \sup_{\eta \geq 0} \sup_{0 \leq t_i < s < t_{i+1} \leq T} \left\{ \max_{0 \leq i < k} \max_{t \in [t_i, t_{i+1}]} \left| x_n^N(t) - x_n^N(s) \right|^2 \right\} \leq \gamma.
\]

To proceed further we need a suitable discretization in time of the process \( x_n^N \).

Let \( H = \{0 = t_0 < \ldots < t_k = T\} \) be a partition of the interval \([0, T]\). Define

\[
z_n^N(t) = e^{At}x_n^N + \sum_{i=0}^{k-1} \int_{t_i \wedge t}^{t_{i+1} \wedge t} e^{A(t-s)}a_n(s, x_n^N(s))\,ds
\]

\[
+ \sum_{i=0}^{k-1} \int_{t_i \wedge t}^{t_{i+1} \wedge t} e^{A(t-s)}b_n(s, x_n^N(s))\,dw_n(s)
\]

\[= z_n^N(t).\]
for any $t \in [0,T]$ and $n \geq 0$, where, as usual, $a \wedge b = \min(a, b)$. (Note that the process $x^N$ depends on $N$ as well.) We have the following estimate:

**Lemma 2.6.** For every $N \in \mathbb{N}$ and $\gamma > 0$ there exists a partition $\Pi = \{0 = t_0 < \ldots < t_k = T\}$ of the interval $[0, T]$ such that

$$\sup_{n \geq 0} \max_{0 \leq i \leq k} \mathbb{E}|x^N(t_i) - x^N(t_{i+1})|^2 \leq \gamma.$$ 

**Proof.** Take $\gamma > 0$ and find a partition $\Pi = \{0 = t_0 < \ldots < t_k = T\}$ such that

$$\text{(2.8)} \text{ holds.}$$

By (2.9) we obtain

$$\mathbb{E}|x^N(t_{j+1}) - x^N(t_{j+1})|^2 \leq 2T \int_0^{t_{j+1}} |e^{a(t_{j+1} - s)}|_{L(H)}^2 \mathbb{E} \left[ \left( \sum_{i=0}^{j} \chi_{[t_i, t_{i+1})}(s) \left( a_n(s, x^N(t_i)) - a_n(s, x^N(t_i)) \right) \right)^2 ds \right] \leq 2\tau \mathbb{E} \left[ \left( \sum_{i=0}^{j} \chi_{[t_i, t_{i+1})}(s) \left( a_n(s, x^N(t_i)) - a_n(s, x^N(t_i)) \right) \right)^2 ds \right]$$

$$\leq K_0 \sum_{i=0}^{j} \int_{t_i}^{t_{i+1}} \mathbb{E}|x^N(t_i) - x^N(t_i)|^2 ds \leq 2K_0 \sum_{i=0}^{j} \int_{t_i}^{t_{i+1}} \mathbb{E}|x^N(t_i) - x^N(t_i)|^2 ds + 2K_0 \sum_{i=0}^{j} \int_{t_i}^{t_{i+1}} \mathbb{E}|x^N(t_i) - x^N(t_i)|^2 ds$$

$$\leq 2K_0 T \gamma + 2K_0 \sum_{i=0}^{j} \int_{t_i}^{t_{i+1}} \mathbb{E}|x^N(t_i) - x^N(t_i)|^2 ds.$$ 

Define

$$f_n(t) = \mathbb{E}|x^N(t_i) - x^N(t_i)|^2, \quad t_i \leq t < t_{i+1}, \; i = 0, \ldots, k - 1.$$ 

The above estimate implies

$$f_n(t) \leq 2K_0 T \gamma + 2K_0 \int_0^t f_n(s) ds,$$

and $K_0$ is independent of $n$, so Lemma 2.6 follows by the Gronwall inequality. $\Box$
The crucial step in the proof of Theorem 1.1 is to establish that the finite dimensional distributions of the process \( x_n \) converge weakly as \( n \to \infty \).

**Proposition 2.7.** Let \( N \in \mathbb{N} \) and a partition \( \Pi = \{0 = t_0 < \ldots < t_k = T\} \) be given. Let \( \nu_n, n \geq 0 \), be Borel probabilities on \( H^{k+1} \) defined by

\[
\nu_n = (x_n^{(t_0)}, \ldots, x_n^{(t_k)})(P_n),
\]

that is

\[
\nu_n(Q) = P_n(\{w; (x_n^{(t_0)}, \ldots, x_n^{(t_k)}) \in Q\})
\]

for any Borel set \( Q \) in \( H^{k+1} \). Then \( \nu_n \to \nu_0 \) weakly in \( H^{k+1} \) as \( n \to \infty \).

**Proof.** The proof proceeds by induction. By assumptions of Theorem 1.1 we have \( x_n^{(t_0)} \to x_0^{(t_0)} \) weakly in \( H \). Assume that for some \( l, 0 \leq l \leq k - 1 \), the convergence

\[
(2.10) \quad \mu_n \equiv \nu_n(P_n) \to \mu_0 \equiv \nu_0(P_0) \quad \text{weakly in } H^{l+1}
\]

has been established, where we set

\[
u_n = (x_n^{(t_0)}, \ldots, x_n^{(t_l)}).
\]

Let us define

\[
\alpha_n: H^{l+1} \to H^{l+2},
\]

\[
(\xi_0, \ldots, \xi_l) \mapsto \left(\xi_0, \ldots, \xi_l, e^{A(t_{l+1}-t_0)}\xi_l + \int_{t_0}^{t_{l+1}} e^{A(t_{l+1}-s)}\alpha_n(s, \xi_l) \, ds\right),
\]

and

\[
B_n: H^{l+1} \to L^1(\Omega^n; H^{l+2}),
\]

\[
(\xi_0, \ldots, \xi_l) \mapsto \left(0, \ldots, 0, \int_{t_l}^{t_{l+1}} e^{A(t_{l+1}-s)}b_n(s, \xi_l) \, dw_n(s)\right).
\]

Obviously,

\[
x_n^{(t_{l+1})} = e^{A(t_{l+1}-t_l)}x_n^{(t_l)} + \int_{t_l}^{t_{l+1}} e^{A(t_{l+1}-s)}\alpha_n(s, x_n^{(t_l)}(s)) \, ds
\]

\[
+ \int_{t_l}^{t_{l+1}} e^{A(t_{l+1}-s)}b_n(s, x_n^{(t_l)}(s)) \, dw_n(s),
\]
thus

\[( x_n(t_0), \ldots, x_n(t_{i+1}) ) = \alpha_n(u_n) + B_n(u_n). \]

Take \( g \in BL(H^{i+2}) \) arbitrary and set

\[ h_n(y) = Eg(\alpha_n(y) + B_n(y)), \quad y \in H^{i+1}. \]

Proposition 2.2 yields

\[ Eg(x_n^H(t_0), \ldots, x_n^H(t_{i+1})) = \int_{H^{i+1}} h_n(\xi) d\mu_n(\xi). \]

It follows

\[ |Eff(x_n^H(t_0), \ldots, x_n^H(t_{i+1})) - \int_{H^{i+1}} h_n(\xi) d\mu_n(\xi)| \leq M_1(n) + M_2(n). \]

To proceed further, we check that \( h_n \in BL(H^{i+1}) \) and

\[ (2.11) \quad \sup_{n \geq 0} \| h_n \|_{BL} < \infty. \]

Indeed,

\[ |h_n(\xi) - h_n(\zeta)| \leq \| g \|_{BL} | \alpha_n(\xi) - \alpha_n(\zeta) + B_n(\xi) - B_n(\zeta)|_{H^{i+1}} \]

\[ \leq K_1 \| g \|_{BL} |\xi - \zeta|_{H^{i+1}} \]

for any \( \xi, \zeta \in H^{i+1} \) by (A3). Therefore, \( M_2(n) \to 0 \) as \( n \to \infty \) due to (2.10). Now, note that

\[ (2.12) \quad \lim_{n \to \infty} h_n(\xi) = h_0(\xi) \quad \text{for any} \ \xi \in H^{i+1}. \]

First,

\[ \alpha_n(\xi) - \alpha_0(\xi) = \left( 0, \ldots, 0, \int_{t_n}^{t_{i+1}} e^{A(t_{i+1} - s)} [a_n(s, \xi) - a_0(s, \xi)] \, ds \right) \]

\[ \stackrel{\text{n \to \infty}}{\to} 0 \quad \text{in} \ H^{i+1} \]

by (1.8). Further,

\[ \int_{t_n}^{t_{i+1}} e^{A(t_{i+1} - s)} b_0(s, \xi) \, dw_n(s) \]
is a centered Gaussian random variable in $H$ with a covariance operator
\[ \int_{t_i}^{t_{i+1}} e^{A(t_{i+1}-t)}h_n(s,\xi)W_n(b_n(s,\xi))e^{A(t_{i+1}-t)} \, ds, \]
so $B_n(\xi) \rightarrow B_0(\xi)$ weakly in $H^{1+2}$ by (1.9) and Proposition 2.1. Hence $\alpha_n(\xi) + B_n(\xi) \rightarrow \alpha_0(\xi) + B_0(\xi)$ weakly in $H^{1+2}$ for any $\xi \in H^{1+1}$ and recalling the definition of $h_n$ and of the weak convergence we see that (2.12) holds.

Take $\delta > 0$ arbitrary, there exists a compact set $K \subset H^{1+1}$ such that
\[ \inf_{n \geq 0} \mu_n(K) \geq 1 - \delta \]
since the measures $\{\mu_n\}$ are weakly convergent. Due to the compactness of $K$, (2.12) and (2.11) the functions $h_n$ converge to $h_0$ uniformly on $K$, hence
\[ \lim_{n \to \infty} \int_K |h_n(\xi) - h_0(\xi)| \, d\mu_n(\xi) = 0. \]
Finally,
\[ \int_{H^{1+1} \setminus K} |h_n(\xi) - h_0(\xi)| \, d\mu_n(\xi) \leq 2 \sup_{n \geq 0} \mu_{n+1} \delta \]
by (2.13), so $M_2(n) \rightarrow 0$ as well and Proposition 2.7 follows.

**Corollary 2.8.** Let $N \in \mathbb{N}$ and a partition $\Lambda = \{0 = s_0 < \ldots < s_q = T\}$ of the interval $[0,T]$ be given. Then
\[ (x_n^0(s_0),\ldots,x_n^0(s_q)) \overset{(P_n)}{\longrightarrow} (x_0^0(s_0),\ldots,x_0^0(s_q)) \quad \text{weakly in } H^{1+1}. \]
In other words, $x_n^0$'s converge in law to $x_0^0$ in the sense of finite dimensional distributions.

**Proof.** Take $\gamma > 0$ arbitrary. According to Lemma 2.6, there exists a partition $\Pi = \{0 = t_0 < \ldots < t_k = T\}$ refining the partition $\Lambda$ and such that
\[ \sup_{n \geq 0} \max_{i=0,\ldots,k} E \left[ \sup_{i=0,\ldots,k} \left| x_n^\Pi(t_i) - x_0^\Pi(t_i) \right|^2 \right] \leq \frac{\gamma^2}{q+1}. \]
In particular,
\[ \sup_{n \geq 0} E \left[ \left( x_n^0(s_0),\ldots,x_n^0(s_q) \right) - \left( x_0^0(s_0),\ldots,x_0^0(s_q) \right) \right]^2_{H^{1+1}} \]
\[ = \sup_{n \geq 0} E \sum_{i=0}^q \left| x_n^0(s_i) - x_0^0(s_i) \right|^2 \leq \gamma^2. \]
Let $g \in BL(H^{s+1})$ be a bounded Lipschitz function, then

$$
|\mathbb{E}g(x_n^N(s_0), \ldots, x_n^N(s_k)) - \mathbb{E}g(x_0^N(s_0), \ldots, x_0^N(s_k))| \\
\leq |\mathbb{E}[g(x_n^N(s_0), \ldots, x_n^N(s_k)) - g(x_0^N(s_0), \ldots, x_0^N(s_k))]| \\
+ |\mathbb{E}[g(x_0^N(s_0), \ldots, x_0^N(s_k)) - \mathbb{E}g(x_0^N(s_0), \ldots, x_0^N(s_k))]| \\
+ |\mathbb{E}[g(x_0^N(s_0), \ldots, x_0^N(s_k)) - g(x_0^N(s_0), \ldots, x_0^N(s_k))]| \\
\equiv I_1(n) + I_2(n) + I_3(0).
$$

We have

$$
I_1(n) \leq \|g\|_{BL} \left( \mathbb{E}[|x_n^N(s_0), \ldots, x_n^N(s_k) - (x_0^N(s_0), \ldots, x_0^N(s_k))|^2_{H^{s+1}}] \right)^{1/2} \\
\leq \|g\|_{BL} \gamma
$$

for any $n \geq 0$ by (2.14). Moreover, Proposition 2.7 yields

$$
(x_n^N(t_0), \ldots, x_n^N(t_k))(P_n) \xrightarrow{n \to \infty} (x_0^N(t_0), \ldots, x_0^N(t_k))(P_0)
$$

weakly in $H^s$, so also

$$
\lim_{n \to \infty} I_2(n) = 0
$$
as $\Pi$ is a refinement of the partition $\Lambda$. Therefore, for any $\gamma > 0$ and $g \in BL(H^{s+1})$ we have

$$
|\mathbb{E}g(x_n^N(s_0), \ldots, x_n^N(s_k)) - \mathbb{E}g(x_0^N(s_0), \ldots, x_0^N(s_k))| \leq (2\|g\|_{BL} + 1) \gamma
$$

for all $n$ sufficiently large, which proves the corollary.

**Proof of Theorem 1.1.** Take an arbitrary $\gamma > 0$. Note that there exists $N \in \mathbb{N}$ such that

$$
\sup_{n > 0} \mathbb{P}_n \left\{ \omega; \sup_{0 < s \leq T} |x_n(t) - x_0^N(t)| > 0 \right\} \leq \gamma.
$$

Indeed, we use again the Prokhorov theorem and the weak convergence of $\eta_n$'s to find a ball $K(0, N) = \{z \in H; |z| \leq N\}$ such that

$$
\sup_{n > 0} \mathbb{P}_n [\eta_n \notin K(0, N)] < \gamma.
$$

Then a standard local uniqueness argument (see e.g. [9], Lemma 4.2) and the continuity of sample paths yield

$$
\mathbb{P}_n \left\{ \eta_n \in K(0, N) \right\} \cap \left\{ \sup_{0 < s \leq T} |x_n(t) - x_0^N(t)| > 0 \right\} = 0,
$$

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hence (2.15) holds true. Using Corollary 2.5 we find a partition \( \{0 = t_0 < \ldots < t_k = T\} \) of the interval \([0, T]\) such that

\[
(2.16) \quad \sup_{n \geq 0} E \left( \max_{i=0}^{k-1} \sup_{t_i \leq t \leq t_{i+1}} \left| x_n^i(t) - x_n^i(t_i) \right| \right)^2 \leq \gamma^2.
\]

Let \( f \in BL(C([0, T], H)) \) be a bounded Lipschitz function. Denote by \( \mathcal{B} \) the space of all functions \( \psi: [0, T] \rightarrow H \) which are right continuous with left limits on \([0, T]\) and continuous on \((0, T) \setminus \{t_1, \ldots, t_{k-1}\} \); endow \( \mathcal{B} \) with the sup-norm. There exists a Lipschitz function \( f^* \in BL(\mathcal{B}) \) such that \( f^* = f \) on \( C([0, T], H) \) and \( \|f^*\|_{BL} = \|f\|_{BL} \) (see e.g. [3], Theorem 6.1.1 and Proposition 11.2.2). Define

\[
f^*: H^{k+1} \rightarrow \mathbb{R}, \quad y = (y_0, \ldots, y_k) \mapsto f^*(y),
\]

where \( y \) is defined by \( y_i(t) = y_i, t_i \leq t < t_{i+1} \). Then \( f^* \in BL(H^{k+1}) \) and \( \|f^*\|_{BL} \leq \|f\|_{BL} \). Further, set \( \bar{x}_n(t) = x_n^i(t), t_i \leq t < t_{i+1}, \ i = 0, \ldots, k-1; \) obviously, \( \bar{x}_n \) is a stochastic process with paths in \( \mathcal{B} \).

Consequently,

\[
\left| E(f(x_n) - E(f(x_0)) \right| \leq E\left| f(x_n) - f(x_n^i) \right| + E\left| f(x_n^i) - f^i(\bar{x}_n) \right| + E\left| f^i(\bar{x}_n) - f^i(\bar{x}_0) \right| + E\left| f^i(\bar{x}_0) - f(x_0) \right| \leq Z_1(n) + Z_2(n) + Z_3(n) + Z_4(0).
\]

Setting \( V(n) = \{ \sup_{0 \leq t \leq T} |x_n(t) - x_n^i(t)| > 0 \} \) we get

\[
Z_1(n) = E_{x \in V(n)} \left| f(x_n) - f(x_n^i) \right| \leq 2\|f\|_{BL} P_n(V(n)) \leq 2\|f\|_{BL} \gamma
\]

for any \( n \geq 0 \) by (2.15). Furthermore,

\[
Z_2(n) = E\left| f^i(\bar{x}_n) - f^i(\bar{x}_0) \right| \leq \|f^i\|_{BL} E \sup_{0 \leq t \leq T} |x_n^i(t) - \bar{x}_n(t)| \leq \|f^i\|_{BL} \gamma
\]

for all \( n \geq 0 \) by (2.16). Finally,

\[
Z_3(n) = E\left| f^i(x_0^i(t_0), \ldots, x_n^i(t_k)) - f^i(x_0^i(t_0), \ldots, \bar{x}_0^i(t_k)) \right| \xrightarrow{n \to \infty} 0
\]

by Corollary 2.8, so we have established that, given arbitrary \( \gamma > 0 \) and \( f \in BL(C([0, T], H)) \), one has

\[
\left| E(f(x_n) - E(f(x_0)) \right| \leq (6\|f\|_{BL} + 1) \gamma
\]

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for all \( n \) sufficiently large. This means that

\[
x_n(\cdot, t_n) \to x_0(\cdot, t_0)
\]

weakly in \( C([0, T], H) \)

for any sequence \( \varepsilon_n \searrow 0 \). Due to the metrizability of the weak convergence, (2.17) is equivalent to the assertion of Theorem 1.1. The proof is complete. \( \square \)

**Proof of Proposition 1.2.** Fix \( x \in H \) and set

\[
K_n(s) = \int_0^s k_n(u) \, du.
\]

Note that by (A2) we have

\[
\sup_{s \in [0,T]} \sup_{t \in [0,T]} |K_n(s)| < \infty, \quad \sup_{s \in [0,T]} \sup_{t \in [0,T]} |K_n(s)| \leq L.
\]

Take \( \delta > 0 \), then

\[
\int_0^t e^{A(t-s)} k_n(s) \, ds = \int_0^{t-\delta} e^{A(t-s)} k_n(s) \, ds + \int_{t-\delta}^t e^{A(t-s)} k_n(s) \, ds
\]

\[
= I_1 + I_2.
\]

First, integrating by parts we obtain

\[
I_1 = e^{A\delta} K_n(t-\delta) + \int_0^{t-\delta} A e^{A(t-s)} K_n(s) \, ds.
\]

The semigroup \( (e^{At}) \) is analytic, so the operators \( A e^{At} \) are continuous and (1.8) yields

\[
\lim_{\varepsilon \to 0^+} e^{A\delta} K_n(t-\delta) = 0
\]

and

\[
\lim_{\varepsilon \to 0^+} A e^{A(t-s)} K_n(s) = 0, \quad s \in [0, t-\delta].
\]

Therefore, for any \( \delta > 0 \) fixed we get \( I_1 \to 0 \) as \( \varepsilon \to 0^+ \) by the dominated convergence theorem. Moreover,

\[
|I_2| \leq \sup_{0 \leq s \leq T} |e^{At}| L \delta,
\]

so \( I_2 \) can be made arbitrarily small (uniformly in \( \varepsilon \)) by choosing \( \delta \) sufficiently small, and the first statement of Proposition 1.2 follows. The second assertion can be proved in a similar way. \( \square \)
The following example shows that the assumptions on the convergence of coefficients adopted in Theorem 1.1 are strictly weaker than those used in Proposition 1.2.

Example 2.9. Let \( A : \text{Dom}(A) \to H \) be a selfadjoint operator such that there exists an orthonormal basis \( \{ \phi_n \}_{n=1}^\infty \) consisting of eigenvectors of the operator \( A \), \( A\phi_n = \lambda_n \phi_n \), where \( 0 > \lambda_n \to -\infty \). As is well known, \( A \) is an infinitesimal generator of an analytic selfadjoint \( C_0 \)-semigroup \( \{ e^{At} \} \) given by

\[
e^{At} = \sum_{n=1}^{\infty} e^{\lambda_n t} \phi_n \otimes \phi_n, \quad t \geq 0.
\]

Let us define \( a_n : [0, T] \times H \to H \) by

\[
a_n(s, x) = f_n, \quad s \in [0, T], \ x \in H, \ n \geq 1.
\]

Then

\[
\left| \int_0^t e^{A(t-s)} a_n(s, x) \, ds \right|_H = \left| \sum_{k=1}^{\infty} \int_0^t e^{\lambda_k(t-s)} (f_k, a_n(s, x)) f_k \, ds \right|_H \\
\leq \int_0^t e^{\lambda_k(t-s)} f_k \, ds = \left| \frac{1}{\lambda_k} (e^{\lambda_k t} - 1) f_k \right| \\
\to \frac{1}{|\lambda_n|} \text{ as } n \to \infty.
\]

On the other hand,

\[
\left| \int_0^t a_n(s, x) \, ds \right|_H = t
\]

for every \( n \geq 1 \).

Further, let us consider a Wiener process \( w \) in \( H \) of the form \( w(t) = \beta(t) f_1 \), where \( \beta \) is a standard one-dimensional Brownian motion. The covariance operator of the process \( w \) is \( W = f_1 \otimes f_1 \) (i.e., \( Wh = (h, f_1) f_1 \) for \( h \in H \)). Obviously, \( W = W^{1/2} \) and \( W f_k = \delta_{1k} f_1 \) for \( k \geq 1 \). Define operators \( b_n(s, x) \in L(H) \) by

\[
b_n(s, x) h \equiv b_n h = \sum_{k=1}^{\infty} (h, f_k) f_{k+n-1}, \quad s \in [0, T], \ x, h \in H, \ n \geq 1.
\]
Noticing that \( b_n f_1 = f_n \) for \( n \geq 1 \) and using the positivity of the operator \( e^{A(t-s)} b_n W b_n^* e^{A(t-s)} \), we obtain

\[
\left| \int_0^t e^{A(t-s)} b_n W b_n^* e^{A(t-s)} \, ds \right|_{\mathcal{V}} = \text{tr} \left( \int_0^t e^{A(t-s)} b_n W b_n^* e^{A(t-s)} \, ds \right) = \sum_{k=1}^{\infty} \left( \left( \int_0^t e^{A(t-s)} b_n W b_k^* e^{A(t-s)} \, ds \right) f_k, f_k \right) = \int_0^t \sum_{k=1}^{\infty} \left( e^{A(t-s)} b_n W b_k^* e^{A(t-s)} f_k, f_k \right) \, ds = \int_0^t \sum_{k=1}^{\infty} \left| W b_k^* e^{A(t-s)} f_k \right|^2 \, ds = \int_0^t \sum_{k=1}^{\infty} \left| e^{A(t-s)} b_n W f_k \right|^2 \, ds = \int_0^t |e^{A(t-s)} f_1|^2 \, ds = \int_0^t |e^{A(t-s)} f_n|^2 \, ds \leq \frac{1}{2|\lambda_n|} \quad n \to \infty.
\]

At the same time,

\[
\left| \int_0^t b_n W b_n^* \, ds \right|_{\mathcal{V}} = t \left| b_n W b_n^* \right|_{\mathcal{V}} = t \text{ tr}(b_n W b_n^*) = t \sum_{j=1}^{\infty} |b_n W f_j|^2 = t |f_n|^2 = t \to 0
\]
as \( n \to \infty \).

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References


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