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_Mathematica Bohemica_, Vol. 117 (1992), No. 2, 207–216


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ASYMPTOTIC PROPERTIES OF SOLUTIONS OF FUNCTIONAL DIFFERENTIAL SYSTEMS

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(Received December 5, 1990)

Summary. In the paper we study the existence of nonoscillatory solutions of the system
\[ x_i^{(n)}(t) = \sum_{j=1}^{2} p_{ij}(t)f_{ij}(x_j(h_{ij}(t))), \quad n \geq 2, \quad i = 1, 2, \]
with the property \( \lim_{t \to \infty} x_i(t)/t^{k_i} = \text{const} \neq 0 \) for some \( k_i \in \{1, 2, \ldots, n-1\}, \ i = 1, 2 \). Sufficient conditions for the oscillation of solutions of the system are also proved.

Keywords: Functional differential system, Schauder-Tychonov fixed point theorem, oscillatory solution, nonoscillatory solutions.

AMS classification: 34K25, 34K05

This paper is concerned with the asymptotic properties of solutions of nonlinear functional differential systems in the form

\[ x_i^{(n)}(t) = \sum_{j=1}^{2} p_{ij}(t)f_{ij}(x_j(h_{ij}(t))), \quad t \geq t_0 \geq 0, \quad i = 1, 2, \quad n \geq 2, \]

under the following standing assumptions:

1. \( p_{ij}, h_{ij}: [t_0, \infty) \to \mathbb{R} \quad (i, j = 1, 2) \) are continuous functions and \( \lim_{t \to \infty} h_{ij}(t) = \infty \) as \( t \to \infty \) \((i, j = 1, 2)\),

2. \( f_{ij}: \mathbb{R} \to \mathbb{R} \quad (i, j = 1, 2) \) are continuous functions and \( uf_{ij}(u) > 0 \) for \( u \neq 0 \) \((i, j = 1, 2)\),

3. \( f_{ij} \quad (i, j = 1, 2) \) are nondecreasing functions.

For any \( t_1 \geq t_0 \) denote
\[ t_2 = \min\{\inf h_{ij}(t); t \geq t_1\}, \quad i, j = 1, 2\}. \]
A function $X(t) = (x_1(t), x_2(t))$ is a solution of (S) if there exists a $t_1 \geq t_0$ such that $X(t)$ is continuous on $[t_2, \infty)$, $n$-times continuously differentiable on $[t_1, \infty)$ and satisfies the system (S) on $[t_1, \infty)$.

By a proper solution of the system (S) we mean a solution $X(t)$ of (S) such that $\sup \{|x_1(t)| + |x_2(t)|: t \geq T\} > 0$ for any $T \geq t_0$. Such a solution is called oscillatory if each of its components has arbitrarily large zeros. A proper solution of (S) is called nonoscillatory (weakly nonoscillatory), if each of its components (one component) is eventually of constant sign on $[T_e, \infty) \subset [t_0, \infty)$.

This paper has two parts. First we prove the existence of nonoscillatory solutions of the system (S) with the property $\lim_{t \to \infty} x_i(t)/t^k_i = \text{const} \neq 0$ for some $k_i \in \{0, 1, \ldots, n-1\}$, $i = 1, 2$. The asymptotic properties of solutions of this type of nonlinear differential equations of higher orders have been studied for example in the papers [1, 3–5].

Secondly, we establish criteria for oscillation of proper solutions of (S).

Denote

$$
\gamma_{ij}(t) = \sup(s: h_{ij}(s) \leq t), \quad t \geq t_0,
$$

$$
\gamma(t) = \max(\gamma_{ij}(t); i, j = 1, 2), \quad t \geq t_0.
$$

**Theorem 1.** Let the conditions (1)–(3) hold and let $k_i \in \{1, 2, \ldots, n-1\}$, $i = 1, 2$. If

$$
(4) \quad \int_{\gamma(t_0)}^{\infty} t^{n-k_i-1} \sum_{j=1}^{2} |p_{ij}(t)| f_{ij}(a_j(h_{ij}(t))^{k_j}) dt < \infty, \quad i = 1, 2
$$

for some $a_j > 0$, $j = 1, 2$, then for any couples $(k_1, k_2)$, $(k_i \in \{1, 2, \ldots, n-1\})$ and $(c_1, c_2)$ ($c_i > 0$, $i = 1, 2$) there exists a nonoscillatory solution $X(t) = (x_1(t), x_2(t))$ of the system (S) such that

$$
(5) \quad \lim_{t \to \infty} x_i(t)/t^{k_i} = c_i, \quad i = 1, 2,
$$

$$
\lim_{t \to \infty} x_i^{(m_i)}(t) = 0 \text{ for } m_i = k_i + 1, \ldots, n-1, \quad i = 1, 2.
$$

**Proof.** Let $a_i$ ($i = 1, 2$) be positive numbers such that (4) holds and $k_i \in \{1, 2, \ldots, n-1\}$, $i = 1, 2$. We put $b_i = a_i/3$, $i = 1, 2$. In view of (2) there exists a $T \geq \gamma(t_0)$ such that

$$
(6) \quad \int_{T}^{\infty} t^{n-k_i-1} \sum_{j=1}^{2} |p_{ij}(t)| f_{ij}(a_j(h_{ij}(t))^{k_j}) dt < b_i, \quad i = 1, 2.
$$
Let \( T_0 = \min\{\inf h_{ij}(t) : t \geq T\}, i, j = 1, 2 \geq t_0 \). We denote by \( C([T_0, \infty)) \) the locally convex space of all vector continuous functions \( X(t) = (x_1(t), x_2(t)) \) defined on \([T_0, \infty)\) which are constant on \([T_0, T]\) with the topology of uniform convergence on any compact subinterval of \([T_0, \infty)\).

We consider a closed, convex subset \( Y \) of \( C([T_0, \infty)) \) defined by

\[
Y = \{X = (x_1, x_2) \in C([T_0, \infty)) ; x_i(t) = 2b_i \frac{T^{k_i}}{k_i!}, t \in [T_0, T] ; \}
\]
\[
b_i \leq x_i \leq 3b_i \frac{T^{k_i}}{k_i!}, t \geq T, i = 1, 2.\]

We define a mapping \( F = (F_1, F_2) : Y \to C([T_0, \infty)) \) by

\[
(F_iX)(t) = \begin{cases} 
2b_i \frac{T^{k_i}}{k_i!}, & t \in [T_0, T], \\
2b_i \frac{T^{k_i}}{k_i!} + (-1)^{n-k_i} \int_T^t \frac{(t-s)^{k_i-1}}{(k_i-1)!} \int_s^\infty \frac{(u-s)^{n-k_i-1}}{(n-k_i-1)!} \\
\times \sum_{j=1}^2 p_{ij}(u) f_{ij} (x_j(h_{ij}(u))) \, du \, ds, & t \geq T, \quad i = 1, 2.
\end{cases}
\]

We shall show that \( F \) is a continuous operator which transforms \( Y \) into a compact of \( Y \).

Ad 1. We prove that \( F(Y) \subseteq Y \). From (8) in view of (3), (6), (7) we have

\[
(F_iX)(t) \leq \begin{cases} 
2b_i \frac{T^{k_i}}{k_i!} + \int_T^t \frac{(t-s)^{k_i-1}}{(k_i-1)!} \int_s^\infty \frac{(u-s)^{n-k_i-1}}{(n-k_i-1)!} \\
\times \sum_{j=1}^2 |p_{ij}(u)| f_{ij} (a_j(h_{ij}(u))) \, du \, ds, & t \geq T, \quad i = 1, 2.
\end{cases}
\]
\[(F_i X)(t) \geq \frac{2b_i t^{k_i}}{k_i!} - \int_T^t \frac{(t-s)^{k_i-1}}{(k_i-1)!} \int_s^\infty \frac{(u-s)^{n-k_i-1}}{(n-k_i-1)!} \times \sum_{j=1}^2 |p_{ij}(u)| f_{ij}(a_j (h_{ij}(u))^{k_j}) \, du \, ds \]
\[\geq \frac{2b_i t^{k_i}}{k_i!} - b_i \int_T^t \frac{(t-s)^{k_i-1}}{(k_i-1)!} \, ds \]
\[\geq \frac{b_i t^{k_i}}{k_i!}, \quad t \geq T, \quad i = 1, 2.\]

Ad 2. We prove that \( F \) is continuous. Let \( X_k = (x_{1k}, x_{2k}) \in Y, \) \( k = 1, 2, \ldots, \) and \( x_{ik} \to x_i (i = 1, 2) \) for \( k \to \infty \) in the space \( C([T_0, \infty)) \). From (8) we than have

\[|(F_i X_k)(t) - (F_i X)(t)| \leq \int_T^t \frac{(t-s)^{k_i-1}}{(k_i-1)!} \int_s^\infty \frac{(u-s)^{n-k_i-1}}{(n-k_i-1)!} \times \sum_{j=1}^2 |p_{ij}(u)||f_{ij}(x_{jk}(h_{ij}(u)) - f_{ij}(x_j(h_{ij}(u)))| \, du \, ds \]
\[\leq t^{k_i} \int T^\infty G_i^k(u) \, du,\]

where we set

\[G_i^k(u) = u^{n-k_i-1} \sum_{j=1}^2 |p_{ij}(u)||f_{ij}(x_{jk}(h_{ij}(u)) - f_{ij}(x_j(h_{ij}(u)))|.\]

It is easy to see that \( \lim_{k \to \infty} G_i^k(u) = 0 \) and \( G_i^k(u) \leq M_i(u) \), where

\[M_i(u) = 2u^{n-k_i-1} \sum_{j=1}^2 |p_{ij}(u)||f_{ij}(a_j (h_{ij}(u))^{k_j}).\]

Using the fact that \( \int_T^\infty M_i(u) \, du < \infty \) and the Lebesgue dominating convergence theorem, from (11) we get \( (F_i X_k)(t) \to (F_i X)(t) \) for \( k \to \infty \) \((i = 1, 2)\) in \( C([T_0, \infty))\). This implies the continuity of \( F = (F_1, F_2) \).

Ad 3. We prove that \( F(Y) \) has a compact closure. From (8), in view of (6), for any \( X \in Y \) we have

\[|(F_i X)'(t)| \leq \frac{3b_i}{k_i-1} t^{k_i-1}, \quad t \geq T, \quad i = 1, 2.\]
Hence $F(Y)$ is equicontinuous on any compact subinterval of $[T_0, \infty)$. Since $F(Y) \subset Y$, $F(Y)$ is uniformly bounded on such subintervals. Therefore by the Arzela-Ascoli theorem $F(Y)$ has a compact closure.

By the Schauder-Tychonov fixed point theorem there exists an $\bar{X} = (\bar{x}_1, \bar{x}_2)$ such that $F\bar{X} = (F_1\bar{X}, F_2\bar{X}) = \bar{X}$. The function $\bar{X}$ satisfies (8) in which $F_iX_i = x_i (i = 1, 2)$.

Differentiating (8) in which $F_iX_i = x_i (i = 1, 2)$ $m_i$-times, $m_i = k_i, \ldots, n - 1$, for $X = (x_1, x_2) = \bar{X}$ we obtain

$$x_i^{(k_i)}(t) = 2b_i + (-1)^{n-k_i}\int_t^\infty \frac{(u-t)^{n-k_i-1}}{(n-k_i-1)!} \sum_{j=1}^2 p_{ij}(u)f_{ij}(x_j(h_{ij}(u))) \, du, \quad t \geq T, \quad i = 1, 2,$$

(12)

$$x_i^{(m_i)}(t) = (-1)^{n-m_i}\int_t^\infty \frac{(u-t)^{n-m_i-1}}{(n-m_i-1)!} \sum_{j=1}^2 p_{ij}(u)f_{ij}(x_j(h_{ij}(u))) \, du, \quad t \geq T, \quad m_i = k_i + 1, \ldots, n - 1, \text{ (if } k_i < 1), \quad i = 1, 2,$$

(13$m_i$)

Differentiating (13$n-1$) we get the system (S). This implies that $X = (x_1, x_2) = \bar{X}$ is a nonoscillatory solution of (S). From (12), (13$m_i$) in view of (4) we get $\lim_{t \to \infty} x_i^{(k_i)}(t) = 2b_i$, $\lim_{t \to \infty} x_i^{(m_i)}(t) = 0$ for $m_i = k_i + 1, \ldots, n - 1, i = 1, 2$. This is equivalent to (5), where $c_i = 2b_i (i = 1, 2)$.

**Theorem 2.** Let the conditions (1)–(3) hold and let

$$\int_0^\infty t^{n-1} \sum_{j=1}^2 |p_{ij}(t)| \, dt < \infty, \quad i = 1, 2,$$

(14)

Then for any couple $(c_1, c_2)$ ($c_i > 0$, $i = 1, 2$) there exists a nonoscillatory solution of the system (S) such that

$$\lim_{t \to \infty} |x_i(t)| = c_i, \quad \lim_{t \to \infty} x_i^{(k)}(t) = 0, \quad k = 1, 2, \ldots, n - 1, \quad i = 1, 2,$$

(15)

**Proof.** Let $c_i > 0$ $(i = 1, 2)$ and $0 < \delta \leq \min(c_1, c_2)$. In view of (2) there exists a $K > 0$ such that for all $(u_1, u_2)$: $|u_i - c_i| \leq \delta (i = 1, 2)$ we have

$$|f_{ij}(u)| \leq K, \quad i, j = 1, 2,$$

(16)
With regard to (14) there exists a $T \geq \gamma(t_0)$ such that

\begin{equation}
\int_{T}^{\infty} \frac{t^{n-1}}{n!} \sum_{j=1}^{2} |p_{ij}(t)| \, dt \leq \frac{\delta}{K}, \quad i = 1, 2.
\end{equation}

Let $T_0$ and $C([T_0, \infty))$ be the same as in the proof of Theorem 1. We consider a closed, convex subset $Y_1$ of $C([T_0, \infty))$ by

$$Y_1 = \{ X = (x_1, x_2) \in C([T_0, \infty)) : |x_i(t) - c_i| \leq \delta, \ t \geq T, \ i = 1, 2 \}.$$ 

We define a mapping $F = (F_1, F_2) : Y_1 \to C([T_0, \infty))$ by

\begin{align}
(F_i X)(t) = c_i + \frac{(-1)^n}{(n-1)!} \int_{T}^{\infty} (s-t)^{n-1} \sum_{j=1}^{2} p_{ij}(t) f_{ij}(x_j(h_{ij}(s))) \, ds, \\
t \in [T_0, T],
\end{align}

$$t \geq T, \ i = 1, 2.$$ 

If we proceed analogously as in the proof of Theorem 1 we can prove without difficulty that $F$ maps $Y_1$ into itself, $F$ is continuous and $F(Y_1)$ has a compact closure. Therefore there exists an $\tilde{X} = (\tilde{x}_1, \tilde{x}_2) \in Y_1$ such that $F \tilde{X} = (F_1 \tilde{X}, F_2 \tilde{X}) = \tilde{X}$. The function $\tilde{X}$ satisfies (18) in which $F_i X = x_i (i = 1, 2)$. We can easily verify that $X = (x_1, x_2) = \tilde{X}$ is a nonoscillatory solution of (S) with the asymptotic behavior (15).

\begin{flushright}
\Box
\end{flushright}

**Theorem 3.** Suppose that (1)–(3) hold and

\begin{equation}
p_{ij}(t) = \sigma_i q_{ij}(t), \quad \sigma_i \in \{-1, 1\}, \quad q_{ij} : [t_0, \infty) \to (0, \infty), \ i, j = 1, 2.
\end{equation}

Let $(k_1, k_2)$ be an arbitrary couple of integers $k_i \in \{0, 1, \ldots, n-1\}$ $(i = 1, 2)$. Then there exists a nonoscillatory solution $(x_1, x_2)$ of the system (S) such that

\begin{equation}
\lim_{t \to \infty} \frac{x_i(t)}{t^{k_i}} = c_i > 0, \quad i = 1, 2,
\end{equation}

if and only if

\begin{equation}
\int_{\gamma(t_0)}^{\infty} t^{n-k_i-1} \sum_{j=1}^{2} q_{ij}(t) f_{ij} \left( a_j \left( h_{ij}(t) \right)^{k_j} \right) \, dt < \infty, \quad i = 1, 2.
\end{equation}

212
for some constants $a_j > 0, j = 1, 2$. 

**Proof.** Let $X = (x_1, x_2)$ be a nonoscillatory solution of (S) which satisfies (20). Without loss of generality we suppose that $x_j(h_{ij}(t)) > 0$ for $t \geq T_1 \geq t_0, i, j = 1, 2$. Then in view of (2) $f_{ij}(x_j(h_{ij}(t))) > 0$ for $t \geq T_1$. From (20) we obtain

\begin{equation}
\lim_{t \to \infty} x_i^{(k_i)}(t) = c_i k_i! > 0, \quad i = 1, 2,
\end{equation}

\begin{equation}
\lim_{t \to \infty} x_i^{(m_i)}(t) = 0, \quad m_i = k_i + 1, \ldots, n_1, \quad i = 1, 2.
\end{equation}

Then integrating the system (S) $(n - k_i - 1)$-times (if $k_i < n - 1$), $i = 1, 2$, from $t \geq T_1$ to $\infty$ and using (22) we have

\begin{equation}
x_i^{(k_i+1)}(t) = (-1)^{n-k_i-1} \sigma_i \int_{T_1}^{\infty} \frac{(s-t)^{n-k_i-2}}{(n-k_i-2)!} \sum_{j=1}^{2} q_{ij}(s)f_{ij}(x_j(h_{ij}(s))) \, ds,
\end{equation}

$t \geq T_1, \quad i = 1, 2$.

Integrating the last equation from $T_1$ to $\infty$ and using (20), after some modifications we obtain

\begin{equation}
\int_{T_1}^{\infty} s^{n-k_i-1} \sum_{j=1}^{2} q_{ij}(s)f_{ij}(x_j(h_{ij}(s))) \, ds < \infty, \quad i = 1, 2.
\end{equation}

On the other hand, by virtue of (20) there exist constants $a_j > 0 (j = 1, 2)$ and $T_2 \geq T_1$ such that $x_j(h_{ij}(t)) \geq a_j(h_{ij}(t))^{k_j}$ for $t \geq T_2 (i, j = 1, 2)$. Then the last inequality, (3) and (23) imply (21).

The “if” part follows from Theorem 1 a Theorem 2.

**Oscillation criteria**

Now we consider the system (S) in the form

(A) $x_i^{(n)}(t) = \sigma_i q_i(t)f_i(x_{3-i}(h_{3-i}(t)))$ \(t \geq t_0, \ i = 1, 2,\) where $\sigma_i \in \{-1, 1\}$.

(B) $q_i: [t_0, \infty) \to (0, \infty), \ i = 1, 2$ are continuous functions,

(C) $h_i$ and $f_i, i = 1, 2$ satisfy (1) and (2), respectively,

(D) for any $b > 0$ there exists $\delta > 0$ such that

\[ \inf \{f_i(u)| |u| \geq b\} \geq \delta, \quad i = 1, 2. \]

In the sequel we use Kiguradze’s lemma. \(\Box\)

**Lemma [2].** Let $u \in C^n[t_0, \infty)$ be such that $(-1)^\nu u(t)u^{(n)}(t) < 0$ for $t \geq t_0,$ $\nu \in \{1, 2\}$. Then there exist an integer $\ell \in \{0, 1, \ldots, n\}$, where $\ell + n + \nu$ is odd, and $T \geq t_0$ such that

\[ u(t)u^{(k)}(t) > 0 \text{ for } k = 0, 1, \ldots, \ell, \quad t \geq T, \]

\[ (-1)^{\ell+k} u(t)u^{(k)}(t) > 0 \text{ for } k = \ell + 1, \ldots, n, \quad t \geq T. \]
Remark. Let \( X = (x_1, x_2) \) be a weakly nonoscillatory solution of (A). Then in view of (24), (25) it follows, from (A) that \( X \) is a nonoscillatory solution.

**Theorem 4.** Suppose that \( \sigma_1 \sigma_2 = -1 \) and

\[
\int_{t_0}^{\infty} q_i(t) \, dt = \infty, \quad i = 1, 2.
\]

Then every proper solution \((x_1(t), x_2(t))\) of (A) is oscillatory when \( n \) is odd, and for \( n \) even it is either oscillatory or \( x_1(t)x_2(t) < 0 \) and, moreover, for \( \sigma_j = 1, \sigma_{3-j} = -1 \) \((j = 1, 2)\) \(|x_j(t)|\) is increasing while \( x_{3-j}^{(k)}(t)\), \((k = 0, 1, \ldots, n)\) tend monotonically to zero as \( t \to \infty \).

**Proof.** Suppose that the system (A) has a weakly nonoscillatory solution \((x_1(t), x_2(t))\). Then in view of Remark it is a nonoscillatory solution. Without loss of generality we suppose that \( \sigma_1 = 1, \sigma_2 = -1 \).

I. Let \( n \) be odd. 1) Suppose that \( x_1(t) > 0, x_2(t) > 0 \) for \( t \geq t_1 \). (The proof in the case \( x_1(t) < 0, x_2(t) < 0 \) is similar.) Then from the system (A) with regard to (24), (25) we obtain \( x_1^{(n)}(t) > 0, x_2^{(n)}(t) < 0 \) for \( t \geq t_2 \geq \gamma(t_1) \). Then by Lemma we get \( x_1'(t) > 0 \) and then \( x_1(t) > b_1 \) for \( t \geq t_3 \geq t_2 \) and some \( b_1 > 0 \). Therefore in view of (26) there exists \( \delta_1 > 0 \) such that \( f_2(x_1(h_1(t))) \geq \delta_1 \) for \( t \geq t_4 \geq \gamma(t_3) \). Then from (A) we get \( x_2^{(n)}(t) \leq -\delta_1 q_2(t), \quad t \geq t_4 \). From the last inequality, in view of (27) we obtain \( x_2^{(n-1)}(t) \to -\infty \) as \( t \to \infty \). The inequalities \( x_2^{(n)}(t) < 0, x_2^{(n-1)}(t) < 0 \) for \( t \geq t_5 \geq t_4 \) imply that \( x_2(t) < 0 \) for all large \( t \). This contradicts the assumption \( x_2(t) > 0 \) for \( t \geq t_1 \).

2) Let \( x_1(t) > 0, x_2(t) < 0 \) for \( t \geq t_1 \). (The proof in the case \( x_1(t) < 0, x_2(t) > 0 \) is similar.) Then the system (A) in view of (24), (25) implies \( x_1^{(n)}(t) < 0, i = 1, 2, t \geq t_2 \geq \gamma(t_1) \). Because \( x_2(t)x_2^{(n)}(t) > 0 \) for \( t \geq t_2 \), by Lemma we get \( x_2'(t) < 0 \) and then \( x_2(t) \leq -a_2 \) for \( t \geq t_3 \geq t_2 \) and some \( a_2 > 0 \). Therefore in view of (26) there exists \( \delta_2 > 0 \) such that \( f_1(x_2(h_2(t))) \leq -\delta_2 \) for \( t \geq t_4 \geq \delta(t_3) \). Then from (A) with regard to (27) we get \( x_1^{(n-1)}(t) < 0 \) for \( t \geq t_5 \). From \( x_1^{(n)}(t) < 0, x_1^{(n-1)}(t) < 0 \) for \( t \geq t_5 \) we obtain \( x_1(t) < 0 \) for all large \( t \). This contradicts the assumption \( x_1(t) > 0 \) for \( t \geq t_1 \).

II. Let \( n \) be even. 1) Suppose that \( x_1(t) > 0, x_2(t) > 0 \) for \( t \geq t_1 \). (The proof in the case \( x_1(t) < 0, x_2(t) < 0 \) is similar.) Then in view of (24), (25) the system (A) implies \( x_1^{(n)}(t) > 0, x_2^{(n)}(t) < 0 \) for \( t \geq t_2 \geq \gamma(t_1) \) and by Lemma \( x_2'(t) > 0 \) and then \( x_2(t) \geq b_3 \) for \( t \geq T_2 \geq t_2 \) and some \( b_3 > 0 \). Therefore in view of (26) there exists \( \delta_3 > 0 \) such that \( f_1(x_2(h_2(t))) \geq \delta_3 \) for \( t \geq T_3 \geq \gamma(T_2) \). Then from (A) with regard to (27) we get \( x_1^{(n-1)}(t) \to -\infty \) as \( t \to \infty \). Therefore in view of (26) there exists \( \delta_4 > 0 \).
such that \( f_2(x(h_1(t))) \geq \delta_4 \) for \( t \geq T_4 \geq \gamma(T_3) \). Further we proceed analogously as in the case I-1) we obtaining \( x_2(t) < 0 \) for large \( t \), which contradicts \( x_2(t) > 0 \) for \( t \geq t_1 \).

2) Suppose that \( x_1(t) > 0 \), \( x_2(t) < 0 \) for \( t \geq t_1 \). (The proof in the case \( x_1(t) < 0 \), \( x_2(t) > 0 \) is similar). Then in view of (24), (25) from (A) we get \( x_i^{(n)}(t) < 0 \), \( i = 1, 2 \), for \( t \geq t_2 = \gamma(t_1) \). Using Lemma, we have \( x_i'(t) > 0 \) and either i) \( x_i''(t) < 0 \), \( x_i''(t) < 0 \), or ii) \( x_i'(t) > 0 \) for \( t \geq t_3 \geq t_2 \). In the case i) we proceed in the same way as in the case I-2), obtaining a contradiction to the assumption \( x_1(t) > 0 \) for \( t \geq t_1 \). Now we consider the case ii). The component \( x_2(t) \) is increasing and

\[ \lim_{t \to \infty} x_2(t) = -b \leq 0. \]

If we suppose that \( b > 0 \), we proceed in the same way as in the case i) arriving at a contradiction. Therefore \( b = 0 \), i.e. \( \lim_{t \to \infty} x_2(t) = 0 \). This in view of Lemma implies \( \lim_{t \to \infty} x_2^{(k)}(t) = 0 \) for \( k = 0, 1, \ldots, n \).

The proof of Theorem 4 is complete.

Acknowledgement. The authors wish to thank the referee for his helpful suggestions.

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Súhrn

ASYMPTOTICKÉ VLASTNOSTI RIEŠENÍ
FUNKCIONÁLNO-DIFERENCIÁLNYCH SYSTÉMOV

ANATOLIJ F. IVANOV, PAVOL MARUŠIAK

V práci je študovaná existencia neoscilatorických riešení systému

\[ x_i^{(n)}(t) = \sum_{j=1}^{2} p_{ij}(t)f_{ij}\left(x_j\left(h_{ij}(t)\right)\right), \quad n \geq 2, \quad i = 1, 2, \]

s vlastnosťami \( \lim_{t \to \infty} x_i(t)/t^{k_i} = \text{const.} \neq 0 \) pre nejaké \( k_i \in \{1, 2, \ldots, n - 1\}, \quad i = 1, 2. \) Dalej sú dokázané postačujúce podmienky pre to, aby systém mal oscilatorické riešenie.