ASYMPTOTIC PROPERTIES OF SOLUTIONS OF FUNCTIONAL DIFFERENTIAL SYSTEMS

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Summary. In the paper we study the existence of nonoscillatory solutions of the system
t_{i}^{(n)}(t) = \sum_{j=1}^{2} p_{ij}(t) f_{ij}(x_{j}(h_{ij}(t))), \quad n \geq 2, \quad i = 1, 2

with the property \( \lim_{t \to \infty} x_{i}(t)/t^{k_{i}} = \text{const} \neq 0 \) for some \( k_{i} \in \{1, 2, \ldots, n - 1\}, \quad i = 1, 2 \). Sufficient conditions for the oscillation of solutions of the system are also proved.

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This paper is concered with the asymptotic properties of solutions of nonlinear functional differential systems in the form

\[
(S) \quad x_{i}^{(n)}(t) = \sum_{j=1}^{2} p_{ij}(t) f_{ij}(x_{j}(h_{ij}(t))), \quad t \geq t_{0} \geq 0, \quad i = 1, 2, \quad n \geq 2,
\]

under the following standing assumptions:

(1) \( p_{ij}, h_{ij} : [t_{0}, \infty) \to \mathbb{R} \) \( (i, j = 1, 2) \) are continuous functions and \( \lim_{t \to \infty} h_{ij}(t) = \infty \)

as \( t \to \infty \) \( (i, j = 1, 2) \),

(2) \( f_{ij} : \mathbb{R} \to \mathbb{R} \) \( (i, j = 1, 2) \) are continuous functions and \( u f_{ij}(u) > 0 \) for \( u \neq 0 \)

\((i, j = 1, 2)\),

(3) \( f_{ij} (i, j = 1, 2) \) are nondecreasing functions.

For any \( t_{1} \geq t_{0} \) denote

\[
t_{2} = \min\{(\inf h_{ij}(t) ; t \geq t_{1}), \quad i, j = 1, 2\}.
\]
A function \( X(t) = (x_1(t), x_2(t)) \) is a solution of (S) if there exists a \( t_1 \geq t_0 \) such that \( X(t) \) is continuous on \( [t_2, \infty) \), \( n \)-times continuously differentiable on \( [t_1, \infty) \) and satisfies the system (S) on \( [t_1, \infty) \).

By a proper solution of the system (S) we mean a solution \( X(t) \) of (S) such that \( \sup\{|x_1(t)| + |x_2(t)|: t \geq T\} > 0 \) for any \( T \geq t_0 \). Such a solution is called oscillatory if each of its components has arbitrarily large zeros. A proper solution of (S) is called nonoscillatory (weakly nonoscillatory), if each of its components (one component) is eventually of constant sign on \( [T_x, \infty) \subset [t_0, \infty) \).

This paper has two parts. First we prove the existence of nonoscillatory solutions of the system (S) with the property \( \lim_{t \to \infty} x_i(t)/t^{k_i} = \text{const} \neq 0 \) for some \( k_i \in \{0, 1, \ldots, n-1\}, i = 1, 2 \). The asymptotic properties of solutions of this type of nonlinear differential equations of higher orders have been studied for example in the papers [1, 3-5].

Secondly, we establish criteria for oscillation of proper solutions of (S).

Denote

\[
\gamma_{ij}(t) = \sup\{s: h_{ij}(s) \leq t\}, \quad t \geq t_0,
\]

\[
\gamma(t) = \max(\gamma_{ij}(t); i, j = 1, 2), \quad t \geq t_0.
\]

**Theorem 1.** Let the conditions (1)–(3) hold and let \( k_i \in \{1, 2, \ldots, n-1\}, i = 1, 2 \). If

\[
(4) \quad \int_{\gamma(t_0)}^{\infty} t^{n-k_i-1} \sum_{j=1}^{2} |p_{ij}(t)|f_{ij}(a_j(h_{ij}(t))^{k_j}) \, dt < \infty, \quad i = 1, 2
\]

for some \( a_j > 0, j = 1, 2 \), then for any couples \( (k_1, k_2), (k_i \in \{1, 2, \ldots, n-1\}) \) and \( (c_1, c_2) (c_i > 0, i = 1, 2) \) there exists a nonoscillatory solution \( X(t) = (x_1(t), x_2(t)) \) of the system (S) such that

\[
(5) \quad \lim_{t \to \infty} x_i(t)/t^{k_i} = c_i, \quad i = 1, 2,
\]

\[
\lim_{t \to \infty} x_i^{(m_i)}(t) = 0 \text{ for } m_i = k_i + 1, \ldots, n - 1, \quad i = 1, 2.
\]

**Proof.** Let \( a_i \ (i = 1, 2) \) be positive numbers such that (4) holds and \( k_i \in \{1, 2, \ldots, n-1\}, i = 1, 2 \). We put \( b_i = a_i/3, i = 1, 2 \). In view of (2) there exists a \( T \geq \gamma(t_0) \) such that

\[
(6) \quad \int_{T}^{\infty} t^{n-k_i-1} \sum_{j=1}^{2} |p_{ij}(t)|f_{ij}(a_j(h_{ij}(t))^{k_j}) \, dt < b_i, \quad i = 1, 2.
\]
Let $T_0 = \min\{\inf h_{ij}(t): t \geq T\}, i, j = 1, 2 \geq t_0$. We denote by $C([T_0, \infty))$ the locally convex space of all vector continuous functions $X(t) = (x_1(t), x_2(t))$ defined on $[T_0, \infty)$ which are constant on $[T_0, T]$ with the topology of uniform convergence on any compact subinterval of $[T_0, \infty)$.

We consider a closed, convex subset $Y$ of $C([T_0, \infty))$ defined by

\[ Y = \{X = (x_1, x_2) \in C([T_0, \infty)); x_i(t) = 2b_i \frac{T^{k_i}}{k_i!}, t \in [T_0, T]: b_i \frac{T^{k_i}}{k_i!} \leq x_i \leq 3b_i \frac{T^{k_i}}{k_i!}, t \geq T, i = 1, 2\}. \]

We define a mapping $F = (F_1, F_2): Y \rightarrow C([T_0, \infty))$ by

\[ (F_iX)(t) = \begin{cases} 
2b_i \frac{T^{k_i}}{k_i!}, & t \in [T_0, T], \\
\frac{2b_i t^{k_i}}{k_i!} + (-1)^{n-k_i} \int_T^t \frac{(t-s)^{k_i-1}}{(k_i-1)!} \int_s^\infty \frac{(u-s)^{n-k_i-1}}{(n-k_i-1)!} \\
\times \sum_{j=1}^2 p_{ij}(u) f_{ij}(x_j(h_{ij}(u))) \, du \, ds, & t \geq T, \quad i = 1, 2.
\end{cases} \]

We shall show that $F$ is a continuous operator which transforms $Y$ into a compact of $Y$.

Ad 1. We prove that $F(Y) \subset Y$. From (8) in view of (3), (6), (7) we have

\[ (F_iX)(t) \leq \frac{2b_i t^{k_i}}{k_i!} + \int_T^t \frac{(t-s)^{k_i-1}}{(k_i-1)!} \int_s^\infty \frac{(u-s)^{n-k_i-1}}{(n-k_i-1)!} \\
\times \sum_{j=1}^2 |p_{ij}(u)| f_{ij}(a_j(h_{ij}(u)))^{k_j} \, du \, ds \\
\leq \frac{2b_i t^{k_i}}{k_i!} + b_i \int_T^t \frac{(t-s)^{k_i-1}}{(k_i-1)!} \, ds \\
\leq \frac{3b_i t^{k_i}}{k_i!}, \quad t \geq T, \quad i = 1, 2. \]
\[(10_i) \quad (F_t X)(t) \leq \frac{2b_i t^{k_i}}{k_i!} - \int_T^t \frac{(t-s)^{k_i-1}}{(k_i-1)!} \int_s^\infty \frac{(u-s)^{n-k_i-1}}{(n-k_i-1)!} ds \times \sum_{j=1}^2 |p_{ij}(u)|f_{ij}\left(a_j h_{ij}(u)^{k_j}\right) du ds \]
\[\geq \frac{2b_i t^{k_i}}{k_i!} - b_i \int_T^t \frac{(t-s)^{k_i-1}}{(k_i-1)!} ds \]
\[\geq \frac{b_i t^{k_i}}{k_i!}, \quad t \geq T, \quad i = 1, 2.\]

Ad 2. We prove that \( F \) is continuous. Let \( X_k = (x_{1k}, x_{2k}) \in Y, \ k = 1, 2, \ldots, \) and \( x_{ik} \to x_i \ (i = 1, 2) \) for \( k \to \infty \) in the space \( C([T_0, \infty)) \). From (8) we than have
\[
|(F_t X_k)(t) - (F_t X)(t)| \leq \int_T^t \frac{(t-s)^{k_i-1}}{(k_i-1)!} \int_s^\infty \frac{(u-s)^{n-k_i-1}}{(n-k_i-1)!} \times \sum_{j=1}^2 |p_{ij}(u)|f_{ij}\left(x_{jk}(h_{ij}(u)) - f_{ij}\left(x_j(h_{ij}(u))\right)\right) du ds \leq \frac{t^{k_i}}{T} \int G_i^k(u) du,
\]
where we set
\[
G_i^k(u) = u^{n-k_i-1} \sum_{j=1}^2 |p_{ij}(u)|f_{ij}\left(x_{jk}(h_{ij}(u)) - f_{ij}\left(x_j(h_{ij}(u))\right)\right).
\]

It is easy to see that \( \lim_{k \to \infty} G_i^k(u) = 0 \) and \( G_i^k(u) \leq M_i(u), \) where
\[
M_i(u) = 2u^{n-k_i-1} \sum_{j=1}^2 |p_{ij}(u)|f_{ij}\left(a_j h_{ij}(u)^{k_j}\right).
\]

Using the fact that \( \int_T^\infty M_i(u) du < \infty \) and the Lebesgue dominating convergence theorem, from (11_i) we get \( (F_t X_k)(t) \to (F_t X)(t) \) for \( k \to \infty \) \((i = 1, 2)\) in \( C([T_0, \infty)) \). This implies the continuity of \( F = (F_1, F_2) \).

Ad 3. We prove that \( F(Y) \) has a compact closure. From (8), in view of (6), for any \( X \in Y \) we have
\[
|(F_t X)'(t)| \leq \frac{3b_i}{k_i-1} t^{k_i-1}, \quad t \geq T, \quad i = 1, 2.
\]
Hence \( F(Y) \) is equicontinuous on any compact subinterval of \([T_0, \infty)\). Since \( F(Y) \subset Y \), \( F(Y) \) is uniformly bounded on such subintervals. Therefore by the Arzela-Ascoli theorem \( F(Y) \) has a compact closure.

By the Schauder-Tychonov fixed point theorem there exists an \( \bar{X} = (\bar{x}_1, \bar{x}_2) \) such that \( F \bar{X} = (F_1 \bar{X}, F_2 \bar{X}) = \bar{X} \). The function \( \bar{X} \) satisfies (8) in which \( F_i \bar{X} = \bar{x}_i \) \((i = 1, 2)\).

Differentiating (8) in which \( F_i \bar{X} = \bar{x}_i \) \((i = 1, 2)\) \(m_i\)-times, \( m_i = k_i, \ldots, n - 1 \), for \( X = (x_1, x_2) = \bar{X} \) we obtain

\[
\begin{align*}
(12) \quad x_i^{(k_i)}(t) & = 2b_i + (-1)^{n-k_i} \int_t^\infty \frac{(u-t)^{n-k_i-1}}{(n-k_i-1)!} \\
& \times \sum_{j=1}^2 p_{ij}(u) f_{ij}(x_j(h_{ij}(u))) \, du, \quad t \geq T, \quad i = 1, 2,
\end{align*}
\]

\[
(13_{m_i}) \quad x_i^{(m_i)}(t) = (-1)^{n-m_i} \int_t^\infty \frac{(u-t)^{n-m_i-1}}{(n-m_i-1)!} \sum_{j=1}^2 p_{ij}(u) f_{ij}(x_j(h_{ij}(u))) \, du,
\]

\[
t \geq T, \quad m_i = k_i + 1, \ldots, n - 1, \quad \text{(if } k_i < 1), \quad i = 1, 2,
\]

Differentiating \((13_{n-1})\) we get the system \((S)\). This implies that \( \bar{X} = (x_1, x_2) = \bar{X} \) is a nonoscillatory solution of \((S)\). From \((12), (13_{m_i})\) in view of \((4)\) we get \( \lim_{t \to \infty} x_i^{(k_i)}(t) = 2b_i, \lim_{t \to \infty} x_i^{(m_i)}(t) = 0 \) for \( m_i = k_i + 1, \ldots, n - 1, \quad i = 1, 2 \). This is equivalent to \((5)\), where \( c_i = 2b_i \) \((i = 1, 2)\).

**Theorem 2.** Let the conditions \((1)-(3)\) hold and let

\[
\int_0^\infty t^{n-1} \sum_{j=1}^2 |p_{ij}(t)| \, dt < \infty, \quad i = 1, 2.
\]

Then for any couple \((c_1, c_2) \) \((c_i > 0, \quad i = 1, 2)\) there exists a nonoscillatory solution of the system \((S)\) such that

\[
\lim_{t \to \infty} |x_i(t)| = c_i, \quad \lim_{t \to \infty} x_i^{(k)}(t) = 0, \quad k = 1, 2, \ldots, n - 1, \quad i = 1, 2.
\]

**Proof.** Let \( c_i > 0 \) \((i = 1, 2)\) and \( 0 < \delta \leq \min(c_1, c_2) \). In view of \((2)\) there exists a \( K > 0 \) such that for all \((u_1, u_2)\): \(|u_i - c_i| \leq \delta \) \((i = 1, 2)\) we have

\[
|f_{ij}(u_j)| \leq K, \quad i, j = 1, 2.
\]
With regard to (14) there exists a $T \geq \gamma(t_0)$ such that
\begin{equation}
\int_T^\infty t^{n-1} \sum_{j=1}^2 |p_{ij}(t)| \, dt \leq \frac{\delta}{K}, \quad i = 1, 2.
\end{equation}

Let $T_0$ and $C([T_0, \infty))$ be the same as in the proof of Theorem 1. We consider a closed, convex subset $Y_1$ of $C([T_0, \infty))$ by
\[ Y_1 = \{ X = (x_1, x_2) \in C([T_0, \infty)): |x_i(t) - c_i| \leq \delta, \ t \geq T, \ i = 1, 2 \}. \]

We define a mapping $F = (F_1, F_2): Y_1 \rightarrow C([T_0, \infty))$ by
\begin{align*}
(F_1X)(t) &= c_i + \frac{(-1)^n}{(n-1)!} \int_T^\infty (s-t)^{n-1} \sum_{j=1}^2 p_{ij}(t)f_{ij}(x_j(h_{ij}(s))) \, ds, \\
& \quad t \in [T_0, T], \\
(F_2X)(t) &= c_i + \frac{(-1)^n}{(n-1)!} \int_T^\infty (s-t)^{n-1} \sum_{j=1}^2 p_{ij}(t)f_{ij}(x_j(h_{ij}(s))) \, ds, \\
& \quad t \geq T, \ i = 1, 2.
\end{align*}

If we proceed analogously as in the proof of Theorem 1 we can prove without difficulty that $F$ maps $Y_1$ into itself, $F$ is continuous and $F(Y_1)$ has a compact closure. Therefore there exists an $\bar{X} = (\bar{x}_1, \bar{x}_2) \in Y_1$ such that $F\bar{X} = (F_1\bar{X}, F_2\bar{X}) = \bar{X}$. The function $\bar{X}$ satisfies (18) in which $F_iX = x_i (i = 1, 2)$. We can easily verify that $X = (x_1, x_2) = \bar{X}$ is a nonoscillatory solution of $\mathcal{S}$ with the asymptotic behavior (15). \hfill \Box

**Theorem 3.** Suppose that (1)-(3) hold and
\begin{equation}
p_{ij}(t) = \sigma_i q_{ij}(t), \quad \sigma_i \in \{-1, 1\}, \quad q_{ij} : [t_0, \infty) \rightarrow (0, \infty), \quad i, j = 1, 2.
\end{equation}

Let $(k_1, k_2)$ be an arbitrary couple of integers $k_i \in \{0, 1, \ldots, n-1\} (i = 1, 2)$. Then there exists a nonoscillatory solution $(x_1, x_2)$ of the system (S) such that
\begin{equation}
\lim_{t \to \infty} \frac{x_i(t)}{t^{k_i}} = c_i > 0, \quad i = 1, 2,
\end{equation}
if and only if
\begin{equation}
\int_{\gamma(t_0)}^\infty t^{n-k_i-1} \sum_{j=1}^2 q_{ij}(t)f_{ij}(a_j(h_{ij}(t))^{k_j}) \, dt < \infty, \quad i = 1, 2.
\end{equation}
for some constants \( a_j > 0, j = 1, 2 \).

Proof. Let \( X = (x_1, x_2) \) be a nonoscillatory solution of (S) which satisfies (20). Without loss of generality we suppose that \( x_j(h_{ij}(t)) > 0 \) for \( t \geq T_1 \geq t_0, i, j = 1, 2 \). Then in view of (2) \( f_{ij}(x_j(h_{ij}(t))) > 0 \) for \( t \geq T_1 \). From (20) we obtain

\[
\lim_{t \to -\infty} x_i^{(k_i)}(t) = c_i k_i! > 0, \quad i = 1, 2,
\]

\[
\lim_{t \to -\infty} x_i^{(m_i)}(t) = 0, \quad m_i = k_i + 1, \ldots, n_1, \quad i = 1, 2.
\]

Then integrating the system (S) \((n - k_i - 1)\)-times (if \( k_i < n - 1 \), \( i = 1, 2 \), from \( t \geq T_1 \) to \( \infty \) and using (22) we have

\[
x_i^{(k_i+1)}(t) = (-1)^{n-k_i-1} \sigma_i \int_t^{\infty} \frac{(s-t)^{n-k_i-2}}{(n-k_i-2)!} \sum_{j=1}^2 q_{ij}(s) f_{ij}(x_j(h_{ij}(s))) \, ds,
\]

\[
t \geq T, \quad i = 1, 2.
\]

Integrating the last equation from \( T_1 \) to \( \infty \) and using (20), after some modifications we obtain

\[
\int_{T_1}^{\infty} s^{n-k_i-1} \sum_{j=1}^2 q_{ij}(s) f_{ij}(x_j(h_{ij}(s))) \, ds < \infty, \quad i = 1, 2.
\]

On the other hand, by virtue of (20) there exist constants \( a_j > 0 (j = 1, 2) \) and \( T_2 \geq T_1 \) such that \( x_j(h_{ij}(t)) \geq a_j(h_{ij}(t))^{k_j} \) for \( t \geq T_2 \) \((i, j = 1, 2) \). Then the last inequality, (3) and (23) imply (21).

The "if" part follows from Theorem 1 a Theorem 2.

Oscillation criteria

Now we consider the system (S) in the form

(A) \( x_i^{(n)}(t) = \sigma_i q_i(t) f_i(x_{3-i}(h_{3-i}(t))) \) \( t \geq t_0, i = 1, 2, \) where \( \sigma_i \in \{-1, 1\} \).

(24) \( q_i : [t_0, \infty) \to (0, \infty), i = 1, 2 \) are continuous functions,

(25) \( h_i \) and \( f_i, i = 1, 2 \) satisfy (1) and (2), respectively,

(26) for any \( b > 0 \) there exists \( \delta > 0 \) such that

\[
\inf\{f_i(u) ; |u| \geq b\} \geq \delta, \quad i = 1, 2.
\]

In the sequel we use Kiguradze’s lemma. \( \square \)

Lemma [2]. Let \( u \in C^n[t_0, \infty) \) be such that \((-1)^\nu u(t)u^{(n)}(t) < 0 \) for \( t \geq t_0, \) \( \nu \in \{1, 2\} \). Then there exist an integer \( \ell \in \{0, 1, \ldots, n\} \), where \( \ell + n + \nu \) is odd, and \( T \geq t_0 \) such that

\[
u(t)u^{(k)}(t) > 0 \text{ for } k = 0, 1, \ldots, \ell, \quad t \geq T,
\]

\[
(-1)^{\ell+k} u(t)u^{(k)}(t) > 0 \text{ for } k = \ell + 1, \ldots, n, \quad t \geq T.
\]
Remark. Let \( X = (x_1, x_2) \) be a weakly nonoscillatory solution of (A). Then in view of (24), (25) it follows, from (A) that \( X \) is a nonoscillatory solution.

Theorem 4. Suppose that \( \sigma_1\sigma_2 = -1 \) and

\[
(27) \quad \int_{t_0}^{\infty} q_i(t) \, dt = \infty, \quad i = 1, 2.
\]

Then every proper solution \((x_1(t), x_2(t))\) of (A) is oscillatory when \( n \) is odd, and for \( n \) even it is either oscillatory or \( x_1(t)x_2(t) < 0 \) and, moreover, for \( \sigma_j = 1, \sigma_{3-j} = -1 \) \((j = 1, 2)\) \(|x_j(t)|\) is increasing while \( x_{3-j}^{(k)}(t)\), \((k = 0, 1, \ldots, n)\) tend monotonically to zero as \( t \to \infty \).

Proof. Suppose that the system (A) has a weakly nonoscillatory solution \((x_1(t), x_2(t))\). Then in view of Remark it is a nonoscillatory solution. Without loss of generality we suppose that \( \sigma_1 = 1, \sigma_2 = -1 \).

I. Let \( n \) be odd. 1) Suppose that \( x_1(t) > 0, x_2(t) > 0 \) for \( t \geq t_1 \). (The proof in the case \( x_1(t) < 0, x_2(t) < 0 \) is similar.) Then from the system (A) with regard to (24), (25) we obtain \( x_1^{(n)}(t) > 0, x_2^{(n)}(t) < 0 \) for \( t \geq t_2 \geq \gamma(t_1) \). Then by Lemma we get \( x_1'(t) > 0 \) and then \( x_1(t) > b_1 \) for \( t \geq t_3 \geq t_2 \) and some \( b_1 > 0 \). Therefore in view of (26) there exists \( \delta_1 > 0 \) such that \( f_2(x_1(h_1(t))) \geq \delta_1 \) for \( t \geq t_4 \geq \gamma(t_3) \). Then from (A) we get \( x_2^{(n)}(t) \leq -\delta_1 q_2(t), t \geq t_4 \). From the last inequality, in view of (27) we obtain \( x_2^{(n-1)}(t) \to -\infty \) as \( t \to \infty \). The inequalities \( x_2^{(n)}(t) < 0, x_2^{(n-1)}(t) < 0 \) for \( t \geq t_5 \geq t_4 \) imply that \( x_2(t) < 0 \) for all large \( t \). This contradicts the assumption \( x_2(t) > 0 \) for \( t \geq t_1 \).

2) Let \( x_1(t) > 0, x_2(t) < 0 \) for \( t \geq t_1 \). (The proof in the case \( x_1(t) < 0, x_2(t) > 0 \) is similar.) Then the system (A) in view of (24), (25) implies \( x_1^{(n)}(t) > 0, i = 1, 2, t \geq t_2 \geq \gamma(t_1) \). Because \( x_2(t)x_2^{(n)}(t) > 0 \) for \( t \geq t_2 \), by Lemma we get \( x_2'(t) < 0 \) and then \( x_2(t) \leq a_2 \) for \( t \geq t_3 \geq t_2 \) and some \( a_2 > 0 \). Therefore in view of (26) there exists \( \delta_2 > 0 \) such that \( f_1(x_2(h_2(t))) \leq -\delta_2 \) for \( t \geq t_4 \geq \delta(t_3) \). Then from (A) with regard to (27) we get \( x_1^{(n-1)}(t) < 0 \) for \( t \geq t_5 \geq t_4 \). From \( x_1^{(n)}(t) < 0, x_1^{(n-1)}(t) < 0 \) for \( t \geq t_5 \) we obtain \( x_1(t) < 0 \) for all large \( t \). This contradicts the assumption \( x_1(t) > 0 \) for \( t \geq t_1 \).

II. Let \( n \) be even. 1) Suppose that \( x_1(t) > 0, x_2(t) > 0 \) for \( t \geq t_1 \). (The proof in the case \( x_1(t) < 0, x_2(t) < 0 \) is similar.) Then in view of (24), (25) the system (A) implies \( x_1^{(n)}(t) > 0, x_2^{(n)}(t) < 0 \) for \( t \geq t_2 \geq \gamma(t_1) \) and by Lemma \( x_2'(t) < 0 \) and then \( x_2(t) \geq b_3 \) for \( t \geq T_2 \geq t_2 \) and some \( b_3 > 0 \). Therefore in view of (26) there exists \( \delta_3 > 0 \) such that \( f_1(x_2(h_2(t))) \geq \delta_3 \) for \( t \geq T_3 \geq \gamma(T_2) \). Then from (A) with regard to (27) we get \( x_1^{(n-1)}(t) \to -\infty \) as \( t \to \infty \). Therefore in view of (26) there exists \( \delta_4 > 0 \)
such that \( f_2(x(h_1(t))) \geq \delta \) for \( t \geq T \geq \gamma(T_3) \). Further we proceed analogously as in the case 1-1) we obtaining \( x_2(t) < 0 \) for large \( t \), which contradicts \( x_2(t) > 0 \) for \( t \geq t_1 \).

2) Suppose that \( x_1(t) > 0, x_2(t) < 0 \) for \( t > t_1 \). (The proof in the case \( x_1(t) < 0, x_2(t) > 0 \) is similar). Then in view of (24), (25) from (A) we get \( x_1^{(n)}(t) < 0, i = 1, 2, \) for \( t \geq t_2 = \gamma(t_1) \). Using Lemma, we have \( x_1'(t) > 0 \) and either i) \( x_2'(t) < 0, x_2'(t) < 0, \) or ii) \( x_2'(t) > 0 \) for \( t \geq t_3 \geq t_2 \). In the case i) we proceed in the same way as in the case 1-2), obtaining a contradiction to the assumption \( x_1(t) > 0 \) for \( t \geq t_1 \). Now we consider the case ii). The component \( x_2(t) \) is increasing and \( \lim_{t \to \infty} x_2(t) = -b \leq 0 \). If we suppose that \( b > 0 \), we proceed in the same way as in the case i) arriving at a contradiction. Therefore \( b = 0 \), i.e. \( \lim_{t \to \infty} x_2(t) = 0 \). This in view of Lemma implies \( \lim_{t \to \infty} x_2^{(k)}(t) = 0 \) for \( k = 0, 1, \ldots, n \).

The proof of Theorem 4 is complete. \( \square \)

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References

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V práci je študovaná existencia neosilatorických riešení systému

\[ x_i^{(n)}(t) = \sum_{j=1}^{2} p_{ij}(t) f_{ij} \left( x_j \left( h_{ij}(t) \right) \right), \quad n \geq 2, \quad i = 1, 2, \]

s vlastnosťami \( \lim_{t \to \infty} x_i(t)/t^{k_i} = \text{const.} \neq 0 \) pre nejaké \( k_i \in \{1, 2, \ldots, n - 1\}, \quad i = 1, 2 \). Dalej sú dokázané postačujúce podmienky pre to, aby systém mal oscilatorické riešenie.