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CENTRALLY DETERMINED STATES ON VON NEUMANN ALGEBRAS

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Summary. It is shown that every von Neumann algebra whose centre determines the state space is already abelian.

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AMS classification: 46L30, 46L50

The following question was posed in [4]: Is every von Neumann algebra with centrally determined state space abelian? The aim of this note is to establish a positive answer to this question.

Let $A$ be an arbitrary von Neumann algebra and let $Z$ be its centre. Let $\mathcal{P}(A)$ and $\mathcal{P}(Z)$ stand for the orthomodular lattices of all projections in $A$ and $Z$, respectively (see [6]). Let us call a mapping $\eta: A \to Z$ a centre state if it is positive, $\eta(C) = C$ and $\eta(CA) = C\eta(A)$ for every $C \in Z$ and $A \in A$ (see [1]). Further, let us call a mapping $s: \mathcal{P}(A) \to (0, 1)$ a state if $s(I) = 1$ ($I$ is an identity in $A$) and $s\left(\sum_{n \in \mathbb{N}} P_n\right) = \sum_{n \in \mathbb{N}} s(P_n)$, whenever $(P_n)$ is sequence of mutually orthogonal elements of $\mathcal{P}(A)$. Finally, let us say that $A$ has a centrally determined state space (see [2, 3]) if states $s_1$ and $s_2$ on $\mathcal{P}(A)$ coincide whenever they agree on $\mathcal{P}(Z)$.

Theorem. A von Neumann algebra $A$ has a centrally determined state space if and only if it is abelian.

Proof. The sufficiency is obvious. Let us take up the necessity. Suppose that $A$ is not abelian. Looking for a contradiction let us assume that $A$ has centrally determined state space. Let us choose $A \in A \setminus Z$. According to [1, Lemma 8.2.3, p. 512] $A$ admits an ultraweakly continuous centre state $\eta: A \to Z$, that is, $A$ admits
such a central state \( \eta \) which is additive with respect to any system of mutually orthogonal projections. Then, obviously \( \eta(A) \neq A \). Let \( \omega \) be a normal state of \( A \) such that \( \omega(\eta(A)) \neq \omega(A) \). Put \( \tilde{\omega} = \omega \circ \eta \). Then \( \tilde{\omega} \) is a normal state of \( A \) again and we have \( \omega|Z = \tilde{\omega}|Z, \omega(A) \neq \tilde{\omega}(A) \). However, using the spectral theorem, we see that \( \omega \) and \( \tilde{\omega} \) do not coincide on \( \mathcal{P}(A) \). We have obtained two distinct states \( \omega|\mathcal{P}(A) \) and \( \tilde{\omega}|\mathcal{P}(A) \) which coincide on the centre \( \mathcal{P}(Z) \). This is a contradiction and the proof is complete. \( \square \)

It is easy to observe that the latter theorem holds even in more general situation, i.e., for instance it holds for any \( C^* \)-algebra \( A \) whose projections generate a dense subspace in \( A \). It should be also noted that our result may be relevant to the noncommutative measure theory on von Neumann algebras. Namely, our theorem complemented with results in [2, 3] implies that the classical version of the Radon-Nikodým theorem holds exactly in the classical (i.e. commutative) case (compare also with [5]).

References


Souhrn

CENTRÁLNĚ DETERMINOVANÉ STAVY NA VON NEUMANNOVÝCH ALGEBRÁCH

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Je ukázáno, že každá von Neumannova algebra, jejíž centrum určuje stavový prostor, je abelovská.

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