Miroslava Růžičková
Comparison theorems for differential equations of neutral type


Persistent URL: [http://dml.cz/dmlcz/125913](http://dml.cz/dmlcz/125913)

Terms of use:

© Institute of Mathematics AS CR, 1997

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library [http://dml.cz](http://dml.cz)
COMPARISON THEOREMS FOR DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

MIHOSLAVA RUŽIČKOVÁ, Žilina

(Received January 30, 1996)

Summary. We are interested in comparing the oscillatory and asymptotic properties of the equations $L_n [x(t) - P(t) x(g(t))] + \delta f(t, x(h(t))) = 0$ with those of the equations $M_n [x(t) - P(t) x(g(t))] + \delta Q(t) y(r(t))) = 0$.

Keywords: neutral differential equations, oscillatory (nonoscillatory) solution, property A, property B, quasi-derivatives

MSC 1991: 34K40, 34K25

1. INTRODUCTION

We consider neutral differential equations of the form

\begin{equation}
L_n [x(t) - P(t) x(g(t))] + \delta f(t, x(h(t))) = 0,
\end{equation}

where $n \geq 2$, $\delta = +1$ or $-1$ and the operator $L_n$ is defined recursively by

\[ L_0 u(t) = u(t), \quad L_k u(t) = \frac{1}{a_k(t)} [L_{k-1} u(t)]', \quad k = 1, 2, \ldots, n, \quad a_n = 1. \]

The following conditions are assumed to hold throughout the paper:

(a) $a_i \in C[[t_0, \infty), (0, \infty)]$, $t_0 \geq 0$ and $\int_{t_0}^\infty a_i(t) \, dt = \infty$, $i = 1, 2, \ldots, n - 1$;

(b) $P \in C[[t_0, \infty), \mathbb{R}]$ and satisfies $|P(t)| \leq \lambda$ on $[t_0, \infty)$ for some constant $\lambda < 1$;

(c) $g \in C[[t_0, \infty), (0, \infty)]$ is increasing, $g(t) < t$ for $t \geq t_0$ and $\lim_{t \to \infty} g(t) = \infty$;

(d) $h \in C[[t_0, \infty), (0, \infty)]$ and $\lim_{t \to \infty} h(t) = \infty$;
We classify the possible nonoscillatory solutions of (A) in a similar way as in the paper [5].

Let \( x(t) \) be a nonoscillatory solution of (A). From (A) and (e) it follows that the function

\[
y(t) = x(t) - P(t) x(g(t))
\]

has to be eventually of constant sign, so that either

\[
x(t) y(t) > 0 \quad \text{(2)}
\]

or

\[
x(t) y(t) < 0 \quad \text{(3)}
\]

for all sufficiently large \( t \). Assume first that (2) holds. Then the function \( y(t) \) satisfies \( \delta y(t) \) \( L_i y(t) < 0 \) eventually and the well-known Kiguradze's lemma (see [5]) implies that there is an integer \( \ell \in \{0, 1, \ldots, n\} \) and a \( t_1 > t_0 \) such that \((-1)^{n-\ell} = 1\) and for every \( t \geq t_1 \)

\[
y(t) L_i y(t) > 0, \quad 0 \leq i \leq \ell,
\]

\[
(-1)^{n-i} y(t) L_i y(t) > 0, \quad 1 \leq i \leq n
\]

holds.
A function $y(t)$ satisfying (4) is said to be a nonoscillatory function of degree $t$. The set of all solutions $x(t)$ of (A) satisfying (2) and (4) will be denoted by $N_0^+$. Now assume that (3) holds. Then $y(t)$ satisfies $-\delta y(t) L_n y(t) < 0$ for all large $t$ and so it is a function of degree $t$ for some $\ell \in \{0, 1, \ldots, n\}$ with $(-1)^{n-\ell} = 1$. The totality of nonoscillatory solutions $x(t)$ of (A) which satisfy (3) and (4) will be denoted by $N_\ell^+$. Consequently, if we denote by $N'$ the set of all possible nonoscillatory solutions of (A), then (see [5])

$$N' = N_1^+ \cup N_2^+ \cup \ldots \cup N_{n-1}^+ \cup N_0^- \text{ for } \delta = 1 \text{ and } n \text{ even,}$$

$$N' = N_0^+ \cup N_2^+ \cup \ldots \cup N_{n-1}^+ \text{ for } \delta = 1 \text{ and } n \text{ odd,}$$

$$N' = N_1^+ \cup N_3^+ \cup \ldots \cup N_{n}^+ \cup N_0^- \text{ for } \delta = -1 \text{ and } n \text{ even,}$$

$$N' = N_0^+ \cup N_2^+ \cup \ldots \cup N_{n}^+ \cup N_0^- \text{ for } \delta = -1 \text{ and } n \text{ odd.}$$

The class $N_0^-$ must be removed from (5) provided $P(t)$ is either oscillatory or eventually negative, because in this case equation (A) cannot possess a nonoscillatory solution $x(t)$ satisfying (3).

It is now clear that the oscillation of all proper solutions of (A) is equivalent to the situation in which $N' = \emptyset$.

**Definition 1.** Equation (A) is said to have property $A$ if for $\delta = 1$ and $n$ even all proper solutions are oscillatory while for $\delta = 1$ and $n$ odd $N' = N_0^+$.

**Definition 2.** Equation (A) is said to have property $B$ if for $\delta = -1$ and $n$ even $N' = N_0^+ \cup N_2^+$ while for $\delta = -1$ and $n$ odd $N' = N_n^+$.

### 3. Comparison Theorems

We are interested in comparing the oscillatory and asymptotic properties of equations (A) with those of the equations (B)

$$M_n [x(t) - P(t)x[q(t)]] + \delta Q(t)y[r(t)] = 0,$$

where $n \geq 2$, $\delta = +1$ or $-1$, $M_0 u(t) = u(t), M_k u(t) = \frac{1}{b_k(t)} [M_{k-1} u(t)], k = 0, 1, 2, \ldots, n, b_n = 1$, and the following conditions are fulfilled:

1. $b_i \in C[[t_0, \infty), (0, \infty)], t_0 \geq 0$ and $\int b_i(t) dt = \infty, i = 1, 2, \ldots, n - 1$;
2. $Q, r \in C[[t_0, \infty), (0, \infty)]$ and $\lim_{t \to \infty} r(t) = \infty$;
(e) \( q \in C[\mathbb{R}, \mathbb{R}] \) is nondecreasing, \( xq(x) > 0 \) for \( x \neq 0 \) and
\[
x y q(xy) \geq K xy q(x)q(y) \quad \text{for each } x, y \ (0 < K = \text{constant});
\]

(f) \( h(t) \geq r(t) \) for \( t \geq t_0 \);

(g) \( a_i(t) \geq b_i(t) \) for \( t \geq t_0, 1 \leq i \leq n - 1 \).

The following notation will be needed:
\( g^{-1}(t) \) is the inverse function of \( g(t) \):
\[
s = \max \{1, K q (\frac{1}{x})\};
\]
\[
\alpha(t) = \int_{t_0}^t \frac{a_1(z_1)}{1} \int_{t_0}^z a_2(z_2) \ldots \int_{t_0}^{z_{n-2}} a_{n-1}(z_{n-1}) dz_{n-1} \ldots dz_3;
\]
\[
\beta(t) = \int_{t_0}^t \frac{a_1(z_1)}{1} \int_{t_0}^z a_2(z_2) \ldots \int_{t_0}^{z_{n-2}} a_{n-2}(z_{n-2}) dz_{n-2} \ldots dz_3;
\]
\( R_0 = (-\infty, 0) \cup (0, \infty) \);
\( C(R) = \{ F : R \to \mathbb{R} | F \text{ is continuous and } x F(x) > 0 \text{ for } x \neq 0 \} \);
\( C_p(R_0) = \{ F \in C(R) | F \text{ is of bounded variation on every interval } [a, b] \subseteq R_0 \} \).

**Lemma 1.** [9] Suppose that \( z(t) \) is a nonoscillatory solution of equation (B).

i) Let \( P(t) \) be eventually positive and let \( x(t) y(t) > 0 \) (\( y(t) \) is defined by (1)). Then \( z(t) \) is a member of \( N_{\infty}^+ \) if and only if \( y(t) \) is a solution of degree 1 of
\[\left\{ M_n y(t) + Q(t) q(y(r(t))) \right\} \text{sgn } y(t) \leq 0,\]
whereby
\[|y(t)| \leq |z(t)| \quad \text{for large } t.\]

ii) Let \( P(t) \) be eventually positive and let \( x(t) y(t) < 0 \). Then \( z(t) \) is a member of \( N_{\infty}^+ \) if and only if \( v(t) = -y(t) \) is a solution of degree 0 of
\[\left\{ - M_n v(t) + S Q(t) q(v(g^{-1}(r(t)))) \right\} \text{sgn } v(t) \leq 0,\]
where \( 0 < S = K q (\frac{1}{x}) = \text{constant} \), whereby
\[\frac{1}{x} |v(g^{-1}(t))| \leq |z(t)| \quad \text{for large } t.\]

iii) Suppose that \( P(t) \) is eventually negative or that \( P(t) \) is oscillatory and satisfies
\[P(t) P(q(t)) \geq 0 \quad \text{for large } t.\]
Then \( x(t) \) is a member of \( \mathcal{N}^\ell \) with \( \ell \geq 1 \) if and only if \( y(t) \) is a solution of degree \( \ell \) of

\[
\left\{ \delta \, M_\alpha \, y(t) + M \, Q(t) \, q(y(r(t))) \right\} \, \text{sgn} \, y(t) \leq 0,
\]

where \( 0 < M = K \, q(1 - \lambda) = \text{constant} \), whereby

\[
|y(t)| \geq (1 - \lambda) \, |y(t)| \quad \text{for large } t.
\]

**Lemma 2.** [7] Suppose \( F \in C(\mathbb{R}) \). Then \( F \in C^\ell(\mathbb{R}_0) \) if and only if \( F(x) = G(x) \, H(x) \) for all \( x \in \mathbb{R}_0 \), where \( G : \mathbb{R}_0 \to [0, \infty) \) is nondecreasing on \((-\infty, 0)\) and nonincreasing on \((0, \infty)\) and \( H : \mathbb{R}_0 \to \mathbb{R} \) is nondecreasing on \( \mathbb{R}_0 \).

Remark. \( G, H \) are called a pair of continuous components of \( F \).

We also assume that there exists a continuous function \( Z : [t_0, \infty) \to [0, \infty) \) such that

\[
f(t, x) \, \text{sgn} \, x \geq Z(t) \, F(x) \, \text{sgn} \, x \quad \text{for } (t, x) \in [t_0, \infty) \times \mathbb{R}.
\]

In the following two comparison theorems we compare equation (A) with the special cases of equation (B), namely, when \( M_\alpha = L_\alpha \) and \( h = r \).

**Theorem 1.** Let \( \delta = 1 \). Suppose that (13) holds and let \( G \) and \( H \) be a pair of continuous components of \( F \) with \( H \) being the nondecreasing one.

i) Assume that \( P(t) \) is eventually negative or that \( P(t) \) is oscillatory and satisfies

\[
H((1 - \lambda) \, r) \, \text{sgn} \, x \geq q(x) \, \text{sgn} \, x \quad \text{for } x \in \mathbb{R},
\]

\[
b_i(t) \equiv a_i(t), \quad 1 \leq i \leq n - 1,
\]

\[
h(t) = v(t),
\]

\[
Z(t) \, G\left( \pm (1 - \lambda) \, c \, a(h(t)) \right) \geq M \, Q(t) \quad \text{for every large } c > 0 \text{ and all large } t
\]

(where \( M = K \, q(1 - \lambda) \)) imply that equation (A) has property \( A \) if equation (B) has property \( A \).

ii) Assume that \( P(t) \) is eventually positive. Then the conditions (15), (16)

\[
H(x) \, \text{sgn} \, x \geq q(x) \, \text{sgn} \, x \quad \text{for } x \in \mathbb{R},
\]

\[
Z(t) \, G\left( \pm c \, a(h(t)) \right) \geq a \, Q(t) \quad \text{for every large } c > 0 \text{ and all large } t
\]

imply that equation (A) has property \( A \) if equation (B) has property \( A \).
Proof. We present the proof for \( n \) even.

i) According to (5), \( \mathcal{N}_{\ell}^+ = \{1, 3, \ldots, n-1\} \) and \( \mathcal{N}_{\ell}^- \) are the possible classes of nonoscillatory solutions of (A) with \( \delta = 1 \) and even \( n \). In the case when \( P(t) \) is eventually negative or oscillatory, \( \mathcal{N}_{\ell}^- \) is necessarily empty. Suppose that \( \mathcal{N}_{\ell}^+ \neq \emptyset \) for some \( \ell \in \{1, 3, \ldots, n-1\} \) and let \( x \in \mathcal{N}_{\ell}^+ \) be a solution of (A). Without loss of generality we may assume that \( x \) is eventually positive. Then from (4), we observe that

\[
L_n y(t) > 0 \quad \text{and} \quad L_{\alpha-1} y(t) < 0 \quad \text{for all large} \ t.
\]

Thus,

\[
L_{\alpha-1} y(t) \leq c_1, \quad c_1 > 0
\]

and hence there exists \( a > 0 \) such that

\[
y(t) \leq a \alpha(t) \quad \text{for all large} \ t
\]

and in view of (d) we have

\[
y(h(t)) \leq a \alpha(h(t)) \quad \text{for all sufficiently large} \ t.
\]

Now, by conditions (c), (12), (13), (14), (17) and Lemma 2 we get

\[
f(t, x(h(t))) \geq f(t, \gamma(h(t))) \geq Z(t)F((1 - \lambda)y(h(t)))
\]

\[
= Z(t)G((1 - \lambda)y(h(t))) H((1 - \lambda)y(h(t)))
\]

\[
\geq Z(t)G((1 - \lambda)a \alpha(h(t))) H((1 - \lambda)y(h(t)))
\]

\[
\geq M Q(t) H((1 - \lambda)y(h(t))) \geq M Q(t)q(y(h(t)))
\]

\[
\geq M Q(t)q(y(h(t)))
\]

and hence the function \( y \) which is of degree \( \ell \) is a solution of the differential inequality (11), in which (15) and (16) hold.

On the other hand, Lemma 1 implies that differential inequality (11), in which (15) and (16) hold, has a solution of degree \( \ell \geq 1 \) if and only if equation (B) with \( M_n = L_n \) and \( h = r \), namely, the equation

\[
L_n [x(t) - P(t)x(g(t))] + \delta Q(t) q(x(h(t))) = 0,
\]

has a solution of degree \( \ell \). We supposed \( 1 \leq \ell \leq n-1 \) and this contradicts the hypothesis that equation (21) is oscillatory.

ii) Let \( \mathcal{N}_{\ell}^+ \neq \emptyset \) for some \( \ell \in \{1, 3, \ldots, n-1\} \). Without loss of generality we may assume that \( x \) is eventually positive. Therefore similarly as above, by conditions (c), (7), (13), (18), (19) and Lemma 2 we get

\[
f(t, x(h(t))) \geq f(t, y(h(t))) \geq Z(t)F(y(h(t)))
\]

\[
= Z(t)G(y(h(t))) H(y(h(t)))
\]

\[
\geq Z(t)G(a \alpha(h(t))) H(y(h(t)))
\]

\[
\geq Q(t) H(y(h(t))) \geq Q(t)q(y(h(t))).
\]
One can see that the function $y$ which is of degree $i \in \{1, 3, \ldots, n - 1\}$ is a solution of the differential inequality (6) in which $M_t = L_t$ and $r = h$.

Applying Lemma 1 we conclude that (21) has a solution of degree $i$. This is a contradiction.

Suppose that $N_t \neq \emptyset$. In this case $x(t)y(t) < 0$. Because $0 < \lambda < 1$ and $H$ is nondecreasing, from (18) we obtain

$$H \left( \frac{1}{\lambda} x \right) \text{sgn} x \geq q(x) \text{sgn} x.$$  

Next, without loss generality, we may assume that $x$ is eventually positive. Then, because $\ell = 0$, we observe from (4) that

$L_0 g(t) < 0$ and $L_1 g(t) > 0$ for all large $t$.

Thus,

$$y(t) \geq -c, \quad c > 0,$$

or

$$-y(t) = u(t) \leq c \text{ for all large } t.$$

Now, by conditions (a), (e), (9), (13), (19), (23) and Lemma 2 we get

$$f(t, x(h(t))) \geq f \left( t, \frac{1}{\lambda} v(g^{-1}(h(t))) \right) \geq Z(t) F \left( \frac{1}{\lambda} v(g^{-1}(h(t))) \right)$$

$$= Z(t) G \left( \frac{1}{\lambda} v(g^{-1}(h(t))) \right) H \left( \frac{1}{\lambda} v(g^{-1}(h(t))) \right)$$

$$\geq Z(t) G(\alpha(h(t))) H \left( \frac{1}{\lambda} v(g^{-1}(h(t))) \right)$$

$$\geq S Q(t) H \left( \frac{1}{\lambda} v(g^{-1}(h(t))) \right) \geq S Q(t) q \left( v(g^{-1}(h(t))) \right)$$

for sufficiently large $t$. Therefore similarly as above, applying Lemma 1 we get a contradiction. The proof in the case when $n$ is odd is similar and will be omitted.

\[ \square \]

**Theorem 2.** Let $\delta = -1$. Suppose that (13) holds and let $G$ and $H$ be a pair of continuous components of $F$ with $H$ being the nondecreasing one.

1) Assume that $P(t)$ is eventually negative or that $P(t)$ is oscillatory and satisfies (10). If (14), (15), (16) and

$$Z(t) G(\pm(1 - \lambda) c \beta(h(t))) \geq M Q(t) \text{ for every large } c > 0 \text{ and all large } t$$

(25)
hold, then equation (A) has property B if equation (B) has property B.

ii) Assume that \( P(t) \) is eventually positive. Then the conditions (15), (16), (18) and

\[
Z(t) G \left( \pm c \beta(h(t)) \right) \geq s Q(t) \text{ for every large } c > 0 \text{ and all large } t
\]

implies that equation (A) has property B if equation (B) has property B.

Proof of Theorem 2 is similar to that of Theorem 1 and will be omitted. \( \square \)

The following theorems are intended to relax conditions (15), (16) in the previous result.

**Theorem 3.** Let \( \delta = 1 \) and let \( G, H \) be a pair of continuous components of \( F \) with \( H \) being the nondecreasing one. Suppose that (13), (14) hold.

i) Assume that \( P(t) \) is eventually negative or that \( P(t) \) is oscillatory and satisfies (10). Then the condition (17) implies that equation (A) has property B if equation (B) has property B.

ii) Assume that \( P(t) \) is eventually positive. Then the condition (19) implies that equation (A) has property B if equation (B) has property B.

Proof. Let \( n \) be even. i) Let equation (B) have property A. By Lemma 1 inequality (11) has property A and by Theorem 1 in [11] inequality (11) with \( M_n = L_n \) and \( r = h \) has property A as well. Theorem 1 now shows that equation (A) has property A.

The proof in the other cases can be done in an analogous way, so we omit it. \( \square \)

**Theorem 4.** Let \( \delta = -1 \) and let \( G, H \) be a pair of continuous components of \( F \) with \( H \) being the nondecreasing one. Suppose that (13), (14) hold.

i) Assume that \( P(t) \) is eventually negative or that \( P(t) \) is oscillatory and satisfies (10). Then the condition (25) implies that equation (A) has property B if equation (B) has property B.

ii) Assume that \( P(t) \) is eventually positive. Then the condition (26) implies that equation (A) has property B if equation (B) has property B.

Proof of Theorem 4 is similar to that of Theorem 3 and we omit it. \( \square \)
References


Author’s address: M. Růžičková, Department of Appl. Mathematics, University of Transport and Communications, J. M. Hurbana 15, 010 26 Žilina, Slovakia, e-mail: miir@vt.utc.sk