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Persistent URL: http://dml.cz/dmlcz/125917

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CIRCULAR DISTANCE IN DIRECTED GRAPHS

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(Received November 21, 1994)

Summary. Circular distance \(d°(x,y)\) between two vertices \(x, y\) of a strongly connected directed graph \(G\) is the sum \(d(x,y) + d(y,x)\), where \(d\) is the usual distance in digraphs. Its basic properties are studied.

Keywords: strongly connected digraph, circular distance, directed cactus

MSC 1991: 05C38, 05C20

In an undirected graph the distance between two vertices is usually defined as the length of the shortest path connecting these vertices. This distance is a metric on the vertex set of the graph. Analogously in a directed graph (usually the strong connectedness is supposed) the distance \(d(x,y)\) from a vertex \(x\) to a vertex \(y\) is defined as the length of the shortest directed path from \(x\) to \(y\). In general, \(d(x,y)\) thus defined is not a metric, because it is not symmetric. In this paper we define a certain distance in a digraph which is a metric.

Let \(G\) be a strongly connected directed graph, let \(x, y\) be two vertices of \(G\). The circular distance \(d°(x,y)\) between the vertices \(x, y\) in the graph \(G\) is defined as

\[
d°(x,y) = d(x,y) + d(y,x),
\]

where \(d\) denotes the usual distance in digraphs (see above). In other words, \(d°(x,y)\) is the length of the shortest directed walk going from \(x\) to \(y\) and then back to \(x\).

Note that in the walk mentioned, vertices and edges may repeat. In the graph in Fig. 1 such shortest walk for \(x\) and \(y\) contains all edges of the graph and the edge \(e\) occurs twice in it.

The following proposition is evident.

**Proposition 1.** The circular distance \(d°(x,y)\) is a metric on the vertex set \(V(G)\) of the graph \(G\).
The properties of the circular distance are considerably different from the properties of the usual distance in graphs.

The length of the shortest cycle (directed circuit) in the graph \( G \) will be called the \textit{directed girth} of \( G \) and denoted by \( g(G) \).

**Proposition 2.** Let \( x, y \) be two distinct vertices of a strongly connected graph \( G \), let \( g(G) \) be the directed girth of \( G \). Then

\[
d^c(x, y) \geq g(G).
\]

**Proof.** Let \( P_1 \) (or \( P_2 \)) be the shortest path from \( x \) to \( y \) (or from \( y \) to \( x \), respectively). The circular distance \( d^c(x, y) \) is equal to the sum of lengths of \( P_1 \) and \( P_2 \). The union of \( P_1 \) and \( P_2 \) must contain a cycle; the length of this cycle is greater than or equal to \( g(G) \) and less than or equal to the sum of lengths of \( P_1 \) and \( P_2 \); this implies the assertion. \( \square \)

Analogously as for the usual distance, we may introduce the circular radius \( \delta^c(G) \) and the circular diameter \( \delta^c(G) \). For each vertex \( x \) of \( G \) we define the \textit{circular elongation} \( e^c(x) \) as the maximum of \( d^c(x, y) \) for all \( y \in V(G) \). Then the minimum of \( e^c(x) \) for all \( x \in V(G) \) is the \textit{circular radius} \( \delta^c(G) \) of \( G \). The set of vertices \( x \) for which \( e^c(x) = \delta^c(G) \) is called the \textit{circular center} \( C^c(G) \) of \( G \). The maximum of \( d^c(x, y) \) over all pairs \( x, y \) of vertices of \( G \) is the \textit{circular diameter} \( \delta^c(G) \) of \( G \).

In the case of infinite graphs it may happen that the maximum of \( d^c(x, y) \) does not exist. Then we put \( \delta^c(G) = \infty \) and also \( \delta^c(G) = \infty \). In the sequel we shall consider only finite radii and diameters.

The following proposition can be proved in the same way as the analogous statement for the \textit{usual distance} in graphs; it follows from the triangle inequality.
Proposition 3. For the circular radius $\phi^*(G)$ and the circular diamter $\delta^*(G)$ of a strongly connected directed graph $G$ the following inequality holds:

$$
\phi^*(G) \leq \delta^*(G) \leq 2\phi^*(G)
$$

Now we have a theorem.

Theorem 1. Let $r$, $d$ be positive integers, $2 \leq r \leq d \leq 2r$. Then there exists a strongly connected directed graph $G$ such that $\phi^*(G) = r$, $\delta^*(G) = d$.

Proof. If $r = d$, then $G$ is the cycle of length $r$. In it $d^*(x, y) = r$ for any two distinct vertices $x, y$.

If $d = r + 1$, distinguish the cases $r = 2$ and $r \geq 3$. If $r = 2$, then let $V(G) = \{u, v_1, v_2\}$ and let the edges of $G$ be $u v_1, v_1 u, v_2 u, v_1 v_2$ (Fig. 2). We have $d^*(u, v_1) = d^*(u, v_2) = 2$, $d^*(v_1, v_2) = 3$, $e^*(u) = 2$, $e^*(v_1) = e^*(v_2) = 3$ and thus $\phi^*(G) = 2$, $\delta^*(G) = 3$. If $r \geq 3$, then let $V(G) = \{v_0, v_1, \ldots, v_{r-1}, w\}$. Let the edges be $v_i v_{i+1}$ for $i = 0, \ldots, r - 2$, $v_{r-1} v_0$, $v_0 w$ and $w v_i$ for $i = 1, \ldots, r - 1$. (Fig. 3 for $r = 8$.) We have $d^*(v_i, w) = r + 1 = d$, $d^*(v_0, v_i) = r$, $d^*(v_i, v_0) = r$ for $i = 2, \ldots, r - 1$. Further we have $d^*(v_0, w) = 3 \leq r$, $d^*(v_i, w) = r - i + 2 \leq r$ for $i = 2, \ldots, r - 1$. Finally, $d^*(v_i, v_j) \leq r$ for any $i$ and $j$, because $v_0, \ldots, v_{r-1}$ form a cycle of length $r$. We have $e^*(v_1) = e^*(w) = d$, $e^*(v_0) = e^*(v_i) = r$ for $i = 2, \ldots, r - 1$. Hence $\delta^*(G) = d$, $\phi^*(G) = r$.

Fig. 2

If $d \geq r + 2$, let the graph $G$ consist of two cycles $C_1$, $C_2$ with exactly one common vertex $a$; let the length of $C_1$ be $r$ and let the length of $C_2$ be $d - r$. Let $a_1$ (or $a_2$) be an arbitrary vertex of $C_1$ (or $C_2$, respectively) different from $a$. Then $d^*(a, a_1) = r$, $d^*(a, a_2) = d - r \leq r$, $d^*(a_1, a_2) = d$. This implies $e^*(a) = r$, $e^*(a_1) = e^*(a_2) = d$ and again $\delta^*(G) = d$, $\phi^*(G) = r$. 

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If to the graph $G$ for the case $d = r + 1$, $r \geq 3$ we add the edge $vw_0$ (Fig. 4), we obtain a graph $G'$ such that the circular center $C^o(G') = \{v_0, v_1, \ldots, v_{r-1}\}$, while the center $C(G')$ for the usual distance $d(x, y)$ is $\{w\}$ and thus $C^o(G') \cap C(G') = \emptyset$. We have a proposition.

![Graph](image)

**Proposition 4.** The circular center $C^o(G)$ and the usual center $C(G)$ of a digraph $G$ may be disjoint.

Note that always $d^o(x, y) \neq 1$; this follows from the definition. Evidently also $\rho^o(G) \neq 1$ and $\delta^o(G) \neq 1$.

**Theorem 2.** Let $(M, m)$ be a metric space such that the set $M$ is finite and the metric $m$ attains only integral values. Then there exists a strongly connected directed graph $G$ such that $M \subseteq V(G)$ and $d^o(x, y) = m(x, y) + 1$ for any two distinct vertices $x, y$ of $M$. Moreover, all vertices of $V(G) - M$ have indegree 1 and outdegree 1.

**Proof.** Choose an arbitrary total ordering $<$ on $M$. For any two vertices $x, y$ of $M$ such that $x < y$ we form the edge $xy$; in this way we obtain a tournament with the vertex set $M$. Further, for any $x$ and $y$ of $M$ such that $x < y$ we add a directed path $P(x, y)$ of length $m(x, y)$ from $y$ to $x$. The inner vertices of any path $P(x, y)$ are not in $M$ and any two such paths have no inner vertex in common. The graph thus obtained is $G$. We see that all vertices of $V(G) - M$ have indegree 1 and outdegree 1. Consider two vertices $x, y$ of $M$ such that $x < y$ and let $d$ denote the usual distance in a digraph. Then evidently $d(x, y) = 1$. The path $P(x, y)$ is the shortest path from $y$ to $x$, because any other path from $y$ to $x$ must contain at least one vertex $z \in M$; then its length is at least $m(y, z) + m(z, x)$ and by the triangle inequality this is greater than or equal to $m(y, x)$. Therefore $d(y, x) = m(x, y)$ and $d^o(x, y) = m(x, y) + 1$.

A certain analogue of trees are directed cacti. A directed cactus is a graph in which each block is a cycle [1].
The following proposition is easy to prove.

**Proposition 5.** Let \( x, y \) be two distinct vertices of a directed cactus \( G \). Then there exists exactly one directed path \( P(x, y) \) from \( x \) to \( y \) in \( G \).

Now we prove a theorem.

**Theorem 3.** If \( x, y \) are two distinct vertices of a directed cactus \( G \), then \( d°(x, y) \) is equal to the sum of lengths of all cycles in \( G \) which have common edges with the path \( P(x, y) \).

**Proof.** We will proceed by induction according to the number \( k \) of blocks which contain edges of \( P(x, y) \). If \( k = 1 \), then \( x \) and \( y \) are in the same block (cycle) \( B \) and this block is the (edge-disjoint) union of \( P(x, y) \) and \( P(y, x) \), therefore \( d°(x, y) \) is equal to the length of the cycle \( B \). Now let \( k \geq 2 \) and suppose that for \( k - 1 \) the assertion is true. Let the first edge of \( P(x, y) \) be in the block \( B_1 \) and let \( a \) be the terminal vertex of the last edge of \( P(x, y) \) being in \( B \). Then \( a \) is an articulation between \( B_1 \) and another block \( B_2 \) which contains the edge of \( P(x, y) \) outgoing from \( a \). The path \( P(a, y) \) is part of \( P(x, y) \) and there are \( k - 1 \) blocks containing edges of \( P(a, y) \), namely all those containing edges of \( P(x, y) \) except \( B_1 \). By the induction hypothesis \( d°(a, y) \) is the sum of lengths of these blocks. Not only \( P(x, y) \), but also \( P(y, x) \) goes through \( a \) and therefore \( d°(x, y) = d°(x, a) + d°(a, y) \), which is the sum of lengths of all cycles which contain edges of \( P(x, y) \).

Now we prove a theorem which concerns circular centers of directed cacti.

**Theorem 4.** The circular center of a finite directed cactus \( G \) either consists of one vertex, or is equal to the vertex set of one block of \( G \).

**Proof.** Let \( d°(G) = r \). First suppose that the circular center \( C°(G) \) contains two vertices \( u_1, u_2 \) which are not contained in the same block. Then there exists an articulation \( a \) of \( G \) which separates (in the same sense as in an undirected graph) the vertices \( u_1, u_2 \). By \( V_1 \) (or \( V_2 \)) we denote the set of vertices of \( G \) which are separated by \( a \) from \( u_2 \) and not from \( u_1 \) (or from \( u_1 \) and not from \( u_2 \), respectively). By \( V_3 \) we denote the set of vertices of \( G \) which are separated by \( a \) from both \( u_1, u_2 \). Suppose that there exists a vertex \( v \) such that \( d°(a, v) > r \). If \( v \in V_1 \cup V_3 \), then

\[
d°(u_2, v) = d°(u_2, a) + d°(a, v) > d°(u_2, a) + r > r;
\]

we have a contradiction with the assumption that \( r \) is the circular radius and \( u_2 \in C°(G) \). If \( v \in V_2 \), then

\[
d°(u_4, v) = d°(u_1, a) + d°(a, v) > d°(u_1, a) + r > r;
\]
again we have a contradiction. Evidently $V(G) = V_1 \cup V_2 \cup V_3 \cup \{a\}$ and therefore $d^c(a, x) < r$ for all $x \in V(G)$. Then $\phi^c(G) < r$, which is again a contradiction. We have proved that $C^c(G)$ must be a subset of the vertex set of a block of $G$. Let $B$ be such a block; it is a cycle. Let its length be $b$. If $B = G$, then evidently each vertex of $B$ belongs to the circular center and $C^c(G) = G = B$. If not, then $r > b$. For each $x \in V(B)$ let $W(x)$ be the set of all vertices of $G$ which are separated by $x$ from all other vertices of $B$. The sets $W(x)$ for all $x \in V(B)$ and the set $V(B)$ are pairwise disjoint and their union is $V(G)$. Let $p$ be the number of vertices $x \in V(B)$ with the property that there exists a vertex $y \in W(x)$ such that $d^c(x, y) \geq r - b$. Suppose $p = 0$. Let $v \in C^c(G) \subseteq V(B)$. If $x = v$, then $d^c(v, x) = 0 < r$. If $x \in V(B) - \{v\}$, then $d^c(v, x) = b < r$. If $x \in W(v)$, then $d^c(v, x) < r$ according to the assumption. If $x \in V(G) - (V(B) \cup W(v))$, there exists $y \in V(B) - \{v\}$ such that $x \in W(y)$. Then

$$d^c(v, x) = d^c(v, y) + d^c(y, x) = b + d^c(y, x) < b + r - b = r.$$ 

This is a contradiction with the assumption that $C^c(G) \subseteq V(B)$. Therefore $p \neq 0$. Suppose $p = 1$ and let $w$ be a vertex of $V(B)$ such that there exists $y \in W(w)$ for which $d^c(w, y) \geq r - b$. We may assume that $y$ is the vertex of $W(w)$ with the maximum circular distance from $w$. If $d^c(w, y) > r - b$, then each vertex of $V(B) - \{w\}$ has the circular distance from $y$ equal to $b + d^c(w, y) > r$. As we have supposed $C^c(G) \subseteq V(B)$, we have $C^c(G) = \{w\}$. If $d^c(w, y) = r - b$, then the circular distance of each vertex of $W(w)$ from $w$ is at most $r - b$ and the circular distance of any other vertex from $w$ is less than $r$; we have a contradiction with the assumption that $\phi^c(G) = r$. Finally, suppose $p \geq 2$. Let $w_1, w_2$ be two distinct vertices of $V(B)$ such that there exist vertices $y_1, y_2$ with $d^c(w_1, y_1) \geq r - b$, $d^c(w_2, y_2) \geq r - b$. If $d^c(w_1, y_1) > r - b$, then only $w_1$ can be in $C^c(G)$. The case $d^c(w_2, y_2) > r - b$ is analogous. Therefore $d^c(w_1, y_1) = d^c(w_2, y_2) = r - b$ and there exists no vertex in $W(w_1)$ with the circular distance from $w_1$ greater than $r - b$ and no vertex in $W(w_2)$ with the circular distance from $w_2$ greater than $r - b$ for each vertex $u \in V(B) - \{w_1\}$ we have $d^c(w_1, u) = r$ and for each vertex $u \in V(B) - \{w_2\}$ we have $d^c(w_2, u) = r$. In no set $W(x)$ for $x \in V(B)$ there is a vertex whose circular distance from $x$ would be greater than $r - b$; this can be proved in the same way as for $x = w_1$. Therefore for each $v \in V(G)$ and $u \in V(B)$ we have $d^c(u, x) \leq r$ and $C^c(G) = V(B)$. □

In Fig. 5 we see a directed cactus in which the circular center is a one-element set; in Fig. 6 we see a directed cactus in which the circular center is the vertex set of a block. In both the figures the vertices of the circular center are black.
References


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