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NATURAL TRANSFORMATIONS OF THE COVELOCITIES FUNCTORS INTO SOME NATURAL BUNDLES

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Summary. In this paper are determined all natural transformations of the natural bundle of (q, r)-covelocities over n-manifolds into such a linear natural bundle over n-manifolds which is dual to the restriction of a linear bundle functor, if $n \geqslant q$.

Keywords: Natural bundle, covelocities functors

AMS classification: 58A20, 53A55

1. Throughout the paper manifolds are assumed to be paracompact, finite dimensional, without boundary, second countable and of class C^∞. Maps will be assumed to be C^∞, unless the smoothness should be proved.

A class of well-known functors in differential geometry can be constructed as follows. Given integers $q, r \geqslant 1$ and an n-manifold M, we put $T^{r*}_q M = J^r(M, \mathbb{R}^q)_0$, the set of all r-jets of M into $\mathbb{R}^q$ with target 0. One can see that $T^{r*}_q M$ with the source projection is a vector bundle over M. We call $T^{r*}_q M$ the (q, r)-covelocities bundle of M. Every embedding $f\colon M \to N$ between n-manifolds induces a vector bundle homomorphism $T^{r*}_q f\colon T^{r*}_q M \to T^{r*}_q N$, $T^{r*}_q f(j^r_x \gamma) = j^r_{f(x)}(\gamma \circ f^{-1})$. One easily verifies that the rule $M \to T^{r*}_q M$, $f \to T^{r*}_q f$, is a linear bundle in dimension n in the sense of A. Nijenhuis, [5].

Let $\mathcal{M}_n$ or $\mathcal{M}$ be the category of all n-manifolds or all manifolds and embeddings or maps, respectively. Let $\nu\beta$ be the category of all vector bundles and vector bundle homomorphisms. A linear natural bundle $E\colon \mathcal{M}_n \to \nu\beta$ will be called ***admissible*** iff there exists a linear bundle functor $F\colon \mathcal{M} \to \nu\beta$ in the sense of I. Kolář and J. Slovák, [3], such that $E = (F|\mathcal{M}_n)^*$, i.e.

(1) $EM = (FM)^*$, the dual vector bundle of FM, for every $M \in \mathcal{M}_n$, and

(2) $Ef = (Ff^{-1})^*: EM \to EN$ for every embedding $f: M \to N$.

In particular, $T_q^{r*}: \mathcal{M}_n \to \nu\beta$ is admissible, for T_q^{r*} is isomorphic to $(T_q^r|\mathcal{M}_n)^*$, where $T_q^r: \mathcal{M} \to \nu\beta$ is the linear (q,r)-velocities bundle functor described in [2]. Of course, the tensor product (or the fiber product, the symmetric tensor product, the antisymmetric tensor product etc.) of a finite number of admissible natural bundles is admissible.

2. Let $E: \mathcal{M}_n \to \nu\beta$ be an admissible natural bundle. Let $F: \mathcal{M} \to \nu\beta$ be a linear bundle functor such that $E = (F|\mathcal{M}_n)^*$. Let r, $q \geqslant 1$ be integers such that $n \geqslant q$. We denote by $\mathrm{Adm}(E, r, q)$ the vector subspace

$$\{\omega \in (F_0\mathbf{R}^q)^*: \text{ for all } f: \mathbf{R}^n \to \mathbf{R}^q \ (j_0^r f = j_0^r p \Longrightarrow \omega \circ F_0 p = \omega \circ F_0 f)\},$$

where $p: \mathbf{R}^n = \mathbf{R}^q \times \mathbf{R}^{n-q} \to \mathbf{R}^q$ is the projection. By $\mathrm{Trans}(T_q^{r*}, E)$ we denote the vector space of all natural transformations of T_q^{r*} into E.

For any $\omega \in \mathrm{Adm}(E, r, q)$ and $M \in \mathcal{M}_n$ we define $T_M^\omega: T_q^{r*}M \to EM$ by $T_M^\omega(j_x^r\gamma) = \omega \circ F_x\gamma$, where $j_x^r\gamma \in T_q^{r*}M$ and $x \in M$.

Lemma 2.1. *If* $\omega \in \mathrm{Adm}(E, r, q)$, *then* $T^\omega = \{T_M^\omega\} \in \mathrm{Trans}(T_q^{r*}, E)$.

P r o o f. First we prove that T_M^ω is well defined. Let γ_1, $\gamma_2: M \to \mathbf{R}^q$ be such that $j_x^r\gamma_1 = j_x^r\gamma_2 \in T_q^{r*}M$. We consider two cases:

(1) $\mathrm{rank}(d_x\gamma_1) = q$. Then there exists an embedding $\varphi: \mathbf{R}^n \to M$ such that $\mathrm{germ}_0(\gamma_1 \circ \varphi) = \mathrm{germ}_0(p)$. Since $j_0^r(\gamma_2 \circ \varphi) = j_0^r(p)$ and $\omega \in \mathrm{Adm}(E, r, q)$, then $T_M^\omega(j_x^r\gamma_1) = \omega \circ F_0 p \circ F_x\varphi^{-1} = \omega \circ F_0(\gamma_2 \circ \varphi) \circ F_x\varphi^{-1} = T_{\mathcal{M}}^\omega(j_x^r\gamma_2)$.

(2) $\mathrm{rank}(d_x\gamma_1) < q$. Let $h: M \to \mathbf{R}^q$ be such that $h(x) = 0$ and $\mathrm{rank}(d_x h) = q$. Then there exists a sequence $t_m \in \mathbf{R}$, $m = 1, 2, \ldots$, such that $\mathrm{rank}\,(d_x(\gamma_1 + t_m h)) = q$ for all m and $t_m \to 0$ as $m \to \infty$. By the regularity condition of F (see [3])

$$T_{\mathcal{M}}^\omega(j_x^r(\gamma_i + t_m h)) = \omega \circ F_x(\gamma_i + t_m h) \to T_M^\omega(j_x^r\gamma_i)$$

as $m \to \infty$ for $i = 1, 2$. By virtue of the first case $T_{\mathcal{M}}^\omega(j_x^r(\gamma_1 + t_m h)) = T_M^\omega(j_x^r(\gamma_2 + t_m h))$ for all m. Therefore $T_{\mathcal{M}}^\omega(j_x^r\gamma_1) = T_M^\omega(j_x^r\gamma_2)$.

Hence T_M^ω is well-defined. For every embedding $f: M \to N$ we have

$$T_N^\omega \circ T_q^{r*}f = Ef \circ T_M^\omega \tag{2.1}$$

as $T_{\mathcal{N}}^\omega \circ T_q^{r*}f(j_x^r\gamma) = T_N^\omega(j_{f(x)}^r(\gamma \circ f^{-1})) = \omega \circ F_x\gamma \circ F_{f(x)}f^{-1} = Ef \circ T_M^\omega(j_x^r\gamma)$ for every $j_x^r\gamma \in T_q^{r*}M$.

It remains to show that T_M^ω is of class C^∞. By (2.1) it is sufficient to verify that $T_{\mathbf{R}^n}^\omega|(T_q^{r*})_0\mathbf{R}^n$ is of class C^∞. By the well-known Boman theorem, [1], it is sufficient to show that $T^{\omega}_{\mathbf{R}^n} \circ \tau$ is of class C^∞ for any $\tau\colon \mathbf{R} \to (T^{r*}_q)_0\mathbf{R}^n$ of class C^∞. Consider $\tau\colon \mathbf{R} \to (T^{r*}_q)_0\mathbf{R}^n$. Let $\gamma\colon \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^q$ be of class C^∞ such that $\tau(t) = j^r_0\big(\gamma(t,.)\big)$ for all $t \in \mathbf{R}$. Let $v \in E_0\mathbf{R}^n$. Then

$$\mathbf{R} \ni t \to T^{\omega}_{\mathbf{R}^n}\big(\tau(t)\big)(v) = \omega\big(F_0\big(\gamma(t,.)\big)(v)\big) \in \mathbf{R}$$

(and then $T^{\omega}_{\mathbf{R}^n} \circ \tau$) is of class C^∞ because of the regularity condition for F. □

3. Let E, F, r, q be as in Item 2. The main result is

Theorem 3.1. *The function*

$$I\colon \operatorname{Adm}(E,r,q) \to \operatorname{Trans}(T^{r*}_q, E), \quad I(\omega) = T^\omega$$

is a linear isomorphism. The inverse isomorphism is given by $S(T) = T_{\mathbf{R}^n}(j^r_0 p) \circ F_0 i$, *where* $p\colon \mathbf{R}^n \to \mathbf{R}^q$ *is as in Item 2 and* $i\colon \mathbf{R}^q \to \mathbf{R}^n$ *is given by* $i(t) = (t, 0)$.

Proof. First we prove that S is well-defined, i.e. $S(T) \in \operatorname{Adm}(E,r,q)$ for every $T \in \operatorname{Trans}(T^{r*}_q, E)$. Let $f\colon \mathbf{R}^n \to \mathbf{R}^q$ be such that $j^r_0 f = j^r_0 p$. There exists a sequence $t_m \in \mathbf{R}$, $m = 1, 2, \dots$, such that $\operatorname{rank}\big(d_0(i \circ f + t_m \operatorname{id}_{\mathbf{R}^n})\big) = n$ for all m and $t_m \to 0$ as $m \to \infty$. Then $T_{\mathbf{R}^n}(j^r_0 p) \circ F_0(i \circ f + t_m \operatorname{id}_{\mathbf{R}^n}) = E(i \circ f + t_m \operatorname{id}_{\mathbf{R}^n})^{-1} \circ T_{\mathbf{R}^n}(j^r_0 p) = T_{\mathbf{R}^n}\big(j^r_0\big(p \circ (i \circ f + t_m \operatorname{id}_{\mathbf{R}^n})\big)\big)$ for all m,

$$T_{\mathbf{R}^n}(j^r_0 p) \circ F_0(i \circ f + t_m \operatorname{id}_{\mathbf{R}^n}) \to S(T) \circ F_0 f, \quad \text{and}$$
$$T_{\mathbf{R}^n}\big(j^r_0\big(p \circ (i \circ f + t_m \operatorname{id}_{\mathbf{R}^n})\big)\big) \to T_{\mathbf{R}^n}(j^r_0 f)$$

as $m \to \infty$. Then $S(T) \circ F_0 f = T_{\mathbf{R}^n}(j^r_0 f) = T_{\mathbf{R}^n}(j^r_0 p) = S(T) \circ F_0 p$. Hence S is well-defined. Moreover, we have proved that

$$T_{\mathbf{R}^n}(j^r_0 p) = S(T) \circ F_0 p = T_{\mathbf{R}^n}(j^r_0 p) \circ F_0 i \circ F_0 p \tag{3.1}$$

for any $T \in \operatorname{Trans}(T^{r*}_q, E)$.

It is obvious that S is linear. We have $S \circ I(\omega) = T^{\omega}_{\mathbf{R}^n}(j^r_0 p) \circ F_0 i = \omega$ for any $\omega \in E_0\mathbf{R}^n$, i.e. $S \circ I = \operatorname{id}$. It remains to prove that $I \circ S = \operatorname{id}$. Consider $T \in \operatorname{Trans}(T^{r*}_q, E)$. Let $\omega = S(T)$. Then $I \circ S(T) = T^\omega$. We have to show that $T^\omega = T$. We see that $T^{\omega}_{\mathbf{R}^n}(j^r_0 p) \circ F_0 i = \omega \circ F_0 p \circ F_0 i = S(T) = T_{\mathbf{R}^n}(j^r_0 p) \circ F_0 i$. Then by (3.1) it follows that

$$T_{\mathbf{R}^n}(j^r_0 p) = T_{\mathbf{R}^n}(j^r_0 p) \circ F_0 i \circ F_0 p = T^{\omega}_{\mathbf{R}^n}(j^r_0 p) \circ F_0 i \circ F_0 p = T^{\omega}_{\mathbf{R}^n}(j^r_0 p).$$

Let $j^r_x\gamma \in T^{r*}_q M$. If $\operatorname{rank}(d_x\gamma) = q$, then there exists an embedding $\varphi\colon \mathbf{R}^n \to M$ such that $\operatorname{germ}_0(\gamma \circ \varphi) = \operatorname{germ}_0(p)$, and then $T_M(j^r_x\gamma) = E\varphi^{-1} \circ T_{\mathbf{R}^n}\big(j^r_0(\gamma \circ \varphi)\big) = E\varphi^{-1} \circ T_{\mathbf{R}^n}(j^r_0 p) = E\varphi^{-1} \circ T^{\omega}_{\mathbf{R}^n}(j^r_0 p) = T^{\omega}_M(j^r_x\gamma)$. Then $T_M = T^{\omega}_M$ on dense subset in $T^{r*}_q M$. Therefore $T_M = T^{\omega}_M$. □

4. Let E, F, r, q be as in Item 2. We see that $\mathrm{Adm}(E,r,q) = (F_0\mathbf{R}^q)^*$, if E is of order $\leqslant r$. Then we have the following corollary of Theorem 3.1.

Corollary 4.1. $\dim\big(\mathrm{Trans}(T_q^{r*}, E)\big) = \dim(F_0\mathbf{R}^q)$, *if* $\mathrm{ord}(E) \leqslant r$.

As an application of Corollary 4.1 we describe $\mathrm{Trans}(T_q^{r*}, \otimes^k T_1^{s*})$, where s, k are natural and $s \leqslant r$.

By Corollary 4.1 $\dim\big(\mathrm{Trans}(T_q^{r*}, \otimes^k T_1^{s*})\big) = \big(\mathrm{card}(A)\big)^k$, where $A = \{\alpha \in (N \cup \{0\})^q : 1 \leqslant |\alpha| \leqslant s\}$. On the other hand for every $(\alpha^1, \dots, \alpha^k) \in A^k$ we have $T^{(\alpha^1,\dots,\alpha^k)} \in \mathrm{Trans}(T_q^{r*}, \otimes^k T_1^{s*})$ given by

$$T_M^{(\alpha^1,\dots,\alpha^k)}(j_x^r\gamma) = j_x^s(\gamma^{\alpha^1}) \otimes \dots \otimes j_x^s(\gamma^{\alpha^k}),$$

where $j_x^r\gamma \in T_q^{r*}M$, $M \in \mathcal{M}_n$. It is easy to verify that $T^{(\alpha^1,\dots,\alpha^k)}, (\alpha^1, \dots, \alpha^k) \in A^k$, are linearly independent, and then they form a basis in $\mathrm{Trans}(T_q^{r*}, \otimes^k T_1^{s*})$, provided $n \geqslant q$. (In [4], J. Kurek proved this fact for $k = 1$.)

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