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GRASSMAN MANIFOLD V_3^4 IN THE PROJECTIVE SPACE P_7 WITH CHARACTERISTICS CONSISTING OF A QUADRIC AND TWO PLANES

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Summary. Some results in the geometry of four-parametric manifolds of three-dimensional spaces in the projective space P_7 are found. The properties of such a manifold V_3^4 with characteristics consisting of a quadric and two planes are studied. The properties of the manifold dual to V_3^4 are found.

Some results in the geometry of linear spaces from [1], [2], [3], [4] are used. The notation of the quantities is the same as in [4].

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Let us consider a four-parametric manifold V_3^4 in the projective seven-dimensional space P_7. Let the three-dimensional linear spaces P_3 be the generators of V_3^4. With each space P_3 we associate a frame consisting of independent points $A_i \in P_3$, $i = 1,2,3,4$, $\bar{A}_i \notin P_3$. The fundamental equations of the moving frame are

$$\begin{aligned} \mathrm{d}A_i &= \omega_i^s A_s + \tilde{\omega}_i^k \bar{A}_k \\ \mathrm{d}\bar{A}_i &= \hat{\omega}_i^k A_k + \bar{\omega}_i^k \bar{A}_k, \end{aligned} \qquad k, s = 1,2,3,4$$

where the forms ω_i^s, $\bar{\omega}_i^k$, $\hat{\omega}_i^k$, $\tilde{\omega}_i^k$ satisfy the structure equations of the space P_7 and $\tilde{\omega}_i^k$ are principal forms.

In all calculations we set $i, j, k, r, s = 1,2,3,4$. The tangent space of V_3^4 at a point $M \in P_3$ (the space $T(V_3^4, M)$) is the linear space determined by the points A_i and the points $\mathrm{d}M$. The point M is called a focal point of order q of V_3^4 if the dimension of $T(V_3^4, M)$ is equal to $7-q$ $(0 < q < 4)$. We assume that Ω is a principal form. Ω is the torsal form corresponding to the focal point $M \in P_3$ of order q if the dimension of the subspace $\Omega = 0$ of $T(V_3^4, M)$ is less than $7 - q$. We shall assume that there are two focal points of order 3 in each space P_3. We locate A_1, A_2 at these points. We locate A_3, A_4 at the focal points in general of order 1 of the space P_3.

The forms $\tilde{\omega}_1^k$ are linear combinations of one principal form only. We denote this form by ω_1. Similarly $\tilde{\omega}_2^k$ are linear combinations of one principal form. We denote this form by ω_2. We assume that ω_1, ω_2 are linearly independent and that they are torsal forms of A_3, A_4, too. The forms $\tilde{\omega}_3^k$ are linear combinations of three principal forms ω_1, ω_2, ω_3, the forms $\tilde{\omega}_4^k$ are linear combinations of three principal forms ω_1, ω_2, ω_4. We assume that ω_1, ω_2, ω_3, ω_4 are linearly independent. Hence we obtain

$$(1)\qquad \begin{aligned} \mathrm{d}A_1 &= \omega_1^k A_k + \omega_1 \varphi_1^{k1} \bar{A}_k \\ \mathrm{d}A_2 &= \omega_2^k A_k + \omega_2 \varphi_2^{k2} \bar{A}_k \\ \mathrm{d}A_3 &= \omega_3^k A_k + \omega_1 \varphi_3^{k1} \bar{A}_k + \omega_2 \varphi_3^{k2} \bar{A}_k + \omega_3 \varphi_3^{k3} \bar{A}_k \\ \mathrm{d}A_4 &= \omega_4^k A_k + \omega_1 \varphi_4^{k1} \bar{A}_k + \omega_2 \varphi_4^{k2} \bar{A}_k + \omega_4 \varphi_4^{k4} \bar{A}_k \end{aligned}$$

where φ_i^{kr} are functions of the parameters of the frame.

The characteristic of the space P_3 is the set of the focal points of V_3^4 in P_3. We find the equation of this characteristic. Let $M = x^i A_i$. Then

$$\begin{aligned} \mathrm{d}M = {} & \omega_1[x^1\varphi_1^{k1}\bar{A}_k + x^3\varphi_3^{k1}\bar{A}_k + x^4\varphi_4^{k1}\bar{A}_k] + \omega_2[x^2\varphi_2^{k2}\bar{A}_k + x^3\varphi_3^{k2}\bar{A}_k + x^4\varphi_4^{k2}\bar{A}_k] \\ & + \omega_3 x^3\varphi_3^{k3}\bar{A}_k + \omega_4 x^4 \varphi_4^{k4}\bar{A}_k + [\ldots] \end{aligned}$$

where $[\ldots]$ are linear combinations of the points A_1, A_2, A_3, A_4. M is a focal point of P_3 if

$$x^3x^4(x^1\varphi_1^{k1} + x^3\varphi_3^{k1} + x^4\varphi_4^{k1}, x^2\varphi_2^{k2} + x^3\varphi_3^{k2} + x^4\varphi_4^{k2}, \varphi_3^{k3}, \varphi_4^{k4}) = 0$$

holds. The rows of the determinant on the left hand side of the equation are determined by $k = 1, 2, 3, 4$. This equation implies

$$\begin{aligned} & x^3x^4\{x^1x^2(\varphi_1^{k1},\varphi_2^{k2},\varphi_3^{k3},\varphi_4^{k4}) + x^1x^3(\varphi_1^{k1},\varphi_3^{k2},\varphi_3^{k3},\varphi_4^{k4}) + x^1x^4(\varphi_1^{k1},\varphi_4^{k2},\varphi_3^{k3},\varphi_4^{k4}) \\ & + x^2x^3(\varphi_3^{k1},\varphi_2^{k2},\varphi_3^{k3},\varphi_4^{k4}) + (x^3)^2(\varphi_3^{k1},\varphi_3^{k2},\varphi_3^{k3},\varphi_4^{k4}) + x^3x^4\big[(\varphi_3^{k1},\varphi_4^{k2},\varphi_3^{k3},\varphi_4^{k4}) \\ & + (\varphi_4^{k1},\varphi_3^{k2},\varphi_3^{k3},\varphi_4^{k4})\big] + x^2x^4(\varphi_4^{k1},\varphi_2^{k2},\varphi_3^{k3},\varphi_4^{k4}) + (x^4)^2(\varphi_4^{k1},\varphi_4^{k2},\varphi_3^{k3},\varphi_4^{k4})\} = 0. \end{aligned}$$

Proposition 1. *If there exist two focal points A_1, A_2 of order three with different torsal forms ω_1, ω_2 and two focal points A_3, A_4 of order one with common torsal forms ω_1, ω_2 in P_3, then the characteristic of P_3 consists of the planes (A_1, A_2, A_3), (A_1, A_2, A_4) and a quadric. All points of the plane (A_1, A_2, A_3) have the torsal forms ω_1, ω_2, ω_3, all points of the plane (A_1, A_2, A_4) have the torsal forms ω_1, ω_2, ω_4.*

This follows from the preceding equation and (1).

We consider the section of the quadratic characteristic of P_3 with the plane (A_1, A_2, A_3). We obtain the equation

$$x^1x^2(\varphi_1^{k1}, \varphi_2^{k2}, \varphi_3^{k3}, \varphi_4^{k4}) + x^1x^3(\varphi_1^{k1}, \varphi_3^{k2}, \varphi_3^{k3}, \varphi_4^{k4}) + x^2x^3(\varphi_3^{k1}, \varphi_2^{k2}, \varphi_3^{k3}, \varphi_4^{k4}) \\ +(x^3)^2(\varphi_3^{k1}, \varphi_3^{k2}, \varphi_3^{k3}, \varphi_4^{k4}) = 0.$$

We shall assume that A_3 is the pole of the line (A_1, A_2) with respect to this conic. Hence

$$(\varphi_1^{k1}, \varphi_3^{k2}, \varphi_3^{k3}, \varphi_4^{k4}) = 0 \wedge (\varphi_3^{k1}, \varphi_2^{k2}, \varphi_3^{k3}, \varphi_4^{k4}) = 0.$$

The functions φ_3^{k2} are linear combinations of φ_3^{k3}, φ_4^{k4}, φ_1^{k1}, while φ_3^{k1} are linear combinations of φ_3^{k3}, φ_4^{k4}, φ_2^{k2}. Hence

$$\begin{aligned} \varphi_3^{k2} &= \kappa_3^{12}\varphi_1^{k1} + \kappa_3^{32}\varphi_3^{k3} + \kappa_3^{42}\varphi_4^{k4} \\ \varphi_3^{k1} &= \kappa_3^{21}\varphi_2^{k2} + \kappa_3^{31}\varphi_3^{k3} + \kappa_3^{41}\varphi_4^{k4}. \end{aligned} \tag{2}$$

We consider the section of the quadratic characteristic with the plane (A_1, A_2, A_4). We obtain the equation

$$x^1x^2(\varphi_1^{k1}, \varphi_2^{k2}, \varphi_3^{k3}, \varphi_4^{k4}) + x^1x^4(\varphi_1^{k1}, \varphi_4^{k2}, \varphi_3^{k3}, \varphi_4^{k4}) + x^2x^4(\varphi_4^{k1}, \varphi_2^{k2}, \varphi_3^{k3}, \varphi_4^{k4}) \\ +(x^4)^2(\varphi_4^{k1}, \varphi_4^{k2}, \varphi_3^{k3}, \varphi_4^{k4}) = 0$$

We shall assume that A_4 is the pole of the line (A_1, A_2) with respect to this conic. Hence

$$(\varphi_1^{k1}, \varphi_4^{k2}, \varphi_3^{k3}, \varphi_4^{k4}) = 0 \wedge (\varphi_4^{k1}, \varphi_2^{k2}, \varphi_3^{k3}, \varphi_4^{k4}) = 0.$$

The functions φ_4^{k2} are linear combinations of φ_3^{k3}, φ_4^{k4}, φ_1^{k1}, the functions φ_4^{k1} are linear combinations of φ_3^{k3}, φ_4^{k4}, φ_2^{k2}. Hence

$$\begin{aligned} \varphi_4^{k2} &= \kappa_4^{12}\varphi_1^{k1} + \kappa_4^{32}\varphi_3^{k3} + \kappa_4^{42}\varphi_4^{k4} \\ \varphi_4^{k1} &= \kappa_4^{21}\varphi_2^{k2} + \kappa_4^{31}\varphi_3^{k3} + \kappa_4^{41}\varphi_4^{k4}. \end{aligned} \tag{3}$$

We locate the point $\bar{A}_1$ in the space $T(V_3^4, A_1)$, the point $\bar{A}_2$ in the space $T(V_3^4, A_2)$, the point $\bar{A}_3$ in the subspace $\omega_1 = \omega_2 = 0$ of $T(V_3^4, A_3)$, the point $\bar{A}_4$ in the subspace $\omega_1 = \omega_2 = 0$ of $T(V_3^4, A_4)$. It is possible to use the specification

$$\begin{aligned} \varphi_1^{21} = \varphi_1^{31} = \varphi_1^{41} &= 0 \wedge \varphi_1^{11} = 1 \\ \varphi_2^{12} = \varphi_2^{32} = \varphi_2^{42} &= 0 \wedge \varphi_2^{22} = 1 \\ \varphi_3^{13} = \varphi_3^{23} = \varphi_3^{43} &= 0 \wedge \varphi_3^{33} = 1 \\ \varphi_4^{14} = \varphi_4^{24} = \varphi_4^{34} &= 0 \wedge \varphi_4^{44} = 1. \end{aligned} \tag{4}$$

Using the relations (1), (2), (3), (4) we obtain the following equations of the frame:

$$
\begin{aligned}
dA_1 &= \omega_1^k A_k + \omega_1 \bar{A}_1 \\
dA_2 &= \omega_2^k A_k + \omega_2 \bar{A}_2 \\
dA_3 &= \omega_3^k A_k + \kappa_3^{12}\omega_2 \bar{A}_1 + \kappa_3^{21}\omega_1 \bar{A}_2 \\
&\qquad + (\kappa_3^{31}\omega_1 + \kappa_3^{32}\omega_2 + \omega_3)\bar{A}_3 + (\kappa_3^{41}\omega_1 + \kappa_3^{42}\omega_2)\bar{A}_4 \\
dA_4 &= \omega_4^k A_k + \kappa_4^{12}\omega_2 \bar{A}_1 + \kappa_4^{21}\omega_1 \bar{A}_2 \\
&\qquad + (\kappa_4^{31}\omega_1 + \kappa_4^{32}\omega_2)\bar{A}_3 + (\kappa_4^{41}\omega_1 + \kappa_4^{42}\omega_2 + \omega_4)\bar{A}_4 \\
d\bar{A}_i &= \hat{\omega}_i^k A_k + \bar{\omega}_i^k \bar{A}_k .
\end{aligned}
\tag{5}
$$

We determine the exterior derivative of the coefficients at $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$, $\bar{A}_4$ from the first four equations of (5). From (5) and from the structure equations of P_7 we obtain

$$
\begin{aligned}
d\omega_1 &= (\omega_1^1 - \bar{\omega}_1^1)\wedge\omega_1 + (\kappa_3^{12}\omega_1^3 + \kappa_4^{12}\omega_1^4)\wedge\omega_2 \\
d\omega_2 &= (\kappa_3^{21}\omega_2^3 + \kappa_4^{21}\omega_2^4)\wedge\omega_1 + (\omega_2^2 - \bar{\omega}_2^2)\wedge\omega_2 \\
d\omega_3 &= \big[-d\kappa_3^{31} + \kappa_3^{31}(-\omega_1^1 + \omega_3^3 + \bar{\omega}_1^1 - \bar{\omega}_3^3) - \kappa_3^{32}\kappa_3^{21}\omega_2^3 - \kappa_3^{32}\kappa_4^{21}\omega_2^4 \\
&\qquad + \kappa_4^{31}\omega_3^4 - \kappa_3^{21}\bar{\omega}_2^3 - \kappa_3^{41}\bar{\omega}_4^3\big]\wedge\omega_1 \\
&\qquad + \big[-d\kappa_3^{32} + \kappa_3^{32}(-\omega_2^2 + \omega_3^3 + \bar{\omega}_2^2 - \bar{\omega}_3^3) - \kappa_3^{31}\kappa_3^{12}\omega_1^3 - \kappa_3^{31}\kappa_4^{12}\omega_1^4 \\
&\qquad + \kappa_4^{32}\omega_3^4 - \kappa_3^{12}\bar{\omega}_1^3 - \kappa_3^{42}\bar{\omega}_4^3\big]\wedge\omega_2 \\
&\qquad + [\omega_3^3 - \bar{\omega}_3^3]\wedge\omega_3 \\
d\omega_4 &= \big[-d\kappa_4^{41} + \kappa_4^{41}(-\omega_1^1 + \omega_4^4 + \bar{\omega}_1^1 - \bar{\omega}_4^4) - \kappa_4^{42}\kappa_3^{21}\omega_2^3 - \kappa_4^{42}\kappa_4^{21}\omega_2^4 \\
&\qquad + \kappa_3^{41}\omega_4^3 - \kappa_4^{21}\bar{\omega}_2^4 - \kappa_4^{31}\bar{\omega}_3^4\big]\wedge\omega_1 \\
&\qquad + \big[-d\kappa_4^{42} + \kappa_4^{42}(-\omega_2^2 + \omega_4^4 + \bar{\omega}_2^2 - \bar{\omega}_4^4) - \kappa_4^{41}\kappa_3^{12}\omega_1^3 - \kappa_4^{41}\kappa_4^{12}\omega_1^4 \\
&\qquad + \kappa_3^{42}\omega_4^3 - \kappa_4^{12}\bar{\omega}_1^4 - \kappa_4^{32}\bar{\omega}_3^4\big]\wedge\omega_2 \\
&\qquad + [\omega_4^4 - \bar{\omega}_4^4]\wedge\omega_4 .
\end{aligned}
\tag{6}
$$

Using these equations we conclude

(7)

$$
\begin{aligned}
&[\kappa_3^{21}\omega_1^3 + \kappa_4^{21}\omega_1^4 - \bar{\omega}_1^2]\wedge\omega_1 + \omega_1^2\wedge\omega_2 = 0 \\
&[\kappa_3^{31}\omega_1^3 + \kappa_4^{31}\omega_1^4 - \bar{\omega}_1^3]\wedge\omega_1 + [\kappa_3^{32}\omega_1^3 + \kappa_4^{32}\omega_1^4]\wedge\omega_2 + \omega_1^3\wedge\omega_3 = 0 \\
&[\kappa_3^{41}\omega_1^3 + \kappa_4^{41}\omega_1^4 - \bar{\omega}_1^4]\wedge\omega_1 + [\kappa_3^{42}\omega_1^3 + \kappa_4^{42}\omega_1^4]\wedge\omega_2 + \omega_1^4\wedge\omega_4 = 0 \\
&\omega_2^1\wedge\omega_1 + [\kappa_3^{12}\omega_2^3 + \kappa_4^{12}\omega_2^4 - \bar{\omega}_2^1]\wedge\omega_2 = 0 \\
&[\kappa_3^{31}\omega_2^3 + \kappa_4^{31}\omega_2^4]\wedge\omega_1 + [\kappa_3^{32}\omega_2^3 + \kappa_4^{32}\omega_2^4 - \bar{\omega}_2^3]\wedge\omega_2 + \omega_2^3\wedge\omega_3 = 0 \\
&[\kappa_3^{41}\omega_2^3 + \kappa_4^{41}\omega_2^4]\wedge\omega_1 + [\kappa_3^{42}\omega_2^3 + \kappa_4^{42}\omega_2^4 - \bar{\omega}_2^4]\wedge\omega_2 + \omega_2^4\wedge\omega_4 = 0 \\
&[\kappa_3^{12}(\kappa_3^{21}\omega_2^3 + \kappa_4^{21}\omega_2^4) - \omega_3^1 + \kappa_3^{21}\bar{\omega}_2^1 + \kappa_3^{31}\bar{\omega}_3^1 + \kappa_3^{41}\bar{\omega}_4^1]\wedge\omega_1 \\
&\quad + [d\kappa_3^{12} + \kappa_3^{12}(\omega_2^2 - \omega_3^3 - \bar{\omega}_2^2 + \bar{\omega}_1^1) - \kappa_4^{12}\omega_3^4 + \kappa_3^{32}\bar{\omega}_3^1 + \kappa_3^{42}\bar{\omega}_4^1]\wedge\omega_2 + \bar{\omega}_3^1\wedge\omega_3 = 0
\end{aligned}
$$

$$
\begin{aligned}
&[\mathrm{d}\kappa_3^{21} + \kappa_3^{21}(\omega_1^1 - \omega_3^3 - \varpi_1^1 + \varpi_2^2) - \kappa_4^{21}\omega_3^4 + \kappa_3^{31}\varpi_3^2 + \kappa_3^{41}\varpi_4^2] \wedge \omega_1 \\
&+ [\kappa_3^{21}(\kappa_3^{12}\omega_1^3 + \kappa_4^{12}\omega_1^4) - \omega_3^2 + \kappa_3^{12}\varpi_1^2 + \kappa_3^{32}\varpi_3^2 + \kappa_3^{42}\varpi_4^2] \wedge \omega_2 + \varpi_3^2 \wedge \omega_3 = 0 \\
&[\kappa_4^{12}(\kappa_3^{21}\omega_2^3 + \kappa_4^{21}\omega_2^4) - \omega_4^1 + \kappa_4^{21}\varpi_2^1 + \kappa_4^{31}\varpi_3^1 + \kappa_4^{41}\varpi_4^1] \wedge \omega_1 \\
&+ [\mathrm{d}\kappa_4^{12} + \kappa_4^{12}(\omega_2^2 - \omega_4^4 - \varpi_2^2 + \varpi_1^1) - \kappa_3^{12}\omega_4^3 + \kappa_4^{32}\varpi_3^1 + \kappa_4^{42}\varpi_4^1] \wedge \omega_2 + \varpi_4^1 \wedge \omega_4 = 0 \\
&[\mathrm{d}\kappa_4^{21} + \kappa_4^{21}(\omega_1^1 - \omega_4^4 - \varpi_1^1 + \varpi_2^2) - \kappa_3^{21}\omega_4^3 + \kappa_4^{31}\varpi_3^2 + \kappa_4^{41}\varpi_4^2] \wedge \omega_1 \\
&+ [\kappa_4^{21}(\kappa_3^{12}\omega_1^3 + \kappa_4^{12}\omega_1^4) - \omega_4^2 + \kappa_4^{12}\varpi_1^2 + \kappa_4^{32}\varpi_3^2 + \kappa_4^{42}\varpi_4^2] \wedge \omega_2 + \varpi_4^2 \wedge \omega_4 = 0 \\
&[\mathrm{d}\kappa_3^{41} + \kappa_3^{41}(\omega_1^1 - \omega_3^3 - \varpi_1^1 + \varpi_4^4) + \kappa_3^{42}(\kappa_3^{21}\omega_2^3 + \kappa_4^{21}\omega_2^4) - \kappa_4^{41}\omega_3^4 + \kappa_3^{21}\varpi_2^4 + \kappa_3^{31}\varpi_3^4] \wedge \omega_1 \\
&+ [\mathrm{d}\kappa_3^{42} + \kappa_3^{42}(\omega_2^2 - \omega_3^3 - \varpi_2^2 + \varpi_4^4) + \kappa_3^{41}(\kappa_3^{12}\omega_1^3 + \kappa_4^{12}\omega_1^4) - \kappa_4^{42}\omega_3^4 + \kappa_3^{12}\varpi_1^4 + \kappa_3^{32}\varpi_3^4] \wedge \omega_2 \\
&+ \varpi_3^4 \wedge \omega_3 - \omega_3^4 \wedge \omega_4 = 0 \\
&[\mathrm{d}\kappa_4^{31} + \kappa_4^{31}(\omega_1^1 - \omega_4^4 - \varpi_1^1 + \varpi_3^3) + \kappa_4^{32}(\kappa_3^{21}\omega_2^3 + \kappa_4^{21}\omega_2^4) - \kappa_3^{31}\omega_4^3 + \kappa_4^{21}\varpi_2^3 + \kappa_4^{41}\varpi_4^3] \wedge \omega_1 \\
&+ [\mathrm{d}\kappa_4^{32} + \kappa_4^{32}(\omega_2^2 - \omega_4^4 - \varpi_2^2 + \varpi_3^3) + \kappa_4^{31}(\kappa_3^{12}\omega_1^3 + \kappa_4^{12}\omega_1^4) - \kappa_3^{32}\omega_4^3 + \kappa_4^{12}\varpi_1^3 + \kappa_4^{42}\varpi_4^3] \wedge \omega_2 \\
&-\omega_4^3 \wedge \omega_3 + \varpi_4^3 \wedge \omega_4 = 0.
\end{aligned}
$$

The system (7) consists of 12 equations, these equations contain 32 forms ω_i^k, ϖ_i^k $(i \neq k)$,

$$
(8) \qquad
\begin{array}{ll}
\mathrm{d}\kappa_3^{12} + \kappa_3^{12}(\omega_2^2 - \omega_3^3 + \varpi_1^1 - \varpi_2^2) & \mathrm{d}\kappa_3^{41} + \kappa_3^{41}(\omega_1^1 - \omega_3^3 - \varpi_1^1 + \varpi_4^4) \\
\mathrm{d}\kappa_3^{21} + \kappa_3^{21}(\omega_1^1 - \omega_3^3 - \varpi_1^1 + \varpi_2^2) & \mathrm{d}\kappa_3^{42} + \kappa_3^{42}(\omega_2^2 - \omega_3^3 - \varpi_2^2 + \varpi_4^4) \\
\mathrm{d}\kappa_4^{12} + \kappa_4^{12}(\omega_2^2 - \omega_4^4 + \varpi_1^1 - \varpi_2^2) & \mathrm{d}\kappa_4^{31} + \kappa_4^{31}(\omega_1^1 - \omega_4^4 - \varpi_1^1 + \varpi_3^3) \\
\mathrm{d}\kappa_4^{21} + \kappa_4^{21}(\omega_1^1 - \omega_4^4 - \varpi_1^1 + \varpi_2^2) & \mathrm{d}\kappa_4^{32} + \kappa_4^{32}(\omega_2^2 - \omega_4^4 - \varpi_2^2 + \varpi_3^3).
\end{array}
$$

According to the Cartan lemma these forms are principal forms. It follows that the forms

$$
\frac{\kappa_3^{12}\kappa_4^{21}}{\kappa_3^{21}\kappa_4^{12}}\,\mathrm{d}\,\frac{\kappa_3^{21}\kappa_4^{12}}{\kappa_3^{12}\kappa_4^{21}}, \qquad \frac{\kappa_4^{31}\kappa_3^{42}}{\kappa_3^{41}\kappa_4^{32}}\,\mathrm{d}\,\frac{\kappa_3^{41}\kappa_4^{32}}{\kappa_4^{31}\kappa_3^{42}}
$$

are also principal forms. The expressions

$$
J_1 = \frac{\kappa_3^{21}\kappa_4^{12}}{\kappa_3^{12}\kappa_4^{21}}, \qquad J_2 = \frac{\kappa_3^{41}\kappa_4^{32}}{\kappa_4^{31}\kappa_3^{42}}
$$

are invariants.

For the quadratic characteristic of P_3 we obtain

$$
(9) \qquad x^1x^2 - (x^3)^2\kappa_3^{21}\kappa_3^{12} - x^3x^4(\kappa_3^{21}\kappa_4^{12} + \kappa_4^{21}\kappa_3^{12}) - (x^4)^2\kappa_4^{21}\kappa_4^{12} = 0.
$$

If $\kappa_3^{21}\kappa_4^{12} - \kappa_4^{21}\kappa_3^{12} = 0$ (or $J_1 = 1$) holds, then the quadric is singular. We shall eliminate this case.

Proposition 2. *The polar line of (A_1, A_2) with respect to the characteristic quadric of P_3 intersects this quadric at points M_1, M_2. The anharmonic ratio of A_3, A_4, M_2, M_1 is equal to J_1.*

Proof. For the intersections of the line (A_3, A_4) with the quadric (9) we obtain

$$\frac{(x^3)^2}{(x^4)^2}\kappa_3^{21}\kappa_3^{12} + \frac{x^3}{x^4}(\kappa_3^{21}\kappa_4^{12} + \kappa_4^{21}\kappa_3^{12}) + \kappa_4^{21}\kappa_4^{12} = 0.$$

It follows that

$$M_1 = \kappa_4^{12}A_3 - \kappa_3^{12}A_4, \qquad M_2 = -\kappa_3^{21}A_4 + \kappa_4^{21}A_3.$$

The anharmonic ratio of A_3, A_4, M_2, M_1 is equal to J_1. □

We shall find a condition that $M = x^iA_i$ be a focal point of V_3^4 of the second order. It is necessary to study the rank of the following matrix:

$$(10)\qquad \begin{pmatrix} x^1 & x^3\kappa_3^{21} + x^4\kappa_4^{21} & x^3\kappa_3^{31} + x^4\kappa_4^{31} & x^3\kappa_3^{41} + x^4\kappa_4^{41} \\ x^3\kappa_3^{12} + x^4\kappa_4^{12} & x^2 & x^3\kappa_3^{32} + x^4\kappa_4^{32} & x^3\kappa_3^{42} + x^4\kappa_4^{42} \\ 0 & 0 & x^3 & 0 \\ 0 & 0 & 0 & x^4 \end{pmatrix}.$$

If $M = x^iA_i$ is a focal point of order 2, then the rank of this matrix is equal to 2. This is satisfied for all points of the line (A_1, A_2) except the points A_1, A_2. These points are focal points of the third order. Let N_3 be a focal point of the second order in the plane $x^4 = 0$ that does not belong to (A_1, A_2). For this point the matrix

$$\begin{pmatrix} x^1 & x^3\kappa_3^{21} & \kappa_3^{41} \\ x^3\kappa_3^{12} & x^2 & \kappa_3^{42} \end{pmatrix}$$

has rank 1. We find the following conditions:

$$x^1x^2 - (x^3)^2\kappa_3^{12}\kappa_3^{21} = 0 \wedge x^1\kappa_3^{42} - x^3\kappa_3^{12}\kappa_3^{41} = 0 \wedge x^2\kappa_3^{41} - x^3\kappa_3^{21}\kappa_3^{42} = 0.$$

The first relation is the equation of the section of the quadratic characteristic with the plane $x^4 = 0$. From the other relations we find the coordinates of N_3: $N_3\big(\kappa_3^{12}(\kappa_3^{41})^2, \kappa_3^{21}(\kappa_3^{42})^2, \kappa_3^{41}\kappa_3^{42}, 0\big)$. For these coordinates the first relation is satisfied, too. Similarly in the plane $x^3 = 0$ there exists one focal point $N_4\big(\kappa_4^{12}(\kappa_4^{31})^2, \kappa_4^{21}(\kappa_4^{32})^2, 0, \kappa_4^{31}\kappa_4^{32}\big)$ of the second order. N_4 belongs to the section of the quadratic characteristic of P_3 with the plane $x^3 = 0$.

Proposition 3. *In the space P_3 there exist the following focal points of V_3^4 of the second order: all points of the line (A_1, A_2) (except A_1, A_2) and the points N_3, N_4.*

The points N_3, N_4 belong to the characteristic quadric of P_3. The tangent planes of the quadratic characteristic of P_3 at the points N_3, N_4 intersect the line (A_1, A_2) at the points R_1, R_2. For the anharmonic ratio $D(A_1, A_2, R_1, R_2)$ we have

$$D(A_1, A_2, R_1, R_2) = \frac{J_1}{J_2^2}.$$

P r o o f. If a focal point of the second order does not belong to the planes $x^3 = 0$, $x^4 = 0$ then

$$x^3\kappa_3^{12} + x^4\kappa_4^{12} = 0 \wedge x^3\kappa_3^{21} + x^4\kappa_4^{21} = 0 \wedge x^3x^4 \neq 0 \wedge x^1 = 0 \wedge x^2 = 0$$

holds. A non-trivial common solution of these relations exists in the case $\kappa_4^{12}\kappa_3^{21} - \kappa_3^{12}\kappa_4^{21} = 0$. The quadratic characteristic of P_3 is singular. This case has been eliminated. The tangent plane at the point N_3 of the characteristic quadric has the equation

$$x^1\kappa_3^{21}(\kappa_3^{42})^2 + x^2\kappa_3^{12}(\kappa_3^{41})^2 - 2x^3\kappa_3^{12}\kappa_3^{21}\kappa_3^{41}\kappa_3^{42} - x^4\kappa_3^{41}\kappa_3^{42}(\kappa_3^{21}\kappa_4^{12} + \kappa_4^{21}\kappa_3^{12}) = 0.$$

Similarly for the tangent plane of this surface at the point N_4 we have

$$x^1\kappa_4^{21}(\kappa_4^{32})^2 + x^2\kappa_4^{12}(\kappa_4^{31})^2 - x^3\kappa_4^{31}\kappa_4^{32}(\kappa_3^{21}\kappa_4^{12} + \kappa_4^{21}\kappa_3^{12}) - 2x^4\kappa_4^{12}\kappa_4^{21}\kappa_4^{31}\kappa_4^{32} = 0.$$

The points R_1, R_2 have the coordinates

$$R_1(-\kappa_3^{12}(\kappa_3^{41})^2, \kappa_3^{21}(\kappa_3^{42})^2, 0, 0), \qquad R_2(-\kappa_4^{12}(\kappa_4^{31})^2, \kappa_4^{21}(\kappa_4^{32})^2, 0, 0).$$

This implies

$$D(A_1, A_2, R_1, R_2) = \frac{\kappa_3^{21}(\kappa_3^{42})^2\kappa_4^{12}(\kappa_4^{31})^2}{\kappa_3^{12}(\kappa_3^{41})^2\kappa_4^{21}(\kappa_4^{32})^2} = \frac{J_1}{J_2^2}.$$

□

Proposition 4. *The frame associated with V_3^4 by* (5) *depends on two functions of four parameters.*

P r o o f. The system (7) is not involutive. It is necessary to find the prolongation of (7). Using the Cartan lemma we calculate the left members of the exterior products in the equations (7) as linear combinations of the forms ω_1, ω_2, ω_3, ω_4. We denote the coefficients in these combinations by α_{ij}^q; $\alpha_{ij}^q = \alpha_{ji}^q$, $q = 1, 2, \ldots, 12$. q determines the serial number of the equation of the system (7). We obtain a system of equations.

From this system it is possible to calculate the forms ω_i^k, $\bar{\omega}_i^k$, $i \neq k$ and the forms (8). We obtain

(11)

$$
\begin{aligned}
&\omega_1^3 = \alpha^2_{31}\omega_1 && \omega_1^2 = \alpha^1_{21}\omega_1 + \alpha^1_{22}\omega_2\\
&\omega_1^4 = \alpha^3_{41}\omega_1 && \omega_2^1 = \alpha^4_{11}\omega_1 + \alpha^4_{12}\omega_2\\
&\omega_2^3 = \alpha^5_{32}\omega_2 && \bar{\omega}_1^2 = \omega_1[\kappa_3^{21}\alpha^2_{31} - \alpha^1_{11} + \kappa_4^{21}\alpha^3_{41}] - \alpha^1_{12}\omega_2\\
&\omega_2^4 = \alpha^6_{42}\omega_2 && \bar{\omega}_2^1 = -\alpha^4_{21}\omega_1 + \omega_2[-\alpha^4_{22} + \kappa_3^{12}\alpha^5_{32} + \kappa_4^{12}\alpha^6_{42}]
\end{aligned}
$$

$$
\begin{aligned}
&\bar{\omega}_1^3 = \omega_1[\alpha^2_{31}\kappa_3^{31} - \alpha^2_{11} + \kappa_4^{31}\alpha^3_{41}] - [\kappa_3^{32}\alpha^2_{31} + \kappa_4^{32}\alpha^3_{41}]\omega_2 - \alpha^2_{13}\omega_3\\
&\bar{\omega}_1^4 = \omega_1[\kappa_3^{41}\alpha^2_{31} - \alpha^3_{11} + \kappa_4^{41}\alpha^3_{41}] - [\kappa_3^{42}\alpha^2_{31} + \alpha^3_{41}\kappa_4^{42}]\omega_2 - \alpha^3_{14}\omega_4\\
&\bar{\omega}_2^3 = \omega_1[-\kappa_3^{31}\alpha^5_{32} - \kappa_4^{31}\alpha^6_{42}] + \omega_2[-\alpha^5_{22} + \kappa_3^{32}\alpha^5_{32} + \kappa_4^{32}\alpha^6_{42}] - \alpha^5_{23}\omega_3\\
&\bar{\omega}_2^4 = \omega_1[-\kappa_3^{41}\alpha^5_{32} - \kappa_4^{41}\alpha^6_{42}] + \omega_2[-\alpha^6_{22} + \kappa_3^{42}\alpha^5_{32} + \kappa_4^{42}\alpha^6_{42}] - \alpha^6_{24}\omega_4
\end{aligned}
$$

$$
\begin{aligned}
&\bar{\omega}_3^1 = \alpha^7_{31}\omega_1 + \alpha^7_{32}\omega_2 + \alpha^7_{33}\omega_3 && \bar{\omega}_4^1 = \alpha^9_{41}\omega_1 + \alpha^9_{42}\omega_2 + \alpha^9_{44}\omega_4\\
&\bar{\omega}_3^2 = \alpha^8_{31}\omega_1 + \alpha^8_{32}\omega_2 + \alpha^8_{33}\omega_3 && \bar{\omega}_4^2 = \alpha^{10}_{41}\omega_1 + \alpha^{10}_{42}\omega_2 + \alpha^{10}_{44}\omega_4
\end{aligned}
$$

$$
\begin{aligned}
&\omega_3^4 = -\alpha^{11}_{41}\omega_1 - \alpha^{11}_{42}\omega_2 - \alpha^{11}_{43}\omega_3 - \alpha^{11}_{44}\omega_4\\
&\omega_4^3 = -\alpha^{12}_{31}\omega_1 - \alpha^{12}_{32}\omega_2 - \alpha^{12}_{33}\omega_3 - \alpha^{12}_{34}\omega_4\\
&\bar{\omega}_3^4 = \alpha^{11}_{31}\omega_1 + \alpha^{11}_{32}\omega_2 + \alpha^{11}_{33}\omega_3 + \alpha^{11}_{34}\omega_4\\
&\bar{\omega}_4^3 = \alpha^{12}_{41}\omega_1 + \alpha^{12}_{42}\omega_2 + \alpha^{12}_{43}\omega_3 + \alpha^{12}_{44}\omega_4\\
&\omega_3^1 = \omega_1[-\kappa_3^{21}\alpha^4_{21} + \kappa_3^{31}\alpha^7_{31} + \kappa_3^{41}\alpha^9_{41} - \alpha^7_{11}] + \omega_2[\kappa_3^{12}(\kappa_3^{21}\alpha^5_{32} + \kappa_4^{21}\alpha^6_{42})\\
&\quad + \kappa_3^{21}(-\alpha^4_{22} + \kappa_3^{12}\alpha^5_{32} + \kappa_4^{12}\alpha^6_{42}) + \kappa_3^{31}\alpha^7_{23} + \kappa_3^{41}\alpha^9_{42} - \alpha^7_{12}]\\
&\quad + \omega_3[\kappa_3^{31}\alpha^7_{33} - \alpha^7_{13}] + \omega_4[\kappa_3^{41}\alpha^9_{44}]\\
&\omega_3^2 = \omega_1\big[\kappa_3^{21}(\kappa_3^{12}\alpha^2_{31} + \kappa_4^{12}\alpha^3_{41})\\
&\quad + \kappa_3^{12}(\kappa_3^{21}\alpha^2_{31} - \alpha^1_{11} + \kappa_4^{21}\alpha^3_{41}) + \kappa_3^{32}\alpha^8_{31} + \kappa_3^{42}\alpha^{10}_{41} - \alpha^8_{21}]\\
&\quad + \omega_2[-\kappa_3^{12}\alpha^1_{12} + \kappa_3^{32}\alpha^8_{32} + \kappa_3^{42}\alpha^{10}_{42} - \alpha^8_{22}] + \omega_3[\kappa_3^{32}\alpha^8_{33} - \alpha^8_{23}] + \omega_4[\kappa_3^{42}\alpha^{10}_{44}]\\
&\omega_4^1 = \omega_1[-\kappa_4^{21}\alpha^4_{21} + \kappa_4^{31}\alpha^7_{31} + \kappa_4^{41}\alpha^9_{41} - \alpha^9_{11}] + \omega_2[\kappa_4^{12}(\kappa_3^{21}\alpha^5_{32} + \kappa_4^{21}\alpha^6_{42})\\
&\quad + \kappa_4^{21}(-\alpha^4_{22} + \kappa_3^{12}\alpha^5_{32} + \kappa_4^{12}\alpha^6_{42}) + \kappa_4^{31}\alpha^7_{32} + \kappa_4^{41}\alpha^9_{42} - \alpha^9_{12}]\\
&\quad + \omega_3[\kappa_4^{31}\alpha^7_{33}] + \omega_4[\kappa_4^{41}\alpha^9_{44} - \alpha^9_{14}]\\
&\omega_4^2 = \omega_1\big[\kappa_4^{21}(\kappa_3^{12}\alpha^2_{31} + \kappa_4^{12}\alpha^3_{41}) + \kappa_4^{12}(\kappa_3^{21}\alpha^2_{31} - \alpha^1_{11} + \kappa_4^{21}\alpha^3_{41})\\
&\quad + \kappa_4^{32}\alpha^8_{31} + \kappa_4^{42}\alpha^{10}_{41} - \alpha^{10}_{21}]\\
&\quad + \omega_2[-\kappa_4^{12}\alpha^1_{12} + \kappa_4^{32}\alpha^8_{32} + \kappa_4^{42}\alpha^{10}_{42} - \alpha^{10}_{22}] + \omega_3[\kappa_4^{32}\alpha^8_{33}] + \omega_4[\kappa_4^{42}\alpha^{10}_{44} - \alpha^{10}_{24}]\\
&d\kappa_3^{12} + \kappa_3^{12}(\omega_2^2 - \omega_3^3 - \bar{\omega}_2^2 + \bar{\omega}_1^1) = \omega_1[-\kappa_4^{12}\alpha^{11}_{44} - \kappa_3^{32}\alpha^7_{31} - \kappa_3^{42}\alpha^9_{41} + \alpha^7_{21}]\\
&\quad + \omega_2[-\kappa_4^{12}\alpha^{11}_{42} - \kappa_3^{32}\alpha^7_{32} - \kappa_3^{42}\alpha^9_{42} + \alpha^7_{22}] + \omega_3[-\kappa_4^{12}\alpha^{11}_{43} - \kappa_3^{32}\alpha^7_{33} + \alpha^7_{23}]\\
&\quad + \omega_4[-\kappa_4^{12}\alpha^{11}_{44} - \kappa_3^{42}\alpha^9_{44}]\\
&d\kappa_3^{21} + \kappa_3^{21}(\omega_1^1 - \omega_3^3 - \bar{\omega}_1^1 + \bar{\omega}_2^2) = \omega_1[-\kappa_4^{21}\alpha^{11}_{41} - \kappa_3^{31}\alpha^8_{31} - \kappa_3^{41}\alpha^{10}_{41} + \alpha^8_{11}]\\
&\quad + \omega_2[-\kappa_4^{21}\alpha^{11}_{42} - \kappa_3^{31}\alpha^8_{32} - \kappa_3^{41}\alpha^{10}_{42} + \alpha^8_{12}] + \omega_3[-\kappa_4^{21}\alpha^{11}_{43} - \kappa_3^{31}\alpha^8_{33} + \alpha^8_{13}]\\
&\quad + \omega_4[-\kappa_4^{21}\alpha^{11}_{44} - \kappa_3^{41}\alpha^{10}_{44}]
\end{aligned}
$$

$$
\begin{aligned}
&d\kappa_4^{12} + \kappa_4^{12}(\omega_2^2 - \omega_4^4 - \varpi_2^2 + \varpi_1^1) = \omega_1[-\kappa_3^{12}\alpha_{31}^{12} - \kappa_4^{32}\alpha_{31}^{7} - \kappa_4^{42}\alpha_{41}^{9} + \alpha_{21}^{9}] \\
&\quad + \omega_2[-\kappa_3^{12}\alpha_{32}^{12} - \kappa_4^{32}\alpha_{32}^{7} - \kappa_4^{42}\alpha_{42}^{9} + \alpha_{22}^{9}] + \omega_3[-\kappa_3^{12}\alpha_{33}^{12} - \kappa_4^{32}\alpha_{33}^{7}] \\
&\quad + \omega_4[-\kappa_3^{12}\alpha_{34}^{12} - \kappa_4^{42}\alpha_{44}^{9} + \alpha_{24}^{9}] \\
&d\kappa_4^{21} + \kappa_4^{21}(\omega_1^1 - \omega_4^4 - \varpi_1^1 + \varpi_2^2) = \omega_1[-\kappa_3^{21}\alpha_{31}^{12} - \kappa_4^{31}\alpha_{31}^{8} - \kappa_4^{41}\alpha_{41}^{10} + \alpha_{11}^{10}] \\
&\quad + \omega_2[-\kappa_3^{21}\alpha_{32}^{12} - \kappa_4^{31}\alpha_{32}^{8} - \kappa_4^{41}\alpha_{42}^{10} + \alpha_{12}^{10}] + \omega_3[-\kappa_3^{21}\alpha_{33}^{12} - \kappa_4^{31}\alpha_{33}^{8}] \\
&\quad + \omega_4[-\kappa_3^{21}\alpha_{34}^{12} - \kappa_4^{41}\alpha_{44}^{10} + \alpha_{14}^{10}] \\
&d\kappa_3^{41} + \kappa_3^{41}(\omega_1^1 - \omega_3^3 - \varpi_1^1 + \varpi_4^4) = \omega_1[- \kappa_4^{41}\alpha_{41}^{11} \\
&\quad - \kappa_3^{21}(-\kappa_3^{41}\alpha_{32}^{5} - \kappa_4^{41}\alpha_{42}^{6}) - \kappa_3^{31}\alpha_{31}^{11} + \alpha_{11}^{11}] \\
&\quad + \omega_2[- \kappa_3^{42}(\kappa_3^{21}\alpha_{32}^{5} + \kappa_4^{21}\alpha_{42}^{6}) - \kappa_4^{41}\alpha_{42}^{11} \\
&\quad - \kappa_3^{21}(-\alpha_{22}^{6} + \kappa_3^{42}\alpha_{32}^{5} + \kappa_4^{42}\alpha_{42}^{6}) - \kappa_3^{31}\alpha_{32}^{11} + \alpha_{12}^{11}] \\
&\quad + \omega_3[-\kappa_4^{41}\alpha_{43}^{11} - \kappa_3^{31}\alpha_{33}^{11} + \alpha_{13}^{11}] + \omega_4[-\kappa_4^{41}\alpha_{44}^{11} + \kappa_3^{21}\alpha_{24}^{6} - \kappa_3^{31}\alpha_{34}^{11} + \alpha_{14}^{11}] \\
&d\kappa_3^{42} + \kappa_3^{42}(\omega_2^2 - \omega_3^3 - \varpi_2^2 + \varpi_4^4) = \omega_1[- \kappa_3^{41}(\kappa_3^{12}\alpha_{31}^{2} + \kappa_4^{12}\alpha_{41}^{3}) \\
&\quad - \kappa_4^{42}\alpha_{41}^{11} + \kappa_3^{12}(\kappa_3^{41}\alpha_{31}^{2} - \alpha_{11}^{3} + \kappa_4^{41}\alpha_{41}^{3}) - \kappa_3^{32}\alpha_{31}^{11} + \alpha_{21}^{11}] \\
&\quad + \omega_2[- \kappa_4^{42}\alpha_{42}^{11} - \kappa_3^{12}(-\kappa_3^{42}\alpha_{31}^{2} - \alpha_{41}^{3}\kappa_4^{42}) - \kappa_3^{32}\alpha_{32}^{11} + \alpha_{22}^{11}] \\
&\quad + \omega_3[-\kappa_4^{42}\alpha_{43}^{11} - \kappa_3^{32}\alpha_{33}^{11} + \alpha_{23}^{11}] + \omega_4[-\kappa_4^{42}\alpha_{44}^{11} + \kappa_3^{12}\alpha_{14}^{3} + \kappa_3^{32}\alpha_{34}^{11} + \alpha_{24}^{11}] \\
&d\kappa_4^{31} + \kappa_4^{31}(\omega_1^1 - \omega_4^4 - \varpi_1^1 + \varpi_3^3) = \omega_1[- \kappa_3^{31}\alpha_{31}^{12} \\
&\quad - \kappa_4^{21}(-\kappa_3^{31}\alpha_{32}^{5} - \kappa_4^{31}\alpha_{42}^{6}) - \kappa_4^{41}\alpha_{41}^{12} + \alpha_{11}^{12}] \\
&\quad + \omega_2[- \kappa_4^{32}(\kappa_3^{21}\alpha_{32}^{5} + \kappa_4^{21}\alpha_{42}^{6}) - \kappa_3^{31}\alpha_{32}^{12} \\
&\quad - \kappa_4^{21}(-\alpha_{22}^{5} + \kappa_3^{32}\alpha_{32}^{5} + \kappa_4^{32}\alpha_{42}^{6}) - \kappa_4^{41}\alpha_{42}^{12} + \alpha_{12}^{12}] \\
&\quad + \omega_3[-\kappa_3^{31}\alpha_{33}^{12} + \kappa_4^{21}\alpha_{23}^{5} - \kappa_4^{41}\alpha_{43}^{12} + \alpha_{13}^{12}] + \omega_4[-\kappa_3^{31}\alpha_{34}^{12} - \kappa_4^{41}\alpha_{44}^{12} + \alpha_{14}^{12}] \\
&d\kappa_4^{32} + \kappa_4^{32}(\omega_2^2 - \omega_4^4 - \varpi_2^2 + \varpi_3^3) = \omega_1[- \kappa_4^{31}(\kappa_3^{12}\alpha_{31}^{2} + \kappa_4^{12}\alpha_{41}^{3}) \\
&\quad - \kappa_3^{32}\alpha_{31}^{12} - \kappa_4^{12}(\alpha_{31}^{2}\kappa_3^{31} - \alpha_{11}^{2} + \kappa_4^{31}\alpha_{41}^{3}) - \kappa_4^{42}\alpha_{41}^{12} + \alpha_{21}^{12}] \\
&\quad + \omega_2[- \kappa_3^{32}\alpha_{32}^{12} - \kappa_4^{12}(-\kappa_3^{32}\alpha_{31}^{2} - \kappa_4^{32}\alpha_{41}^{3}) - \kappa_4^{42}\alpha_{42}^{12} + \alpha_{22}^{12}] \\
&\quad + \omega_3[-\kappa_3^{32}\alpha_{33}^{12} + \kappa_4^{12}\alpha_{13}^{2} - \kappa_4^{42}\alpha_{43}^{12} + \alpha_{23}^{12}] \\
&\quad + \omega_4[-\kappa_3^{32}\alpha_{34}^{12} - \kappa_4^{42}\alpha_{44}^{12} + \alpha_{24}^{12}].
\end{aligned}
$$

Using (11) we obtain from (6)

(12)

$$
\begin{aligned}
&d\omega_1 = (\omega_1^1 - \varpi_1^1) \wedge \omega_1 + (\alpha_{31}^{2}\kappa_3^{12} + \alpha_{41}^{3}\kappa_4^{12})[\omega_1 \wedge \omega_2] \\
&d\omega_2 = (\omega_2^2 - \varpi_2^2) \wedge \omega_2 - (\alpha_{32}^{5}\kappa_3^{21} + \alpha_{42}^{6}\kappa_4^{21})[\omega_1 \wedge \omega_2] \\
&d\omega_3 = [- d\kappa_3^{31} + \kappa_3^{31}(-\omega_1^1 + \varpi_1^1 + \omega_3^3 - \varpi_3^3)] \wedge \omega_1 \\
&\quad + [- d\kappa_3^{32} + \kappa_3^{32}(-\omega_2^2 + \varpi_2^2 + \omega_3^3 - \varpi_3^3)] \wedge \omega_2 \\
&\quad + [\omega_3^3 - \varpi_3^3] \wedge \omega_3 + [\omega_1 \wedge \omega_2][- \kappa_3^{31}(\alpha_{31}^{2}\kappa_3^{12} + \alpha_{41}^{3}\kappa_4^{12}) + \kappa_3^{32}(\alpha_{32}^{5}\kappa_3^{21} + \alpha_{42}^{6}\kappa_4^{21}) \\
&\quad - \alpha_{41}^{11}\kappa_4^{32} + \alpha_{42}^{11}\kappa_4^{31} - \kappa_3^{12}(\alpha_{31}^{2}\kappa_3^{31} - \alpha_{11}^{2} + \kappa_4^{31}\alpha_{41}^{3}) \\
&\quad + \kappa_3^{21}(-\alpha_{22}^{5} + \kappa_3^{32}\alpha_{32}^{5} + \kappa_4^{32}\alpha_{42}^{6}) + \kappa_3^{41}\alpha_{42}^{12} - \kappa_3^{42}\alpha_{41}^{12}]
\end{aligned}
$$

$$
\begin{aligned}
&+ [\omega_1 \wedge \omega_3][\alpha_{43}^{11}\kappa_4^{31} - \kappa_3^{21}\alpha_{23}^{5} + \kappa_3^{41}\alpha_{43}^{12}] + [\omega_1 \wedge \omega_4][\alpha_{44}^{11}\kappa_4^{31} + \kappa_3^{41}\alpha_{44}^{12}]\\
&+ [\omega_2 \wedge \omega_3][\alpha_{43}^{11}\kappa_4^{32} - \kappa_3^{12}\alpha_{13}^{2} + \kappa_3^{42}\alpha_{43}^{12}] + [\omega_2 \wedge \omega_4][\alpha_{44}^{11}\kappa_4^{32} + \kappa_3^{42}\alpha_{44}^{12}]\\
&\mathrm{d}\omega_4 = \left[-\mathrm{d}\kappa_4^{41} + \kappa_4^{41}(-\omega_1^1 + \bar{\omega}_1^1 + \omega_4^4 - \bar{\omega}_4^4)\right] \wedge \omega_1\\
&+ \left[-\mathrm{d}\kappa_4^{42} + \kappa_4^{42}(-\omega_2^2 + \bar{\omega}_2^2 + \omega_4^4 - \bar{\omega}_4^4)\right] \wedge \omega_2 + [\omega_4^4 - \bar{\omega}_4^4] \wedge \omega_4\\
&+ [\omega_1 \wedge \omega_2]\left[-\kappa_4^{41}(\alpha_{31}^{2}\kappa_3^{12} + \alpha_{41}^{3}\kappa_4^{12}) + \kappa_4^{42}(\alpha_{32}^{5}\kappa_3^{21} + \alpha_{42}^{6}\kappa_4^{21})\right.\\
&+ \alpha_{32}^{12}\kappa_3^{41} - \alpha_{31}^{12}\kappa_3^{42} - \kappa_4^{12}(\kappa_3^{41}\alpha_{31}^{2} - \alpha_{11}^{3} + \kappa_4^{41}\alpha_{41}^{3})\\
&\left.+ \kappa_4^{21}(-\alpha_{22}^{6} + \kappa_3^{42}\alpha_{32}^{5} + \kappa_4^{42}\alpha_{42}^{6}) + \kappa_4^{31}\alpha_{32}^{11} - \kappa_4^{32}\alpha_{31}^{11}\right]\\
&+ [\omega_1 \wedge \omega_3][\alpha_{33}^{12}\kappa_3^{41} + \kappa_4^{31}\alpha_{33}^{11}] + [\omega_1 \wedge \omega_4][\alpha_{34}^{12}\kappa_3^{41} - \alpha_{24}^{6}\kappa_4^{21} + \kappa_4^{31}\alpha_{34}^{11}]\\
&+ [\omega_2 \wedge \omega_3][\alpha_{33}^{12}\kappa_3^{42} + \kappa_4^{32}\alpha_{33}^{11}] + [\omega_2 \wedge \omega_4][\alpha_{34}^{12}\kappa_3^{42} - \kappa_4^{12}\alpha_{14}^{3} + \kappa_4^{32}\alpha_{34}^{11}].
\end{aligned}
$$

Using the structure equations of the projective space P_7, the equations of the frame (5) and the relations (12) we form the exterior derivatives of (11).

Differentiating the expressions for ω_i^k, $i \neq k$, we obtain

(13a)
$$\sum_{s=1}^{4}[\Gamma_{ik}^{s} \wedge \omega_s] + \sum_{p,q=1,\,p\neq q}^{4} H_i^{k(p,q)}[\omega_p \wedge \omega_q] = 0,$$

differentiating the expressions for $\bar{\omega}_i^k$, $i \neq k$ we obtain

(13b)
$$\sum_{s=1}^{4}[\bar{\Gamma}_{ik}^{s} \wedge \omega_s] + \sum_{p,q=1,\,p\neq q}^{4} \bar{H}_i^{k(p,q)}[\omega_p \wedge \omega_q] = 0.$$

Differentiating the expression for $\mathrm{d}\kappa_i^{kr}$ we obtain

(13c)
$$\sum_{s=1}^{4}[\tilde{\Gamma}_i^{kr,s} \wedge \omega_s] + \sum_{p,q=1,\,p\neq q}^{4} K_i^{kr(p,q)}[\omega_p, \omega_q] = 0.$$

Γ_{ik}^{s}, $\bar{\Gamma}_{ik}^{s}$, $\tilde{\Gamma}_i^{kr,s}$ are forms independent of $\omega_1, \omega_2, \omega_3, \omega_4$, while $H_i^{k(p,q)}$, $\bar{H}_i^{k(p,q)}$, $K_i^{kr(p,q)}$ are functions dependent on $\alpha_{m,n}^{q}$, κ_{mn}^{j}. It is possible to write the system (13a), (13b), (13c) in the form

(14)
$$
\begin{aligned}
&\Omega_1^3 \wedge \omega_1 = 0 \qquad \Omega_1^4 \wedge \omega_1 = 0 \qquad \Omega_2^3 \wedge \omega_2 = 0 \qquad \Omega_2^4 \wedge \omega_2 = 0\\
&\Omega_1^{2,1} \wedge \omega_1 + \Omega_1^{2,2} \wedge \omega_2 = 0 \qquad \Omega_2^{1,1} \wedge \omega_1 + \Omega_2^{1,2} \wedge \omega_2 = 0\\
&\bar{\Omega}_1^{2,1} \wedge \omega_1 - \Omega_1^{2,2} \wedge \omega_2 = 0 \qquad -\Omega_2^{12} \wedge \omega_1 + \bar{\Omega}_2^{1,2} \wedge \omega_2 = 0\\
&\bar{\Omega}_1^{3,1} \wedge \omega_1 + (-\kappa_3^{32}\Omega_1^3 - \kappa_4^{32}\Omega_1^4) \wedge \omega_2 - \Omega_1^3 \wedge \omega_3 = 0\\
&(-\kappa_3^{31}\Omega_2^3 - \kappa_4^{31}\Omega_2^4) \wedge \omega_1 + \bar{\Omega}_2^{3,2} \wedge \omega_2 - \Omega_2^3 \wedge \omega_3 = 0\\
&\bar{\Omega}_1^{4,1} \wedge \omega_1 + (-\kappa_3^{42}\Omega_1^3 - \kappa_4^{42}\Omega_1^4) \wedge \omega_2 - \Omega_1^4 \wedge \omega_4 = 0
\end{aligned}
$$

$$(-\kappa_3^{41}\Omega_2^3 - \kappa_4^{41}\Omega_2^4) \wedge \omega_1 + \bar{\Omega}_2^{4,2} \wedge \omega_2 - \Omega_2^4 \wedge \omega_4 = 0$$

$$\bar{\Omega}_3^{1,1} \wedge \omega_1 + \bar{\Omega}_3^{1,2} \wedge \omega_2 + \bar{\Omega}_3^{1,3} \wedge \omega_3 = 0$$

$$\bar{\Omega}_3^{2,1} \wedge \omega_1 + \bar{\Omega}_3^{2,2} \wedge \omega_2 + \bar{\Omega}_3^{2,3} \wedge \omega_3 = 0$$

$$\bar{\Omega}_4^{1,1} \wedge \omega_1 + \bar{\Omega}_4^{1,2} \wedge \omega_2 + \bar{\Omega}_4^{1,4} \wedge \omega_4 = 0$$

$$\bar{\Omega}_4^{2,1} \wedge \omega_1 + \bar{\Omega}_4^{2,2} \wedge \omega_2 + \bar{\Omega}_4^{2,4} \wedge \omega_4 = 0$$

$$\Omega_3^{4,1} \wedge \omega_1 + \Omega_3^{4,2} \wedge \omega_2 + \Omega_3^{4,3} \wedge \omega_3 + \Omega_3^{4,4} \wedge \omega_4 = 0$$

$$\Omega_4^{3,1} \wedge \omega_1 + \Omega_4^{3,2} \wedge \omega_2 + \Omega_4^{3,3} \wedge \omega_3 + \Omega_4^{3,4} \wedge \omega_4 = 0$$

$$\bar{\Omega}_3^{4,1} \wedge \omega_1 + \bar{\Omega}_3^{4,2} \wedge \omega_2 + \bar{\Omega}_3^{4,3} \wedge \omega_3 - \Omega_3^{4,3} \wedge \omega_4 = 0$$

$$\bar{\Omega}_4^{3,1} \wedge \omega_1 + \bar{\Omega}_4^{3,2} \wedge \omega_2 - \Omega_4^{3,4} \wedge \omega_3 + \bar{\Omega}_4^{3,4} \wedge \omega_4 = 0$$

$$\Omega_3^{1,1} \wedge \omega_1 + (\Omega_3^{1,2} + \kappa_3^{12}\kappa_4^{21}\Omega_2^4 + \kappa_3^{12}\kappa_3^{21}\Omega_2^3 + \kappa_3^{21}\bar{\Omega}_2^{1,2} + \kappa_3^{31}\bar{\Omega}_3^{1,2} + \kappa_3^{41}\bar{\Omega}_4^{1,2}) \wedge \omega_2$$
$$+ (\kappa_3^{31}\bar{\Omega}_3^{1,3} - \bar{\Omega}_3^{1,1}) \wedge \omega_3 + \kappa_3^{41}\bar{\Omega}_4^{1,4} \wedge \omega_4 = 0$$

$$(-\Omega_3^{1,2} + \kappa_4^{12}\Omega_3^{4,1} - \kappa_3^{32}\bar{\Omega}_3^{1,1} - \kappa_3^{42}\bar{\Omega}_4^{1,1}) \wedge \omega_1 + \Omega_3^{12,2} \wedge \omega_2$$
$$+ (\kappa_4^{12}\Omega_3^{4,3} - \kappa_3^{32}\bar{\Omega}_3^{1,3} + \bar{\Omega}_3^{1,2}) \wedge \omega_3 + (\kappa_4^{12}\Omega_3^{4,4} - \kappa_3^{42}\bar{\Omega}_4^{1,4}) \wedge \omega_4 = 0$$

$$(\Omega_3^{2,1} + \kappa_3^{12}\bar{\Omega}_1^{2,1} + \kappa_3^{21}\kappa_3^{12}\Omega_1^3 + \kappa_3^{21}\kappa_4^{12}\Omega_1^4 + \kappa_3^{32}\bar{\Omega}_3^{2,1} + \kappa_3^{42}\bar{\Omega}_4^{2,1}) \wedge \omega_1 +$$
$$+ \Omega_3^{2,2} \wedge \omega_2 + (\kappa_3^{32}\bar{\Omega}_3^{2,3} - \bar{\Omega}_3^{2,2}) \wedge \omega_3 + \kappa_3^{42}\bar{\Omega}_4^{2,4} \wedge \omega_4 = 0$$

$$\Omega_3^{21,1} \wedge \omega_1 + (-\Omega_3^{2,1} - \kappa_3^{31}\bar{\Omega}_3^{2,2} - \kappa_3^{41}\bar{\Omega}_4^{2,2} + \kappa_4^{21}\Omega_3^{4,2}) \wedge \omega_2$$
$$+ (\kappa_4^{21}\Omega_3^{4,3} + \bar{\Omega}_3^{2,1} - \kappa_3^{31}\bar{\Omega}_3^{2,3}) \wedge \omega_3 + (\kappa_4^{21}\Omega_3^{4,4} - \kappa_3^{41}\bar{\Omega}_4^{2,4}) \wedge \omega_4 = 0$$

$$\Omega_4^{1,1} \wedge \omega_1 + (\Omega_4^{1,2} + \kappa_4^{21}\bar{\Omega}_2^{1,2} + \kappa_3^{21}\kappa_4^{12}\Omega_2^3 + \kappa_4^{21}\kappa_4^{12}\Omega_2^4 + \kappa_4^{31}\bar{\Omega}_3^{1,2} + \kappa_4^{41}\bar{\Omega}_4^{1,2}) \wedge \omega_2$$
$$+ \kappa_4^{31}\bar{\Omega}_3^{1,3} \wedge \omega_3 + (\kappa_4^{41}\bar{\Omega}_4^{1,4} - \bar{\Omega}_4^{1,1}) \wedge \omega_4 = 0$$

$$(-\Omega_4^{1,2} - \kappa_4^{32}\bar{\Omega}_3^{1,1} - \kappa_4^{42}\bar{\Omega}_4^{1,1} + \kappa_3^{12}\Omega_4^{3,1}) \wedge \omega_1 + \Omega_4^{12,2} \wedge \omega_2$$
$$+ (\kappa_3^{12}\Omega_4^{3,3} - \kappa_4^{32}\bar{\Omega}_3^{1,3} \wedge \omega_3 + +(\kappa_3^{12}\Omega_4^{3,4} - \kappa_4^{42}\bar{\Omega}_4^{1,4} + \bar{\Omega}_4^{1,2}) \wedge \omega_4 = 0$$

$$(\Omega_4^{2,1} + \kappa_3^{12}\kappa_4^{21}\Omega_1^3 + \kappa_4^{12}\kappa_4^{21}\Omega_1^4 + \kappa_4^{32}\bar{\Omega}_3^{2,1} + \kappa_4^{12}\bar{\Omega}_1^{2,1} + \kappa_4^{42}\bar{\Omega}_4^{2,1}) \wedge \omega_1$$
$$+ \Omega_4^{2,2} \wedge \omega_2 + \kappa_4^{32}\bar{\Omega}_3^{2,3} \wedge \omega_3 + (\kappa_4^{42}\bar{\Omega}_4^{2,4} - \bar{\Omega}_4^{2,2}) \wedge \omega_4 = 0$$

$$\Omega_4^{21,1} \wedge \omega_1 + (-\Omega_4^{2,1} + \kappa_3^{21}\Omega_4^{3,2} - \kappa_4^{31}\bar{\Omega}_3^{2,2} - \kappa_4^{41}\bar{\Omega}_4^{2,2}) \wedge \omega_2$$
$$+ (\kappa_3^{21}\Omega_4^{3,3} - \kappa_4^{31}\bar{\Omega}_3^{2,3}) \wedge \omega_3 + (\kappa_3^{21}\Omega_4^{3,4} - \kappa_4^{41}\bar{\Omega}_4^{2,4} + \bar{\Omega}_4^{2,1}) \wedge \omega_4 = 0$$

$$\Omega_3^{41,1} \wedge \omega_1 + (\Omega_3^{41,2} - \kappa_3^{42}\kappa_3^{21}\Omega_2^3 - \kappa_3^{42}\kappa_4^{21}\Omega_2^4 + \kappa_4^{41}\Omega_3^{4,2} - \kappa_3^{21}\bar{\Omega}_2^{4,2} - \kappa_3^{31}\bar{\Omega}_3^{4,2}) \wedge \omega_2$$
$$+ (\kappa_4^{41}\Omega_3^{4,3} - \kappa_3^{31}\bar{\Omega}_3^{4,3} + \bar{\Omega}_3^{4,1}) \wedge \omega_3 + (\kappa_4^{41}\Omega_3^{4,4} + \kappa_3^{21}\Omega_2^4 + \kappa_3^{31}\Omega_3^{4,3} - \Omega_3^{4,1}) \wedge \omega_4 = 0$$

$$(\Omega_3^{41,2} - \kappa_3^{12}\bar{\Omega}_1^{4,1} - \kappa_3^{41}\kappa_3^{12}\Omega_1^3 + \kappa_4^{42}\Omega_3^{41} - \kappa_3^{32}\bar{\Omega}_3^{41} - \kappa_3^{41}\kappa_4^{12}\Omega_1^4) \wedge \omega_1 + \Omega_3^{42,2} \wedge \omega_2$$
$$+ (\kappa_4^{42}\Omega_3^{4,3} - \kappa_3^{32}\bar{\Omega}_3^{4,3} + \bar{\Omega}_3^{4,2}) \wedge \omega_3 + (\kappa_4^{42}\Omega_3^{4,4} + \kappa_3^{12}\Omega_1^4 + \kappa_3^{32}\Omega_3^{4,3} - \Omega_3^{42}) \wedge \omega_4 = 0$$

$$\Omega_4^{31,1} \wedge \omega_1 + (\Omega_4^{31,2} - \kappa_4^{21}\bar{\Omega}_2^{3,2} + \kappa_3^{31}\Omega_4^{3,2} - \kappa_4^{41}\bar{\Omega}_4^{3,2} - \kappa_4^{32}\kappa_3^{21}\Omega_2^3 - \kappa_4^{32}\kappa_4^{21}\Omega_2^4) \wedge \omega_2$$
$$+ (\kappa_3^{31}\Omega_4^{3,3} + \kappa_4^{41}\Omega_4^{3,4} - \Omega_4^{3,1} + \kappa_4^{21}\Omega_2^3) \wedge \omega_3 + (\kappa_3^{31}\Omega_4^{3,4} - \kappa_4^{41}\bar{\Omega}_4^{3,4} + \bar{\Omega}_4^{3,1}) \wedge \omega_4 = 0$$

$$(\Omega_4^{31,2} - \kappa_4^{12}\bar{\Omega}_1^{3,1} + \kappa_3^{32}\Omega_4^{3,1} - \kappa_4^{42}\bar{\Omega}_4^{3,1} - \kappa_4^{31}\kappa_3^{12}\Omega_1^3 - \kappa_4^{31}\kappa_4^{12}\Omega_1^4) \wedge \omega_1 + \Omega_4^{32,2} \wedge \omega_2$$
$$+ (\kappa_3^{32}\Omega_4^{3,3} + \kappa_4^{42}\Omega_4^{3,4} - \Omega_4^{3,2} + \kappa_4^{12}\Omega_1^3) \wedge \omega_3 + (\kappa_3^{32}\Omega_4^{3,4} - \kappa_4^{42}\bar{\Omega}_4^{3,4} + \bar{\Omega}_4^{3,2}) \wedge \omega_4 = 0,$$

where

$$\begin{array}{ll}
& \Omega_1^3, \Omega_1^4, \Omega_2^3, \Omega_2^4, \Omega_1^{2,1}, \Omega_1^{2,2}, \Omega_2^{1,1}, \Omega_2^{1,2}, \Omega_3^{1,1}, \Omega_3^{1,2}, \Omega_3^{2,1}, \Omega_3^{2,2}, \\
& \Omega_4^{1,1}, \Omega_4^{1,2}, \Omega_4^{2,1}, \Omega_4^{2,2}, \Omega_3^{4,1}, \Omega_3^{4,2}, \Omega_3^{4,3}, \Omega_3^{4,4}, \Omega_4^{3,1}, \Omega_4^{3,2}, \Omega_4^{3,3}, \Omega_4^{3,4}, \\
(15) & \bar\Omega_1^{2,1}, \bar\Omega_2^{1,2}, \bar\Omega_1^{3,1}, \bar\Omega_2^{3,2}, \bar\Omega_1^{4,1}, \bar\Omega_2^{4,2}, \bar\Omega_3^{1,1}, \bar\Omega_3^{1,2}, \bar\Omega_3^{2,1}, \bar\Omega_3^{2,2}, \bar\Omega_3^{1,3}, \bar\Omega_3^{2,3}, \\
& \bar\Omega_4^{1,1}, \bar\Omega_4^{1,2}, \bar\Omega_4^{1,4}, \bar\Omega_4^{2,1}, \bar\Omega_4^{2,2}, \bar\Omega_4^{2,4}, \bar\Omega_3^{4,1}, \bar\Omega_3^{4,2}, \bar\Omega_3^{4,3}, \bar\Omega_4^{3,1}, \bar\Omega_4^{3,2}, \bar\Omega_4^{3,4}, \\
& \Omega_3^{12,2}, \Omega_3^{21,1}, \Omega_4^{12,2}, \Omega_4^{21,1}, \Omega_3^{41,1}, \Omega_3^{41,2}, \Omega_3^{42,2}, \Omega_4^{31,1}, \Omega_4^{31,2}, \Omega_4^{32,2}
\end{array}$$

are independent forms of $d\alpha_{ik}^r$, $d\kappa_{ik}^r$, ω_1^1, ω_2^2, ω_3^3, ω_4^4, $\bar\omega_1^1$, $\bar\omega_2^2$, $\bar\omega_3^3$, $\bar\omega_4^4$, ω_1, ω_2, ω_3, ω_4. We introduce some of these forms only:

$$\begin{array}{ll}
& \bar\Omega_3^{1,3} = d\alpha_{33}^7 + \alpha_{33}^7(\omega_3^3 + \bar\omega_1^1 - 2\bar\omega_3^3) - \bar H_3^{1(3,4)}\omega_4 \\
(16) & \bar\Omega_4^{2,4} = d\alpha_{44}^{10} + \alpha_{44}^{10}(\omega_4^4 + \bar\omega_2^2 - 2\bar\omega_4^4) + \bar H_4^{2(3,4)}\omega_3 \\
& \bar\Omega_3^{2,3} = d\alpha_{33}^8 + \alpha_{33}^8(\omega_3^3 + \bar\omega_2^2 - 2\bar\omega_3^3) - \bar H_3^{2(3,4)}\omega_4 \\
& \bar\Omega_4^{1,4} = d\alpha_{44}^9 + \alpha_{44}^9(\omega_4^4 + \bar\omega_1^1 - 2\bar\omega_4^4) + \bar H_4^{1(3,4)}\omega_3.
\end{array}$$

The system (14) involves $q = 58$ forms (15). For the integral elements of (14) we obtain

$$s_1 = 32,\ s_2 = 16,\ s_3 = 8,\ s_4 = 2.$$

All forms (15) are principal forms. Using the Cartan lemma we can calculate the forms (15) as linear combinations of ω_1, ω_2, ω_3, ω_4. These combinations contain $N = 96 = s_1 + 2s_2 + 3s_3 + 4s_4$ parameters. The system (14) is involutive. The solution depends on two functions of four parameters. □

It follows from (16) that

$$\frac{\alpha_{33}^8\alpha_{44}^9}{\alpha_{33}^7\alpha_{44}^{10}}\, d\, \frac{\alpha_{33}^7\alpha_{44}^{10}}{\alpha_{33}^8\alpha_{44}^9}$$

is a principal form. The expression

$$I = \frac{\alpha_{33}^7\alpha_{44}^{10}}{\alpha_{33}^8\alpha_{44}^9}$$

is invariant. We shall study the invariant I.

ω_1, ω_2, ω_3 are the torsal forms of V_3^4 at the point A_3. We assume that $\omega_1 = \omega_2 = \omega_3 = 0$. Then A_3 forms a curve in the space P_3. The tangent t_3 of this curve at the point A_3 is determined by the points A_3, $\kappa_3^{41}\alpha_{44}^9 A_1 + \kappa_3^{42}\alpha_{44}^{10}A_2 - \alpha_{44}^{11}A_4$. This follows from (5), (11). Similarly if $\omega_1 = \omega_2 = \omega_4 = 0$, then A_4 forms a curve in P_3. Its tangent t_4 is determined by the points A_4, $\kappa_4^{31}\alpha_{33}^7 A_1 + \kappa_4^{32}\alpha_{33}^8 A_2 - \alpha_{33}^{12}A_3$. We assume that t_3, t_4 do not lie in the same plane. Consequently

$$\kappa_4^{32}\alpha_{33}^8\kappa_3^{41}\alpha_{44}^9 \neq \kappa_4^{31}\alpha_{33}^7\kappa_3^{42}\alpha_{44}^{10}.$$

We consider the quadric determined by the lines (A_1, A_2), t_3, t_4. Its generator of the second system through A_4 intersects (A_1, A_2) at the point $C_4 = \kappa_3^{41}\alpha_{44}^9 A_1 + \kappa_3^{42}\alpha_{44}^{10} A_2$. Similarly the generator of the second system through A_3 intersects (A_1, A_2) at the point $C_3 = \kappa_4^{31}\alpha_{33}^7 A_1 + \kappa_4^{32}\alpha_{33}^8 A_2$. The anharmonic ratio $D(A_1, A_2, C_3, C_4)$ satisfies the relation

$$D(A_1, A_2, C_3, C_4) = \frac{\kappa_4^{32}\alpha_{33}^8\kappa_3^{41}\alpha_{44}^9}{\kappa_4^{31}\alpha_{33}^7\kappa_3^{42}\alpha_{44}^{10}} = \frac{J_2}{I}.$$

In the above considerations we have studied the manifold V_3^4 whose generators are the spaces P_3. Now we consider the manifold V_2^4 whose generators are the planes $x_4 = 0$. We shall study the characteristics of the generators of V_2^4. The tangent space of the manifold V_2^4 at the point $M = x^1 A_1 + x^2 A_2 + x^3 A_3$ is the linear space determined by the points A_1, A_2, A_3 and the points $\mathrm{d}M$. If the dimension of this space is less than 6, then M is a focal point of V_2^4. It is necessary to study the rank of the matrix

$$\begin{pmatrix} \alpha_{41}^3 x^1 - x^3\alpha_{41}^{11} & x^1 & x^3\kappa_3^{21} & x^3\kappa_3^{31} & x^3\kappa_3^{41} \\ \alpha_{42}^6 x^2 - x^3\alpha_{42}^{11} & \kappa_3^{12}x^3 & x^2 & x^3\kappa_3^{32} & x^3\kappa_3^{42} \\ -x^3\alpha_{43}^{11} & 0 & 0 & x^3 & 0 \\ -x^3\alpha_{44}^{11} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The points of the line (A_1, A_2) except A_1, A_2 are focal points of the second order. A_1, A_2 are focal points of the third order. We assume that $x^3 \neq 0 \wedge \alpha_{44}^{11} \neq 0$ holds. For focal points of the first order we obtain

$$x^1x^2 - (x^3)^2\kappa_3^{12}\kappa_3^{21} = 0 \wedge x^1\kappa_3^{42} - x^3\kappa_3^{12}\kappa_3^{41} = 0 \wedge x^2\kappa_3^{41} - x^3\kappa_3^{21}\kappa_3^{42} = 0.$$

The solution of these equations are the coordinates of a point N_3; $N_3\big(\kappa_3^{12}(\kappa_3^{41})^2, \kappa_3^{21}(\kappa_3^{42})^2, \kappa_3^{41}\kappa_3^{42}, 0\big)$.

Similar considerations are true for the manifold whose generators are the planes $x^3 = 0$.

We shall study the space P_7^* dual to the space P_7. The elements (points) of P_7^* are the hyperplanes of P_7. We denote $T_1 = (A_2, A_3, A_4, \bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4)$, $T_2 = (A_1, A_3, A_4, \bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4)$, $T_3 = (A_1, A_2, A_4, \bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4)$, $T_4 = (A_1, A_2, A_3, \bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4)$, $\bar{T}_1 = (A_1, A_2, A_3, A_4, \bar{A}_2, \bar{A}_3, \bar{A}_4)$, $\bar{T}_2 = (A_1, A_2, A_3, A_4, \bar{A}_1, \bar{A}_3, \bar{A}_4)$, $\bar{T}_3 = (A_1, A_2, A_3, A_4, \bar{A}_1, \bar{A}_2, \bar{A}_4)$, $\bar{T}_4 = (A_1, A_2, A_3, A_4, \bar{A}_1, \bar{A}_2, \bar{A}_3)$. In P_7^* we consider the frame consisting of the points T_1, T_2, T_3, T_4, $\bar{T}_1$, $\bar{T}_2$, $\bar{T}_3$, $\bar{T}_4$. The fundamental equations of this moving frame are

(17)

$$
\begin{aligned}
d\bar{T}_1 &= (\omega_1^1+\omega_2^2+\omega_3^3+\omega_4^4+\varpi_2^2+\varpi_3^3+\varpi_4^4)\bar{T}_1\\
&\quad +\varpi_2^1\bar{T}_2-\varpi_3^1\bar{T}_3+\varpi_4^1\bar{T}_4-\omega_1T_1-\kappa_3^{12}\omega_2T_3+\kappa_4^{12}\omega_2T_4\\
d\bar{T}_2 &= \varpi_1^2\bar{T}_1+(\omega_1^1+\omega_2^2+\omega_3^3+\omega_4^4+\varpi_1^1+\varpi_3^3+\varpi_4^4)\bar{T}_2\\
&\quad +\varpi_3^2\bar{T}_3-\varpi_4^2\bar{T}_4+\kappa_3^{21}\omega_1T_3-\kappa_4^{21}\omega_1T_4-\omega_2T_2\\
d\bar{T}_3 &= -\varpi_1^3\bar{T}_1+\varpi_2^3\bar{T}_2+(\omega_1^1+\omega_2^2+\omega_3^3+\omega_4^4+\varpi_1^1+\varpi_2^2+\varpi_4^4)\bar{T}_3\\
&\quad +\varpi_4^3\bar{T}_4-(\kappa_3^{31}\omega_1+\kappa_3^{32}\omega_2+\omega_3)T_3+(\kappa_4^{31}\omega_1+\kappa_4^{32}\omega_2)T_4\\
d\bar{T}_4 &= \varpi_1^4\bar{T}_1-\varpi_2^4\bar{T}_2+\varpi_3^4\bar{T}_3+(\omega_1^1+\omega_2^2+\omega_3^3+\omega_4^4+\varpi_1^1+\varpi_2^2+\varpi_3^3)\bar{T}_4\\
&\quad +(\kappa_3^{41}\omega_1+\kappa_3^{42}\omega_2)T_3-(\kappa_4^{41}\omega_1+\kappa_4^{42}\omega_2+\omega_4)T_4\\
dT_1 &= -\hat{\omega}_1^1\bar{T}_1+\hat{\omega}_2^1\bar{T}_2-\hat{\omega}_3^1\bar{T}_3+\hat{\omega}_4^1\bar{T}_4+(\omega_2^2+\omega_3^3+\omega_4^4+\varpi_1^1+\varpi_2^2+\varpi_3^3+\varpi_4^4)T_1\\
&\quad +\omega_2^1T_2-\omega_3^1T_3+\omega_4^1T_4\\
dT_2 &= \hat{\omega}_1^2\bar{T}_1-\hat{\omega}_2^2\bar{T}_2+\hat{\omega}_3^2\bar{T}_3-\hat{\omega}_4^2\bar{T}_4+\omega_1^2T_1\\
&\quad +(\omega_1^1+\omega_3^3+\omega_4^4+\varpi_1^1+\varpi_2^2+\varpi_3^3+\varpi_4^4)T_2+\omega_3^2T_3-\omega_4^2T_4\\
dT_3 &= -\hat{\omega}_1^3\bar{T}_1+\hat{\omega}_2^3\bar{T}_2-\hat{\omega}_3^3\bar{T}_3+\hat{\omega}_4^3\bar{T}_4-\omega_1^3T_1+\omega_2^3T_2\\
&\quad +(\omega_1^1+\omega_2^2+\omega_4^4+\varpi_1^1+\varpi_2^2+\varpi_3^3+\varpi_4^4)T_3+\omega_4^3T_4\\
dT_4 &= \hat{\omega}_1^4\bar{T}_1-\hat{\omega}_2^4\bar{T}_2+\hat{\omega}_3^4\bar{T}_3-\hat{\omega}_4^4\bar{T}_4+\omega_1^4T_1-\omega_2^4T_2+\omega_3^4T_3\\
&\quad +(\omega_1^1+\omega_2^2+\omega_3^3+\varpi_1^1+\varpi_2^2+\varpi_3^3+\varpi_4^4)T_4
\end{aligned}
$$

where ω_i^k, ϖ_i^k, $i\neq k$ have the form (11). We denote by V_3^{*4} the four-parametric manifold in P_7^*, whose generators are $(\bar{T}_1,\bar{T}_2,\bar{T}_3,\bar{T}_4)$. Corresponding to each point of this space in P_7 is the hyperplane passing through A_1, A_2, A_3, A_4.

Proposition 5. *The characteristic of the manifold V_3^{*4} consists of the coordinate planes with the vertices at the points $\bar{T}_1$, $\bar{T}_2$, $\bar{T}_3$, $\bar{T}_4$. These points are focal points of the second order. All points of $(\bar{T}_1,\bar{T}_2)$ are also focal points of the second order and have common torsal forms ω_1, ω_2.*

This follows from the equations (17). We consider the correspondence K between the focal points of the manifold V_3^4 and the points of the generators of V_3^{*4}. Corresponding to a focal point of the first order is the tangent space of V_3^4 at this point. Corresponding to the focal points of the second or third orders are all hyperplanes of the space P_7 containing the tangent spaces of V_3^4 at these points. Corresponding to the point A_1 is a linear combination of $\bar{T}_2$, $\bar{T}_3$, $\bar{T}_4$, to the point A_2 a linear combination of $\bar{T}_1$, $\bar{T}_3$, $\bar{T}_4$. Corresponding to points of the line (A_1,A_2) different from A_1, A_2 are linear combinations of $\bar{T}_3$, $\bar{T}_4$. The point $N_3(\kappa_3^{12}(\kappa_3^{41})^2,\kappa_3^{21}(\kappa_3^{42})^2,\kappa_3^{41}\kappa_3^{42},0)$ is a focal point of the second order of V_3^4. The tangent space of V_3^4 in N_3 is determined by A_1, A_2, A_3, A_4, $\kappa_3^{12}\kappa_3^{41}\bar{A}_1+\kappa_3^{21}\kappa_3^{42}\bar{A}_2+\kappa_3^{42}\kappa_3^{41}\bar{A}_4$, $\bar{A}_3$. There exist two linear independent hyperplanes $-\kappa_3^{42}\kappa_3^{41}\bar{T}_2 + \kappa_3^{21}\kappa_3^{42}\bar{T}_4$, $-\kappa_3^{42}\kappa_3^{41}\bar{T}_1 - \kappa_3^{12}\kappa_3^{41}\bar{T}_4$ containing the tangent space of V_3^4 at the point N_3. This implies that corresponding to the point N_3 in K is a linear combination of these hyperplanes. Similarly corresponding to the point $N_4\big(\kappa_4^{12}(\kappa_4^{31})^2, \kappa_4^{21}(\kappa_4^{32})^2, 0, \kappa_4^{31}\kappa_4^{32}\big)$in K is a linear combination of the hyperplanes $\bar{T}_2\kappa_4^{31}\kappa_4^{32} + \bar{T}_3\kappa_4^{32}\kappa_4^{21}$, $\bar{T}_1\kappa_4^{31}\kappa_4^{32} - \bar{T}_3\kappa_4^{12}\kappa_4^{31}$. We assume that a point $M = x^1A_1 + x^2A_2 + x^3A_3 + x^4A_4$ belongs to the plane $x^4 = 0$. We exclude the points of the line (A_1, A_2) and the point N_3. Then the tangent space of V_3^4 at the point M is determined by the points A_1, A_2, A_3, A_4, $x^1\bar{A}_1 + x^3\kappa_3^{21}\bar{A}_2 + x^3\kappa_3^{41}\bar{A}_4$, $\bar{A}_3$, $x^3\kappa_3^{12}\bar{A}_1 + x^2\bar{A}_2 + x^3\kappa_3^{42}\bar{A}_4$ (this follows from (10)). The point corresponding to the point M in K is the point $\bar{T}_1\big[-(x^3)^2\kappa_3^{21}\kappa_3^{42} + x^2x^3\kappa_3^{41}\big] + \bar{T}_2\big[-x^1x^3\kappa_3^{42} + (x^3)^2\kappa_3^{41}\kappa_3^{12}\big] + \bar{T}_4\big[x^1x^2 - (x^3)^2\kappa_3^{12}\kappa_3^{21}\big]$. The coefficient at T_4 is equal to zero if and only if M belongs to the characteristic quadric of P_3. We assume that a point M belongs to the plane $x^3 = 0$. We exclude the points of the line (A_1, A_2) and the point N_4. Then the tangent space of V_3^4 at the point M is determined by the points A_1, A_2, A_3, A_4, $x^1\bar{A}_1 + x^4\kappa_4^{21}\bar{A}_2 + x^4\kappa_4^{31}\bar{A}_3$, $x^4\kappa_4^{12}\bar{A}_1 + x^2\bar{A}_2 + x^4\kappa_4^{32}\bar{A}_3$, $\bar{A}_4$. The point corresponding to the point M in K is $\bar{T}_1\big[(x^4)^2\kappa_4^{21}\kappa_4^{32} - x^2x^4\kappa_4^{31}\big] + \bar{T}_2\big[x^1x^4\kappa_4^{32} - (x^4)^2\kappa_4^{12}\kappa_4^{31}\big] + \bar{T}_3\big[x^1x^2 - (x^4)^2\kappa_4^{12}\kappa_4^{21}\big]$. The coefficient at T_4 is equal to zero if and only if M belongs to the characteristic quadric of P_3. We consider the points of the characteristic quadric of P_3 which do not belong to the planes $x^3 = 0$, $x^4 = 0$. According to (10) the tangent space of V_3^4 at such a point M is determined by the points A_1, A_2, A_3, A_4, $\bar{A}_3$, $\bar{A}_4$, $x^1\bar{A}_1 + (x^3\kappa_3^{21} + x^4\kappa_4^{21})\bar{A}_2$. Geometrically the last point is the same as the point $(x^3\kappa_3^{12} + x^4\kappa_4^{12})\bar{A}_1 + x^2\bar{A}_2$. The point corresponding to the point M in K is $x^1\bar{T}_2 + (\kappa_3^{21}x^3 + x^4\kappa_4^{21})\bar{T}_1$. Geometrically this point is the same as the point $(x^3\kappa_3^{12} + x^4\kappa_4^{12})\bar{T}_2 + x^2\bar{T}_1$. The plane $x^2 = 0$ is the tangent plane of the characteristic quadric of P_3 and intersects this quadric in the lines

$$x^3\kappa_3^{12} + x^4\kappa_4^{12} = 0, \qquad x^3\kappa_3^{21} + x^4\kappa_4^{21} = 0.$$

The first line contains the point $M_1(0, 0, \kappa_4^{12}, -\kappa_3^{12})$, the second the point $M_2(0, 0, \kappa_4^{21}, -\kappa_3^{21})$. The same holds for the plane $x^1 = 0$.

Proposition 6. *Along each generator of the quadric characteristic of P_3 intersecting (A_1, M_1), (A_2, M_2) this quadric has the common tangent space of V_3^4. (We eliminate the points of (A_1, A_2) and the points N_3, N_4.)*

Proof. We consider the point $S_1(t, 0, \kappa_4^{12}, -\kappa_3^{12})$ on the line (A_1, M_1) and similarly the point $S_2(0, \bar{t}, \kappa_4^{21}, -\kappa_3^{21})$ on the line (A_2, M_2). The line (S_1, S_2) has the equations

$$x^1 = \lambda_1 t \qquad x_2 = \lambda_2\bar{t} \qquad x^3 = \lambda_1\kappa_4^{12} + \lambda_2\kappa_4^{21} \qquad x^4 = -\lambda_1\kappa_3^{12} - \lambda_2\kappa_3^{21}.$$

This line belongs to the characteristic quadric of P_3 if

$$t\bar{t} + (\kappa_3^{21}\kappa_4^{12} - \kappa_3^{12}\kappa_4^{21})^2 = 0.$$

We consider an arbitrary point $\hat{S}$ on this line. The tangent space of V_3^4 at the point $\hat{S}$ is determined by $A_1, A_2, A_3, A_4, \lambda_1 t\bar{A}_1 + \left[(\lambda_1\kappa_4^{12}+\lambda_2\kappa_4^{21})\kappa_3^{21} - (\lambda_1\kappa_3^{12}+\lambda_2\kappa_3^{21})\kappa_4^{21}\right]\bar{A}_2$, $\bar{A}_3$, $\bar{A}_4$. The point corresponding to the point $\hat{S}$ in K is $t\bar{T}_2 + \bar{T}_1(\kappa_4^{12}\kappa_3^{21} - \kappa_3^{12}\kappa_4^{21})$. This point is independent of the position on the line of the quadric characteristic of P_3. □

References

[1] *С. П. Фиников*: Теория пар конгруенций, Москва, 1956.
[2] *Л. З. Кругляков*: Основы проективно-дифференциальной геометрии семейства многомерных плоскостей, Томск, 1980.
[3] *A. Švec*: Projective differential geometry of line congruences, Praha, 1956.
[4] *J. Vala*: Special Grassman manifold V_3^4 in the projective space P_7, Časopis pěst. mat. *113* (1988), 36–51.

Souhrn

GRASSMANOVY VARIETY V_3^4 V PROJEKTIVNÍM PROSTORU P_7, JEJICHŽ CHARAKTERISTIKY SE SKLÁDAJÍ Z KVADRIKY A DVOU ROVIN

JOSEF VALA

Výsledky náležejí geometrii čtyřparametrických variet trojrozměrných prostorů v projektivním prostoru P_7. Jsou studovány vlastnosti těch variet V_3^4, jejichž charakteristiky se skládají z kvadriky a dvou rovin. Nalezeny též vlastnosti variety duální k V_3^4.

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