HYPERREFLEXIVE OPERATORS
ON FINITE DIMENSIONAL HILBERT SPACES

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(Received July 19, 1991)

Summary. In this paper a complete characterization of hyperreflexive operators on finite dimensional Hilbert spaces is given.

Keywords: invariant subspace, commutant, reflexivity

AMS classification: 15A21, 47A15

0. INTRODUCTION

Let $H$ be a complex separable Hilbert space, $\mathcal{B}(H)$ the algebra of all continuous linear operators on $H$ and $T \in \mathcal{B}(H)$. We denote by $\{T\}'$ the commutant of $T$ ($X \in \{T\}'$ if and only if $XT = TX$) and by $\{T\}'' = \bigcap\{\{X\}' : XT = TX\}$ the double commutant of $T$. A closed linear subspace $M \subset H$ is called invariant for an algebra $\mathcal{A}$ if it is invariant for every $X \in \mathcal{A}$. If $M$ is invariant for the algebra $\{T\}'$ then $M$ is said to be hyperinvariant for $T$. The lattice of all invariant subspaces of $\mathcal{A}$ is denoted by $\text{Lat } \mathcal{A}$.

Let $\mathcal{M}$ be a set of subspaces of $H$. We denote by $\text{Alg } \mathcal{M}$ the algebra of all $X \in \mathcal{B}(H)$ for which every $M \in \mathcal{M}$ is invariant. The algebra $\mathcal{A} \subset \mathcal{B}(H)$ is called reflexive if $\mathcal{A} = \text{Alg } \text{Lat } \mathcal{A}$. An operator $T \in \mathcal{B}(H)$ is called reflexive if $\text{Alg } T$—the weakly closed algebra generated by $T$ and the identity $I$—is reflexive. We shall write $\text{Lat } T$ instead of $\text{Lat } \text{Alg } T$. In [3] a characterization of reflexive operators on finite dimensional spaces was given. If the commutant of the operator $T$ is reflexive then $T$ is called hyperreflexive. In this paper hyperreflexive operators on finite dimensional spaces are characterized.

\footnote{This research has been partially supported by Grant of Slovak Academy of Sciences GA-SAV 367/91}
In this paper $\oplus$ means the direct (not necessarily orthogonal) sum.

1. THE NILPOTENT CASE

First we consider the nilpotent operators on a finite dimensional space $H$. Let $N \in \mathcal{B}(H)$ be a nilpotent operator of order $n$ (i.e. $N^n = 0, N^{n-1} \neq 0$). Since similarity preserves hyperreflexivity (see [2] for a little more general result) we may assume that the matrix representation of $N$ is

$$N = J(k_1) \oplus J(k_2) \oplus \ldots \oplus J(k_m), \quad n = k_1 \geq k_2 \geq \ldots \geq k_m.$$  

Let the corresponding decomposition of $H$ be

$$H = H_1 \oplus H_2 \oplus \ldots \oplus H_m.$$  

Here $J(k)$ means the $k \times k$ Jordan cell (i.e. each entry on the first subdiagonal is 1, and all other entries are 0). We shall use the following descriptions of $\{N\}'$ and $\text{Lat}\{N\}'$ [4, p. 128]:

$$\mathcal{L} \in \text{Lat}\{N\}' \iff \mathcal{L} = \bigoplus_{j=1}^{m} \ker J(k_j)^{r_i} \text{ for an } m\text{-tuple of integers},$$

$$r_1, \ldots, r_m, \quad r_1 \geq \ldots \geq r_m \geq 0, \quad k_1 - r_1 \geq \ldots \geq k_m - r_m \geq 0.$$  

Let $A \in \mathcal{B}(H)$ have a block decomposition (corresponding to the decomposition (2) of $H$) $A = (A_{ij})$. Then

$$A \in \{N\}' \iff \begin{cases} A_{ii} \in \{J(k_i)\}' \text{ for all } i; \\ \text{for } i < j, \quad A_{ij} = \begin{pmatrix} 0 \\ X \end{pmatrix} \text{ with } X \in \{J(k_j)\}'; \\ \text{for } i > j, \quad A_{ij} = (Y \ 0) \text{ with } Y \in \{J(k_i)\}'. \end{cases}$$

Recall that $\{J(k)\}'$ consists of polynomials in $J(k)$ and thus of lower-triangular matrices with equal entries on each subdiagonal ($a_{i+1,j+1} = a_{ij}, 1 \leq i, j \leq k$). Now, we are able to describe $\text{AlgLat}\{N\}'$:

**Theorem 1.** Let $N \in \mathcal{B}(H)$ be a nilpotent operator of the form (1). Let $A \in \mathcal{B}(H)$ have a block decomposition (corresponding to the decomposition (2) of $H$) $A = (A_{ij})$. Then $A$ belongs to $\text{AlgLat}\{N\}'$ if and only if it has the following form:

$$A_{ij} = \begin{cases} \text{a lower-triangular matrix if } i = j; \\ \begin{pmatrix} 0 \\ X \end{pmatrix} \text{ with lower-triangular } X \text{ if } i < j; \\ (Y \ 0) \text{ with lower-triangular } Y \text{ if } i > j. \end{cases}$$
Proof. Let us recall that for a pair of integers \( k, r; 0 \leq r \leq k \), the space \( \ker J(k)^r \) consists of all vectors \( x = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) for which \( \lambda_1 = \lambda_2 = \ldots = \lambda_{k-r} = 0 \). If \( \mathcal{L} \) is a subspace of the form (3) and if \( x = x_1 \oplus x_2 \oplus \ldots \oplus x_m \in \mathcal{L} \), then for every \( A \) with a block decomposition satisfying (5) and for all pairs \( i, j; 1 \leq i, j \leq m, A_{ij}x_j \in \ker J(k_i)^r \). This shows that \( A \in \text{Alg Lat}\{ \mathcal{N} \}' \).

Now, suppose that \( A = (A_{ij}) \in \text{Alg Lat}\{ \mathcal{N} \}' \). Let the integers \( j \) and \( s \) satisfy \( 1 \leq j \leq m, 0 \leq s \leq k_j \). If

\[
 r_j = \begin{cases} 
 s & \text{if } s \leq k_i, \\
 k_i & \text{if } s > k_i. 
\end{cases}
\]

then \( \mathcal{L} = \bigoplus_{i=1}^{m} \ker J(k_i)^{r_i} \subseteq \text{Lat}\{ \mathcal{N} \}' \).

\[ A\mathcal{L} \subseteq \mathcal{L} \Rightarrow A_{ij} \ker J(k_j)^s \subseteq \ker J(k_i)^s, \quad s = 0, 1, \ldots, k_j, \]

for all pairs \( i, j \) satisfying \( 1 \leq i \leq j \). It follows that \( A_{ii} \) is lower-triangular for all \( i \) and if \( i < j \) then \( A_{ij} = \begin{pmatrix} 0 \\ Y \end{pmatrix} \) with a lower-triangular \( k_j \times k_j \) matrix \( Y \).

Let the integers \( i \) and \( s \) satisfy \( 1 \leq i \leq m, 0 \leq s \leq k_i \). Setting \( r_j = \max\{s + k_j - k_i, 0\} \) for \( j = 1, 2, \ldots, m \) we obtain \( \mathcal{L} = \bigoplus_{j=1}^{m} \ker J(k_j)^{r_j} \subseteq \text{Lat}\{ \mathcal{N} \}' \), and this yields

\[ A_{ij} \ker J(k_j)^{r_j + k_j - k_i} \subseteq \ker J(k_i)^s \]

for \( j < i \) and every \( s; 0 \leq s \leq k_i \). It follows that \( A_{ij} = (X, 0) \) with a lower-triangular \( k_i \times k_i \) matrix \( X \). This completes the proof. \( \square \)

Corollary. A nilpotent operator \( N \in \mathcal{B}(H) \) is hyperreflexive if and only if \( N = 0 \).

Proof. This is an obvious consequence of the descriptions (4) and (5) of the commutant of \( N \) and of the algebra \( \text{Alg Lat}\{ \mathcal{N} \}' \), respectively. \( \square \)

We shall need the following simple result:

Lemma 2. The double commutant of every nilpotent operator \( N \) consists of polynomials in \( N \).

Proof. We use the models (1), (2) and (4) of \( N \) and of its commutant, respectively. Since \( \{N\}'' \subseteq \{N\}' \), \( A \in \{N\}'' \) has a block decomposition \((A_{ij})\) satisfying (4). For \( p = 1, 2, \ldots, m \) let \( B_p \in \{N\}' \) have the block \( B_{pp} = I \) and all other blocks \( B_{ij} = 0 \).

\[ AB_p = B_p A \Rightarrow A_{ij} = 0 \text{ for all } i \neq j. \]
There exist polynomials \( p_1, p_2, \ldots, p_m \) such that \( p_i(J(k_i)) = A_{ii} \) \( (i = 1, 2, \ldots, m) \). Setting \( C_j \in \{N\}' \) for \( j \in \{2, 3, \ldots, m\} \) to be the matrix with the block \( C_{ij} = \begin{pmatrix} 0 \\ I \end{pmatrix} \) and all other blocks 0 we obtain from \( AC_j = C_j A \)
\[
p_j(J(k_j)) = p_1(J(k_j)), \quad j = 1, 2, \ldots, m
\]
and so \( A = p_1(N) \).

2. General operators in finite dimensional spaces

The investigation of a general linear operator \( T \in \mathcal{B}(H) \) can be reduced to the investigation of the nilpotent operators similarly as in [2] and [4].

Theorem 3. Let \( T \in \mathcal{B}(H) \) have the minimum polynomial \( m_T(\lambda) = \prod_{i=1}^n (\lambda - \lambda_i)^{m_i} \). Let \( H_i = \ker(T - \lambda_i I)^{m_i} \).

Then the following assertions hold:

(i) \( H_i \in \Lat(T)' \) for \( i = 1, 2, \ldots, n \),

(ii) \( T_i = T|_{H_i} \) is hyperreflexive if and only if \( m_i = 1 \),

(iii) \( H = H_1 \oplus H_2 \oplus \ldots \oplus H_n \),

(iv) \( T \) is hyperreflexive if and only if all operators \( T_i \) are hyperreflexive.

Therefore \( T \) is hyperreflexive if and only if it is similar to a diagonal operator.

Proof. The assertion (i) is an easy consequence of the fact that
\[
(T - \lambda_i I)^{m_i} \in \{T\}''.
\]

To prove (ii) let us observe that \( \{T_i\}' = \{T_i - \lambda_i I\}' \). The operator \( T_i - \lambda_i \) is nilpotent and so the assertion (ii) follows from the corollary of Theorem 1.

The rest of this theorem is a consequence of the following lemma:

Lemma 4. Let \( H_1, H_2 \) be finite dimensional spaces. Let \( H = H_1 \oplus H_2 \) and let \( X \in \mathcal{B}(H_1), Y \in \mathcal{B}(H_2) \). Then the following assertions are equivalent:

(1) The minimum polynomials \( m_X, m_Y \) of \( X, Y \) are relatively prime.
(2) \( \text{Alg}(X \oplus Y) = \text{Alg} X \oplus \text{Alg} Y \).
(3) \( \text{Lat}(X \oplus Y) = \text{Lat} X \oplus \text{Lat} Y \).
(4) \( \text{Alg Lat}(X \oplus Y) = \text{Alg Lat} X \oplus \text{Alg Lat} Y \).
(5) \( \{X \oplus Y\}' = \{X\}' \oplus \{Y\}' \).
(6) \( \text{Lat}\{X \oplus Y\}' = \text{Lat}\{X\}' \oplus \text{Lat}\{Y\}' \).
(7) \( \text{Alg Lat}\{X \oplus Y\}' = \text{Alg Lat}\{X\}' \oplus \text{Alg Lat}\{Y\}' \).
(8) \( \{X \oplus Y\}'' = \{X\}'' \oplus \{Y\}'' \).

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Proof. The proof of this theorem can be found e.g. in [4] and [6]. Let us recall the proof of (1) \( \iff \) (2).

If (1) holds then there exist polynomials \( a, b \) such that
\[
1 = am_x + bm_y.
\]
Therefore
\[
I \oplus 0 = (b \cdot m_y)(X \oplus Y) \in \text{Alg}(X \oplus Y) \quad \text{and}
0 \oplus I = (a \cdot m_x)(X \oplus Y) \in \text{Alg}(X \oplus Y).
\]
This shows that (2) holds.

If (2) holds then there exists a polynomial \( q \) such that
\[
q(X \oplus Y) = I \oplus 0.
\]
Let us denote by \( p \) the least common divisor of \( m_x \) and \( m_y \), and let \( f = m_x / p \). Since
\[
q(Y) = 0 \quad p \text{ divides } q, \quad \text{consequently } m_x = p \cdot f \text{ divides } q \cdot f.
\]
It follows that
\[
f(X) = I \cdot f(X) = q(X)f(X) = (q \cdot f)(X) = 0
\]
and so \( p = 1 \) and (1) holds.

By a simple computation [6] it can be also proved that (n) implies (n + 1) for \( n = 2, 3, \ldots, 7 \). Using Lemma 2 we obtain easily that the assertions (2) and (8) are identical. This completes the proof of the lemma.

3. Algebraic operators in separable spaces

The preceding results can be easily proved also for algebraic operators in infinite dimensional Hilbert spaces. To show this let us suppose that \( H \) is a complex separable (infinite dimensional) Hilbert space and that \( T \in \mathcal{B}(H) \) is algebraic, i.e. there exists a polynomial \( p \) with complex coefficients such that \( p(T) = 0 \). In this case the notion of minimum polynomial makes sense and Lemma 3 remains true (with the same proof).

First we consider a nilpotent operator \( T \in \mathcal{B}(H) \). Let \( n \geq 1 \) be an integer such that \( T^n = 0 \) and \( T^{n-1} \neq 0 \). We may assume that \( \|T\| \leq 1 \) (if \( \|T\| > 1 \), we replace \( T \) by \( T/\|T\| \)). So \( T \) is a contraction of class \( C_0 \) in the sense of [5, Chap. III.4] and we may use the theory of Jordan models of \( C_0 \)-contractions [1, Theorem III.5.1].

Let \( H^2 \) denote the Hardy space of analytic functions in the unit circle. The minimal function of \( T \) is \( m(\lambda) = \lambda^n \). By [2, Theorem B] \( T \) is hyperreflexive if and only if the operator \( S(m) = P_m S|H(m) \) is hyperreflexive. Here \( H(m) = (mH^2)^\perp \), \( P_m \) is the orthogonal projection onto \( H(m) \) and \( (Su)(\lambda) = \lambda u(\lambda) \) is the unilateral shift. In our case \( m(\lambda) = \lambda^n \) and \( S(m) \) is a nilpotent operator on the n-dimensional Hilbert space \( H(m) \) with \( S(m)^{n-1} \neq 0 \). By the corollary of Theorem 1 this is possible only if \( n = 1 \), i.e. \( T = 0 \). Therefore the following analogue of Theorem 3 holds:

Theorem 5. Let \( T \in \mathcal{B}(H) \) be an algebraic operator having the minimum polynomial \( m_T(\lambda) = \prod_{i=1}^n (\lambda - \lambda_i)^{m_i} \). Let \( H_i = \ker(T - \lambda_i I)^{m_i} \).
Then the following assertions hold:

(i) $H_i \in \text{Lat}\{T\}'$ for $i = 1, 2, \ldots, n$,
(ii) $T_i = T|_{H_i}$ is hyperreflexive if and only if $m_i = 1$,
(iii) $H = H_1 \oplus H_2 \oplus \ldots \oplus H_n$,
(iv) $T$ is hyperreflexive if and only if all operators $T_i$ are hyperreflexive.

REMARKS AND OPEN PROBLEMS

1. In [2] a characterization of reflexive operators (for $\dim H < \infty$) was given. It follows from this characterization and from Theorem 3 that if $T \in \mathcal{A}(H)$ is hyperreflexive, then it is also reflexive. The other implication is not true, e.g. the operator
$$
\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
$$
is reflexive, but it is not hyperreflexive.

In general (i.e. for $\dim H = \infty$) it is an open problem whether each hyperreflexive operator must be reflexive.

2. The following question is also a natural open problem:
Is a hyperreflexive quasinilpotent operator necessarily equal to 0?

3. The assertion of Lemma 2 holds for every operator (in a finite dimensional space). This can be proved combining Lemma 2, Theorem 3 and Lemma 4. A slightly different proof can be found in a recent book of R.A. Horn and C.R. Johnson [7, Theorem 4.4.19].

References


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