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ON SELECTIONS OF MULTIFUNCTIONS

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Summary. The purpose of this paper is to introduce a definition of cliquishness for multifunctions and to study the search for cliquish, quasi-continuous and Baire measurable selections of compact valued multifunctions. A correction as well as a generalization of the results of [5] are given.

Keywords: Cliquishness, quasi-continuous and Baire measurable selection.

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Define a multifunction $F\colon R \to R$ (R-the real line with the usual topology) as follows: $F(x) = \{1/x\}$ for $x \neq 0$ and $F(0) = R$. The multifunction F has no quasi-continuous selection. That means Theorem 2 of [5] is not valid. It is easy to see from the proofs of theorems of [5] that they are correct for a compact valued multifunction.

The aim of this paper is to present new proofs of the theorems from [5]. Despite the fact that we consider compact valued multifunctions our assumptions on Y as well as the types of continuity are more general than those of the paper [5]. We hope the results presented here will give a correct and more comprehensive information concerning cliquish, quasi-continuous and Baire measurable selections.

In what follows X is a T_1-Baire topological space while Y is a separable metrizable one. By [2, p. 328 Th. 3], there is a metric d for Y such that (Y,d) is totally bounded. Let (Y°, d°) be a completion of (Y,d). By [2, p. 337, Corollary of Th. 19], Y° is compact. By $S(\varepsilon, A)$ $\big(S^\circ(\varepsilon, A)\big)$ we denote an ε-neighborhood of $A \subset Y$ $(A \subset Y^\circ)$, i.e. $S(\varepsilon, A) = \{y \in Y : d(y,A) < \varepsilon\}$ $\big(S^\circ(\varepsilon, A) = \{y \in Y^\circ : d^\circ(y,A) < \varepsilon\}\big)$, $\varepsilon > 0$. If $A = \{z\}$ we write $S(\varepsilon, z)$ $\big(S^\circ(\varepsilon, z)\big)$. By $\operatorname{int}(B)$ we denote the interior of $B \subset X$.

A multifunction $F\colon X \to \mathscr{K}(Y)$ is a set valued function which assigns to each element x of X a set $F(x) \in \mathscr{K}(Y) = \{A \subset Y : A \text{ is non-empty compact}\}$. A selection of F is any function $f\colon X \to Y$ such that $f(x) \in F(x)$ for any $x \in X$.

The upper (lower) inverse image $F^+(A)$ $(F^-(A))$ is defined for any $A \subset Y$ as $F^+(A) = \{x \in X : F(x) \subset A\}$, $F^-(A) = \{x \in X : F(x) \cap A \neq \emptyset\}$.

Let $\mathscr{B}$ be a family of subsets of X such that $\mathscr{G} \subset \mathscr{B} \subset \mathscr{B}r$ where $\mathscr{G} = \{A \subset X : A \text{ is non-empty open }\}$ and $\mathscr{B}r = \{A \subset X : A \text{ is of the second category having the Baire property}\}$. The following definition introduces a notion of cliquishness of a multifunction.

Definition 1. A multifunction $F\colon X \to \mathscr{K}(Y)$ is said to be $\mathscr{B}$-cliquish at a point $p \in X$ if for any $\varepsilon > 0$ and any neighborhood U of p there is a set $B \in \mathscr{B}$ such that $\bigcap_{x \in B} S(\varepsilon, F(x)) \neq \emptyset$. F is $\mathscr{B}$-cliquish if it is $\mathscr{B}$-cliquish at any point.

Remark 1. (i) The condition $\bigcap_{x \in B} S(\varepsilon, F(x)) \neq \emptyset$ implies that there is a point $y \in Y$ such that $S(\varepsilon, y) \cap F(x) \neq \emptyset$ for any $x \in B$.

(ii) If a single valued function $f\colon X \to Y$ is given, then under the natural interpretation of $f(x)$ as a one point set the above definition for $\mathscr{B} = \mathscr{G}$ is equivalent to the usual definition of cliquishness of a function [1]. As we will show below a function $f\colon X \to Y$ is $\mathscr{B}r$-cliquish if and only if f is Baire measurable (i.e. $f^{-1}(G)$ has the Baire property for any open set $G \subset Y$).

(iii) The set of all points at which F is $\mathscr{B}$-cliquish is closed. Consequently, F is $\mathscr{B}$-cliquish if and only if it is $\mathscr{B}$-cliquish on a dense set.

The next definition recalls a few known notions of continuity which are frequently used in this paper.

Definition 2. A multifunction $F\colon X \to \mathscr{K}(Y)$ is said to be u-$\mathscr{B}$-continuous (l-$\mathscr{B}$-continuous) at a point $p \in X$ if for any open sets V, U with $p \in U$, $F(p) \subset V$ $(F(p) \cap V \neq \emptyset)$ there is a set $B \in \mathscr{B}$ such that $B \subset U \cap F^+(V)$ $(B \subset U \cap F^-(V))$. F is u-$\mathscr{B}$-continuous (l-$\mathscr{B}$-continuous) if it is u-$\mathscr{B}$-continuous (l-$\mathscr{B}$-continuous) at any point [4]. For $\mathscr{B} = \mathscr{G}$ we have the well-known notion of upper (lower) quasi-continuity [6].

F is said to be upper (lower) semi-continuous (briefly u.s.c. (l.s.c)) at a point $p \in X$ if for any open set V such that $F(p) \subset V$ $(F(p) \cap V \neq \emptyset)$ we have $p \in \operatorname{int}(F^+(V))$ $(p \in \operatorname{int}(F^-(V)))$. F is said to be u.s.c. (l.s.c.) if it is u.s.c. (l.s.c.) at any point.

If a single valued function $f\colon X \to Y$ is given, then the notions of u-$\mathscr{B}$-continuity and l-$\mathscr{B}$-continuity coincide and we simply refer to $\mathscr{B}$-continuity of f. The situation is analogous with quasi-continuity and continuity of f.

Lemma 1. *For any $\mathscr{B}$-cliquish multifunction $F\colon X \to \mathscr{K}(Y)$ there is a quasi-continuous function $f\colon X \to Y^\circ$ and residual set $S \subset X$ such that*

(i) *for any $x \in S$, $f(x) \in F(x)$ and f is continuous at x,*

(ii) *for any* $p \in X$, *any neighborhood* U *of* p *and for any* $\varepsilon > 0$ *there is a set* $B \in \mathscr{B}$, $B \subset U$ *such that* $F(x) \cap S^\circ(\varepsilon, f(x)) \neq \emptyset$ *for any* $x \in B$.

Proof. Let $p \in X$. Define $A(p) \subset Y^\circ$ as follows: $A(p) = \{z \in Y^\circ\colon$ for any open sets $U, V (V$ open in $Y^\circ)$ with $p \in U$, $z \in V$ there is a set $B \in \mathscr{B}$, $B \subset U$ such that $F(x) \cap V \neq \emptyset$ for any $x \in B\}$. We will show that $A(p)$ is non-empty. Let $\varepsilon_n > 0$, $\varepsilon_n \to 0$ and let $\mathscr{U}(p)$ be a complete system of neighborhoods of p. Since F is $\mathscr{B}$-cliquish at p, for any $n = 1, 2, \ldots$ and any $U \in \mathscr{U}(p)$ there is a set $B(n, U) \in \mathscr{B}$, $B(n, U) \subset U$ and a point $y(n, U) \in Y$ such that $F(x) \cap S(\varepsilon_n, y(n, U)) \neq \emptyset$ for any $x \in B(n, U)$. Since Y° is compact, there is a point $y \in Y^\circ$ which is an accumulation point of the net $\{y(n, U)\colon n = 1, 2, \ldots, U \in \mathscr{U}(p)\}$. It is clear that $y \in A(p)$.

Since $A(p)$ is closed in Y°, we can define a compact valued multifunction $A\colon X \to \mathscr{K}(Y^\circ)$ assigning to each point $p \in X$ the set $A(p)$. We will show that A is u.s.c. Suppose that A is not u.s.c. at a point p. That means there is an open set $V \supset A(p)$ (V open in Y°) such that for any $U \in \mathscr{U}(p)$ there is a point $p(U) \in U$ such that $A(p(U)) \setminus V \neq \emptyset$. Let $y(U) \in A(p(U)) \setminus V$. Since Y° is compact, there is a point $y \in Y^\circ \setminus V$ which is an accumulation point of the net $\{y(U)\colon U \in \mathscr{U}(p)\}$. Since $y \notin V$, $y \notin A(p)$. On the other hand, it is easy to see that $y \in A(p)$, which is a contradiction.

Now we will show that $A(p) \subset F(p)$ for any $p \in P$ where P is residual. Define a property H^+ of F at a point $x \in X$ as follows: F has the property H^+ at a point x if for any open set $V \supset F(x)$ there is a neighborhood U of x such that $F^+(V) \cap U \cap H$ is of the second category for any non-empty open set $H \subset U$ (see [4]). By [4, Remark 1.1] there is a residual set $P \subset X$ such that F has the property H^+ at any point of P. Suppose that there is $y \in A(p) \setminus F(p)$, $p \in P$. Let G, H be open disjoint and such that $y \in G$, $F(p) \subset V$. Since F has the property H^+ at p, there is a neighborhood U of p such that $F^+(V) \cap U \cap H$ is of the second category for any non-empty open set $H \subset U$. On the other hand $y \in A(p)$, hence there is $B \in \mathscr{B}$, $B \subset U$ such that $F(x) \cap G \neq \emptyset$ for any $x \in B$. Since X is Baire and B is of the second category having the Baire property, $B \cap F^+(V)$ is of the second category. For $x \in B \cap F^+(V)$ we have $F(x) \subset V$ and $F(x) \cap G \neq \emptyset$, which contradicts $V \cap G = \emptyset$.

By [4, Corollary 1 of Th. 5.3] there is a quasi-continuous selection $f\colon X \to Y^\circ$ of A. By [6, Th. 3.1.1], the set Q of points where f is continuous is residual. Let $S = P \cap Q$. S is residual and for any $x \in S$ the condition (i) holds. Since $f(p) \in A(p)$ for any $p \in X$, the condition (ii) is fulfilled. □

Theorem 1. *A multifunction* $F\colon X \to \mathscr{K}(Y)$ *is* $\mathscr{B}$*-cliquish if and only if* F *has a selection which is* $\mathscr{B}$*-continuous at any* $x \in S$ *where* S *is a residual set.*

Proof. It is evident that if F has a selection being $\mathscr{B}$-continuous on a residual set, then F is $\mathscr{B}$-cliquish on a dense set. By Remark 1 (iii), F is $\mathscr{B}$-cliquish.

Now suppose F is $\mathscr{B}$-cliquish. By Lemma 1, there is a function $f\colon X \to Y^\circ$ satisfying the conditions (i) and (ii) of Lemma 1.

Define a multifunction $G\colon X \to Y^\circ$ as follows: $G(p) = \operatorname{cl}\left(S^\circ\left(\varepsilon(p), f(p)\right)\right) \cap F(p)$ where $\varepsilon(p) = d^\circ\left(f(p), F(p)\right)$ and $\operatorname{cl}\left(S^\circ\left(\varepsilon(p), f(p)\right)\right)$ is the closure in Y° of $S^\circ\left(\varepsilon(p), f(p)\right)$. Since $F(p)$ is compact, G is a non-empty and compact valued multifunction. Moreover, $G(p) = \{f(p)\} \subset F(p)$ for any $p \in S$ by Lemma 1 (i).

We will show that G is u-$\mathscr{B}$-continuous at any $p \in S$. Let $V \supset G(p) = \{f(p)\}$ be open in Y° and let U be a neighborhood of p. Since f is continuous at p, there is an open set $H \subset U$, $p \in H$ and there is $\varepsilon > 0$ such that $f(x) \in S^\circ\left(\varepsilon/4, f(p)\right) \subset S^\circ\left(\varepsilon, f(p)\right) \subset V$ for any $x \in H$. By Lemma 1 (ii), there is a set $B \in \mathscr{B}$, $B \subset H$ such that $F(x) \cap S^\circ\left(\varepsilon/4, f(p)\right) \neq \emptyset$ for any $x \in B$. Hence $d^\circ\left(f(x), F(x)\right) < \varepsilon/2$ for $x \in B$. Since $G(x) = \operatorname{cl}\left(S^\circ\left(d^\circ\left(f(x), F(x)\right), f(x)\right)\right) \cap F(x) \subset S^\circ\left(\varepsilon/2, f(x)\right) \cap F(x) \subset S^\circ\left(\varepsilon, f(p)\right) \cap F(x) \subset V$ for any $x \in B$, G is u-$\mathscr{B}$-continuous at p. Since $G(x) \subset F(x) \subset Y$ for any $x \in X$, G is u-$\mathscr{B}$-continuous at p as a multifunction from X into Y.

Let $g\colon X \to Y$ be a selection of G. Since $G(x) \subset F(x)$ for any $x \in X$, g is a selection of F. $G\colon X \to Y$ is u-$\mathscr{B}$-continuous at any $x \in S$ and $G(x) = \{f(x)\}$ on S, g is $\mathscr{B}$-continuous on the residual set S. □

Corollary 1. *For a function $f\colon X \to Y$ the following conditions are equivalent:*

(i) *f is $\mathscr{B}$-cliquish,*

(ii) *the set of $\mathscr{B}$-continuity points of f is residual,*

(iii) *the set of $\mathscr{B}$-continuity points of f is dense.*

By [3], f is $\mathscr{G}$-cliquish iff the set of continuity points of f is residual. Thus we have

Corollary 2. *The following conditions are equivalent:*

(i) *f is $\mathscr{G}$-cliquish,*

(ii) *the set of quasi-continuity points of f is residual,*

(iii) *the set of quasi-continuity points of f is dense,*

(iv) *the set of continuity points of f is residual.*

By [4, Th. 3.3], $f\colon X \to Y$ is Baire measurable iff the set of $\mathscr{B}r$-continuity points of f is residual. Consequently, we have

Corollary 3. *Let $f\colon X \to Y$. The following conditions are equivalent:*

(i) *f is $\mathscr{B}r$-cliquish,*

(ii) *the set of $\mathscr{B}r$-continuity points of f is residual,*

(ii) *the set of $\mathscr{B}r$-continuity points of f is dense,*

(iv) *f is Baire measurable.*

Corollary 4. *If $F: X \to \mathscr{K}(Y)$ is l-$\mathscr{B}$-continuous on a dense set, then it has a selection f which is $\mathscr{B}$-continuous on a residual set. Consequently, if F is lower quasi-continuous on a dense set (F is lower-Baire measurable, i.e. $F^{-}(G)$ has the Baire property for any open set $G \subset Y$), then F has a $\mathscr{S}$-cliquish (Baire measurable) selection.*

Proof. It follows from the fact that if F is l-$\mathscr{B}$-continuous at x, then it is $\mathscr{B}$-cliquish at x. Proof of the existence of a Baire measurable selection follows from [4, Th. 3.3]. □

Definition 3. A multifunction $F: X \to \mathscr{K}(Y)$ is u-$\mathscr{D}$-continuous at a point $p \in X$ if for any open sets V, U with $F(p) \subset V$, $p \in U$ there is a set $A \subset U$ of the second category such that $A \subset U \cap F^{+}(V)$. F is u-$\mathscr{D}$-continuous if it is u-$\mathscr{D}$-continuous at any point.

Remark 2. Since the set of u-$\mathscr{D}$-continuity points is a subset of the set of points at which F has the property H^{+}, by [4, Remark 1.1] any compact valued multifunction is u-$\mathscr{D}$-continuous on a residual set.

Theorem 2. *Let $F: X \to \mathscr{K}(Y)$ be a u-$\mathscr{D}$-continuous multifunction. F has a quasi-continuous selection if and only if it is $\mathscr{B}$-cliquish.*

Proof. It is clear that if F has a quasi-continuous selection, then F is $\mathscr{B}$-cliquish. Now suppose that F is $\mathscr{B}$-cliquish. By Lemma 1, there is a quasi-continuous function $f: X \to Y^{\circ}$ such that $f(x) \in F(x)$ and f is continuous at x for any $x \in S$ where S is a residual set. Define a multifunction $A: X \to \mathscr{K}(Y^{\circ})$ as follows: $A(p) = \{y \in Y^{\circ}:$ for any open sets U, V (V open in Y°) with $y \in U$, $p \in V$ there is a set $H \in \mathscr{S}$, $H \subset U$ such that $f(H) \subset V\}$. Since f is continuous on S, $A(x) = \{f(x)\}$ for any $x \in S$. Similarlly as in the proof of Lemma 1, we can show that A is u.s.c. and a non-empty and compact valued multifunction.

Now we will show that any selection of A is quasi-continuous. Let g be a selection of A and let $p \in X$ and U, V be open (V open in Y°) with $p \in U$, $g(p) \in V$. Since $g(p) \in A(p)$, there is $H \in \mathscr{S}$ such that $H \subset U$ and $f(H) \subset V$. $A(x) = \{g(x)\} = \{f(x)\}$ for any $x \in S$, hence $g(H \cap S) \subset V$. Thus g is $\mathscr{B}r$-continuous. By [4, Th. 2.5], g is quasi-continuous.

Now it is sufficient to show that $A(p) \cap F(p) \neq \emptyset$ for any $p \in X$. Suppose that $A(p) \cap F(p) = \emptyset$. Hence there are open disjoint sets G, W such that $G \supset A(p)$ and

$W \supset F(p)$. Since A is u.s.c., $p \in \text{int}(A^+(G))$. F is u-$\mathscr{D}$-continuous at p, hence there is a set T of the second category such that $T \subset (\text{int}(A^+(G))) \cap F^+(W)$. Thus for $x \in T \cap S$ we have $A(x) = \{f(x)\} \subset G$ and $F(x) \subset W$. Since $G \cap W = \emptyset$, we have a contradiction to the fact that $f(x) \in F(x)$ for $x \in S$. □

Corollary 5. *If $F : X \to \mathscr{K}(Y)$ is u-$\mathscr{B}$-continuous, then it has a quasi-continuous selection.*

Proof. By [4, Th. 2.1], F is l.s.c. except for a set of the first category. Hence F is $\mathscr{B}$-cliquish and the proof follows from Theorem 2. □

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