CONVEXITIES OF LATTICE ORDERED GROUPS

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Summary. In this paper an injective mapping of the class of all infinite cardinals into the collection of all convexities of lattice ordered groups is constructed; this generalizes an earlier result on convexities of $d$-groups.

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The notion of convexity of lattices has been introduced by E. Fried ([9], p. 225; cf. also [4]). By applying analogous postulates we can define convexities also for other types of ordered algebraic structures.

In the present paper the collection $C(\mathcal{L})$ of all convexities of lattice ordered groups will be investigated.

An injective mapping of the class of all infinite cardinals into the collection $C(\mathcal{L})$ will be constructed; hence $C(\mathcal{L})$ is a proper class. This generalizes a result from [6] concerning convexities of $d$-groups.

The notion of torsion class is due to J. Martinez [8]. For some torsion classes (which have been studied in literature) we shall deal with the question whether they are convexities.

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1. Preliminaries

We shall apply the standard notation for lattice ordered groups. The group operation in a lattice ordered group will be written additively; the commutativity of this operation will not be assumed.

Let \( \mathcal{L} \) be the class of all lattice ordered groups. A nonempty subclass of \( \mathcal{L} \) will be said to be a convexity of lattice ordered groups if it is closed under homomorphic images, convex \( \ell \)-subgroups, and direct products.

We denote by \( \mathcal{C}(\mathcal{L}) \) the collection of all convexities of lattice ordered groups. This collection is partially ordered by inclusion. The least element of \( \mathcal{C}(\mathcal{L}) \) is the class \( X_0 \) consisting of all one-element lattice ordered groups.

Let \( \emptyset \neq X \subseteq \mathcal{L} \). We denote by

- \( HX \) — the class of all homomorphic images of elements of \( X \);
- \( CX \) — the class of all convex \( \ell \)-subgroups of elements of \( X \);
- \( PX \) — the class of all direct products of elements of \( X \).

1.1. Lemma. Let \( \emptyset \neq X \subseteq \mathcal{L} \). Then

(i) \( HCPX \in \mathcal{C}(\mathcal{L}) \);
(ii) for each \( Y \in \mathcal{C}(\mathcal{L}) \) with \( X \subseteq Y \) the relation \( HCPX \subseteq Y \) is valid.

The proof will be omitted. For analogous results concerning convexities of lattices and convexities of \( \ell \)-groups cf. [9], p. 256 and [6].

In view of 1.1. the convexity \( HCPX \) will be said to be generated by \( X \).

The direct product of lattice ordered groups \( A \) and \( B \) will be denoted by \( A \times B \). If \( I \) is any nonempty system of indices and \( G_i \in \mathcal{L} \) for each \( i \in I \), then \( \prod_{i \in I} G_i \) denotes the direct product of the system \( \{G_i\}_{i \in I} \). If \( I = \emptyset \), then we put \( \prod_{i \in I} G_i = \{0\} \).

When no confusion can occur, then for \( j \in I \) the lattice ordered group \( G_j \) will be identified with the \( \ell \)-subgroup of \( \prod_{i \in I} G_i \) consisting of all elements \( g \) of the direct product under consideration such that \( g(i) = 0 \) for each \( i \in I \setminus \{j\} \).

If \( G \in \mathcal{L} \), \( g \in G \) and if \( D \) is an \( \ell \)-ideal of \( G \), then we put \( \overline{X} = x + D \); for \( X \subseteq G \) we set \( \overline{X} = \{x: x \in X\} \).

We will apply below the following well-known results:

1.2. Lemma. Let \( G \in \mathcal{L} \), \( G = A \times B \) and let \( D \) be a convex \( \ell \)-subgroup of \( G \). Then \( D = (A \cap D) \times (B \cap D) \).

1.3. Lemma. Let \( G \), \( A \) and \( B \) be as in 1.2. Let \( D \) be an \( \ell \)-ideal of \( G \). Then \( \overline{G} = G/D = \overline{A} \times \overline{B} \).
2. THE LATTICE ORDERED GROUPS $G_\alpha$

For each infinite cardinal $\alpha$ we denote by $J_\alpha$ the first ordinal having the power $\alpha$. The additive group of all integers with the natural linear order will be denoted by $\mathbb{Z}$. Let $\alpha$ be a fixed infinite cardinal and for each $j \in J_\alpha$ let $P_j = \mathbb{Z}$. Now let $Q'(\alpha)$ be the lexicographic product of the system $\{P_j\}(j \in J_\alpha)$. The $\ell$-subgroup of $Q'(\alpha)$ consisting of all elements $q'$ such that the set $\{j \in J(\alpha) : q'(j) \neq 0\}$ is finite will be denoted by $Q(\alpha)$.

Let $G_\alpha$ be the set of all triples $(x, y, z)$ such that $x, y \in Q(\alpha)$ and $z \in \mathbb{Z}$. For $(x_1, y_1, z_1), (x_2, y_2, z_2) \in G_\alpha$ we put $(x_1, y_1, z_1) \leq (x_2, y_2, z_2)$ if either

(i) $z_1 < z_2$,

or

(ii) $z_1 = z_2$ and $x_1 \leq x_2, y_1 \leq y_2$.

Next we define the binary operation $+$ in $G_\alpha$ as follows.

a) If $z_1$ is even, then we put

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2).$$

b) If $z_1$ is odd, then we define

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + y_2, y_1 + x_2, z_1 + z_2).$$

Then $G_\alpha$ is a non-abelian lattice ordered group. Clearly $\text{card} G_\alpha = \alpha$.

The class of all infinite cardinals will be denoted by $J$.

2.1. Lemma. Let $\alpha, \beta \in J$, $\beta < \alpha$. Then $G_\beta$ does not belong to the class $\text{HCP}(G_\alpha)$.

Proof. By way of contradiction, assume that $G_\beta$ belongs to $\text{HCP}(G_\alpha)$. Thus there exist $B \in \text{CP}(G_\alpha)$ and an $\ell$-ideal $D$ of $B$ such that $B/D$ is isomorphic to $G_\beta$. Next, there is an indexed system $\{A_i\}_{i \in I}$ of lattice ordered groups such that $A_i = G_\alpha$ for each $i \in I$ and $B$ is a convex $\ell$-subgroup of the lattice ordered group $A = \bigoplus_{i \in I} A_i$.

For $a \in A$ we denote by $a_i$ the component of $a$ in the direct factor $A_i$. Let $b \in B \setminus D$, $i \in I$, $b_i = (x_i, y_i, z_i)$. If for each such $b_i$ and each $i \in I$ the relation $z_i = 0$ is valid, then $B/D$ is commutative; since $G_\beta$ fails to be commutative, we obtain a contradiction. Therefore there exists $b \in B \setminus D$ such that $z_i \neq 0$ for some $i \in I$. Denote

$I_1 = \{i \in I : z_i \neq 0\}$, $I_2 = I \setminus I_1$. 

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Hence $I_1 \neq \emptyset$. Put

$$A^1 = \prod_{i \in I_1} A_i \quad (i \in I_1), \quad A^2 = \prod_{i \in I_2} A_i \quad (i \in I_2),$$

$$B^1 = B \cap A^1 \quad B^2 = B \cap A^2.$$

Then $A = A^1 \times A^2$, hence in view of 1.2 and 1.3,

(1) $B/D = B_1 \times B_2$.

There exist elements $b'$ and $b''$ in $A$ such that

$$b' = (x_i, y_i, 0)_{i \in I}, \quad b'' = (0, 0, \ast)_{i \in I}.$$  

For each $i \in I$ we have

$$-|b_i| \leq b'_i \leq |b_i|, \quad -2|b_i| \leq b''_i \leq 2|b_i|,$$

hence both $b'$ and $b''$ belong to $B$.

For each $t \in B$ we put $t = t + D$. Clearly $b = b' + b''$ and $\bar{b} = \bar{b}' + \bar{b}''$. If $\bar{b}'' = \emptyset$ (i.e., $\bar{b} = \bar{b}'$) for all $b \in B$ with the above mentioned properties, then $B/D$ would be abelian, which is impossible. Hence without loss of generality we can suppose that $b = b''$. Further, we can suppose that $b' > 0$.

We have $b \in B$ and $\bar{b} \neq \emptyset$, whence $B''$ is a nonzero lattice ordered group. It is obvious that $G_B$ is directly indecomposable, thus so is $B/D$. Hence (1) yields that $B/D = B_1$. Therefore we can assume without loss of generality that $I = I_1$, whence

(2) $z_i > 0$ for each $i \in I$.

The relation (2) yields that whenever $b^i \in A$ such that $b^i = (x^i, y^i, 0)$ for each $i \in I$, then $-b \leq b^i < b$, whence $b^i \in B$.

If for each $b^i$ with the above mentioned properties the relation $b^i \in D$ holds, then $B/D$ is commutative, which is a contradiction. Hence among the elements $b^i$ under consideration there exists at least one with $b^i \neq D$. Below we deal with this fixed $b^i$.

Let $b^{i_1}$ be the element of $A$ with $b^{i_1} = (x^i, 0, 0)$ for each $i \in I$, similarly, let $b^{i_2} \in A$ such that $b^{i_2} = (0, y^i, 0)$ for each $i \in I$. Then either $b^{i_1}$ or $b^{i_2}$ does not belong to $D$. Without loss of generality we can suppose that $b^{i_1} \neq D$ and that $b^{i_1} > 0$.

Let $I_{11} = \{i \in I: b^{i_1} \neq 0\}$, $I_{12} = I \setminus I_{11}$. Next, we put

$$B^{i_1} = \{t \in B: t_i = 0 \quad \text{for each} \ i \in I_{12}\},$$

$$B^{i_2} = \{t \in B: t_i = 0 \quad \text{for each} \ i \in I_{12}\}.$$
Then $B = B^{11} \times B^{12}$. Hence in view of 1.3,

$$
B/D = B^{11} \times B^{12}.
$$

Clearly $b^{11} \in B^{11} \setminus D$, therefore $b^{11} \in B^{11}$ and $b^{11} \neq \emptyset$. Thus $B^{11} \neq \emptyset$. From the fact that $G_D$ is directly indecomposable we obtain that $B/D = B^{11}$. Now it is obvious that instead of $A$ and $B$ it suffices to take the lattice ordered groups

$$
\prod_{i \in I_1} A_i, \quad B \cap \prod_{i \in I_1} A_i,
$$

respectively. This means that without loss of generality we can suppose the validity of the relation $I = I_1$. Hence $b^{11}_i > 0$ for each $i \in I$. Hence $x^1_i > 0$ for each $i \in I$.

Now we apply the fact that $x^1_i$ belongs to $Q_a$. Let $j(i)$ be the least element of $J_a$ with $x^1_i(j(i)) \neq 0$.

In view of the definition of $J_a$ there exists a monotone injection $\psi_i$ of $J_a$ onto

$$
\{j \in J_a : j \geq j(i)\}.
$$

Let $J_a^0$ be the set of all elements of $J_a$ which are distinct from the least element of $J_a$. We construct the elements $b^j (j \in J_a^0)$ in $A$ as follows. For each $i \in I$ and $j \in J_a$ let $b^j_i = (x^1_i, 0, 0)$ where, for each $j(1) \in J_a$, we have

$$
x^1_i(j(1)) = 1 \quad \text{if } j(1) = \psi_i(j) \quad \text{and}
$$

$$
x^1_i(j(1)) = 0 \quad \text{otherwise}.
$$

Thus $b^j \in B$ for each $j \in J^0_a$.

If $j(1)$ and $j(2)$ are elements of $J_a^0$ with $j(1) < j(2)$, then

$$
|b(1)| < |b(2)| - b(1).
$$

Hence if $b(2) - b(1) \in D$, we would have $b^j \in D$, which is a contradiction. Therefore $b^{(1)} + D$ and $b^{(2)} + D$ are distinct elements of $B/D$. Thus $\text{card}(B/D) \geq \text{card} J^0_a = \alpha$.

On the other hand, $\text{card}(B/D) = \text{card} G_D = \beta$ and so we arrived at a contradiction.

\[ \Box \]

2.2. Theorem. For each infinite cardinal $\alpha$ let $\varphi(\alpha) = \text{HC}(G_{\alpha})$, where $G_{\alpha}$ is as above. Then $\varphi$ is an injective mapping of the class $J$ of all infinite cardinals into the collection of all convexities of lattice ordered groups.

Proof. This is a consequence of 1.1. and 2.1. \[ \Box \]

We apply the notion of a $d$-group in the same sense as in the paper of Kopytov and Dimitrov [7]; cf. also [5]. Convexities of $d$-groups were investigated in [6].

Let $D$ be the class of all $d$-groups and $C(D)$ the collection of all convexities of $d$-groups. Since the class $L$ of all lattice ordered groups is a variety in $D$ (cf. [7])
and since each variety in $\mathcal{D}$ is an element of $C(\mathcal{D})$, we conclude that each convexity of lattice ordered groups is, at the same time, a convexity of $d$-groups. In fact, $C(\mathcal{L})$ is an interval of $C(\mathcal{D})$. Thus 2.2 implies

2.3. Corollary. (Cf. [6].) There exists an injective mapping of the class of all infinite cardinals into the collection $C(\mathcal{D})$.

3. CONCLUDING REMARKS; RADICAL CLASSES AND TORSION CLASSES

3.1. Each variety of lattice ordered groups is a convexity. This is an immediate consequence of the definition of convexity.

3.2. A convexity of lattice ordered groups need not be closed with respect to $t$-subgroups. For example, let $G_\alpha$ and $G_\beta$ be as in Section 2 ($\alpha$ and $\beta$ are infinite cardinals with $\beta < \alpha$). Then $G_\beta$ is isomorphic to an $t$-subgroup of $G_\alpha$, but $G_\beta$ does not belong to the convexity generated by $G_\alpha$.

3.3. A nonempty class $X$ of lattice ordered groups is said to be closed under joins of convex $t$-subgroups if, whenever $G \in \mathcal{L}$ and $\{G_i\}_{i \in I}$ is a system of convex $t$-subgroups of $G$ such that $G_i$ belongs to $X$ for each $i \in I$, then the join $\bigvee_{i \in I} G_i$ also belongs to $X$.

A nonempty class $Y$ of lattice ordered groups is called a radical class [3] if it is closed under isomorphisms, convex $t$-subgroups and joins of convex $t$-subgroups.

A radical class which is closed under direct products is called a product radical class; this notion was studied by Dao Rong Ton [2]. Hence a product radical class which is closed under homomorphic images is a particular case of convexity.

A radical class of lattice ordered groups need not be a convexity. For example, the class of all archimedean lattice ordered groups is a radical class, but it fails to be a convexity (since it is not closed under homomorphic images).

3.4. A radical class which is closed under homomorphic images is called a torsion class (Martinez [8]). A torsion class is a convexity if it is closed under direct products.

The main results of Conrad’s paper [1] consist in a detailed investigation of torsion classes $A, F, F, D, O, R$ and $B$ (for definitions of these classes cf. below; they have been studied also in other papers). Let us consider the question which of these torsion classes are convexities.

The torsion classes under consideration are defined as follows:

$A$—all hyperarchimedean lattice ordered groups;
F—all lattice ordered groups such that each bounded disjoint subset is finite;
F”—all finite valued lattice ordered groups;
D—all lattice ordered groups whose regular subgroups satisfy the descending chain condition;
O—all cardinal sums of linearly ordered groups;
R—all cardinal sums of archimedean linearly ordered groups;
B—all lattice ordered groups such that each prime exceeds a unique minimal prime.

Let $Z$ be as above (cf. Section 2). Then $Z$ belongs to each of the torsion classes under consideration. Let $I$ be an infinite set and for each $i \in I$ let $G_i = Z$, $G = \prod_{i \in I} G_i$.

3.4.1. Suppose that $I = \mathbb{N}$ (the set of all positive integers). Let $f$ and $g$ be elements of $G$ such that $f(n) = n$ and $g(n) = 1$ for each $n \in \mathbb{N}$. Then $f \land g < f \land (n+1)g$ for $\forall n \in \mathbb{N}$. Hence in view of Theorem 1.1. in [1] the lattice ordered group $G$ is not hyperarchimedean. Therefore $A$ is not a convexity.

3.4.2. $G$ does not belong to $F$, hence $F \not\subseteq C(L)$.

3.4.3. Let $g \in G$ be such that $g(i) = 1$ for each $i \in I$. Then $g$ has infinitely many values in $G$, hence $G \not\subseteq F_\infty \not\subseteq C(L)$.

3.4.4. Let $G$ be as in 3.4.1 and for each $j \in I$ let $G_j = \{g \in G : g(i) = 0 \text{ for each } i \in I \text{ with } i < j\}$. Then $G_1 \supset G_2 \supset G_3 \supset \ldots$ and the set $\{G_n\}_{n \in I}$ has no minimal element. Also, all $G_n$ are regular subgroups of $G$. Hence $G \not\subseteq D$ and so $D \not\subseteq C(L)$.

3.4.5. The lattice ordered group $G$ does not belong to $O$, hence $G \not\subseteq R$. Therefore $O \not\subseteq C(L)$ and $R \not\subseteq C(L)$.

3.4.6. Now we will show that $B$ is a convexity. We will apply the following result (cf. [1], p. 492):

(*) For each lattice ordered group $G$ the following are equivalent:

(i) $G \in B$.

(ii) Each pair of incomparable primes in $G$ generates $G$.

Let $I$ be a nonempty set of indices and for each $i \in I$ let $B_i$ be a lattice ordered group belonging to $B$ with $B_i \neq \{0\}$. Put $G = \prod_{i \in I} B_i$. We have to verify that $G$ belongs to $B$ as well.

First we consider the question what is the general form of primes in $G$. Let $H^1$ be a prime in $G$. Let $I(H^1)$ be the set of all $i \in I$ having the property that there is $g \in G \setminus H^1$ such that $g(i) \neq 0$. Then $I(H^1) \neq \emptyset$.
Suppose that \( i(1) \) and \( i(2) \) are distinct elements of \( I(H^1) \). Put \( G_{i(1)} = \{ g' \in G : g'(i(1)) = 0 \} \), and let \( G_{i(2)} \) be defined analogously. Next, let

\[
Q_1 = H^1 + G_{i(1)}, \quad Q_2 = H^1 + G_{i(2)}.
\]

Then \( Q_1 \) and \( Q_2 \) are convex \( \ell \)-subgroups of \( G \) and \( H^1 \subseteq Q_j \) (\( j = 1, 2 \)). We have neither \( Q_1 \subseteq Q_2 \) nor \( Q_2 \subseteq Q_1 \). Nonetheless, since \( H^1 \) is prime, the system of all convex \( \ell \)-subgroups \( Q \) of \( G \) with \( H^1 \subseteq Q \) is linearly ordered; hence we arrive at a contradiction. Therefore \( I(H^1) \) is a one-element set, \( I(H^1) = \{ i(1) \} \). Hence \( G_{i(1)} \subseteq H^1 \) and thus

\[
H^1 = (H^1 \cap G_{i(1)}) + G_{i(1)}.
\]

It is easy to verify that \( H^1 \cap G_{i(1)} \) is a prime subgroup of \( H^1 \).

Conversely, if \( H^1 \in C(G) \), \( i(1) \in I \), \( H^1 \cap G_{i(1)} \) is a prime in \( G_{i(1)} \) and if (4) holds, then \( H^1 \) is a prime in \( G \).

Let \( H^2 \) be a prime in \( G \) such that \( H^1 \) and \( H^2 \) are incomparable. There is \( i(2) \in I \) such that \( I(H^2) = \{ i(2) \} \). Analogously as above we have

\[
H^2 = (H^2 \cap G_{i(2)}) + G_{i(2)}.
\]

We distinguish two cases.

(i) First suppose that \( i(1) \neq i(2) \). Then \( G_{i(1)} + G_{i(2)} = G \), whence the pair \( H^1 \) and \( H^2 \) generates \( G \).

(ii) Next suppose that \( i(1) = i(2) \). Denote

\[
H^3_1 = H^1 \cap G_{i(1)}, \quad H^3_2 = H^2 \cap G_{i(1)}.
\]

Then \( H^3_1 \) and \( H^3_2 \) are incomparable primes in \( G_{i(1)} \). Thus, since \( G_{i(1)} \) belongs to \( B \), in view of (4 *) the pair \( H^3_1 \) and \( H^3_2 \) generates \( G_{i(1)} \). Therefore the pair \( H^1 \) and \( H^2 \) generates \( G \).

By applying (4 *) again we infer that \( G \) belongs to \( B \).

References


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