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Locally inner derivations of standard operator algebras


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Summary. It is proved that every locally inner derivation on a symmetric norm ideal of operators is an inner derivation.

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INTRODUCTION

Let $X$ be a complex Banach space, let $B(X)$ denote the algebra of all bounded linear operators on $X$ and suppose that $A \subset B(X)$ is a subalgebra.

The linear operator $\delta : A \rightarrow B(X)$ is called a derivation if

$$\delta(TS) = T\delta(S) + \delta(T)S \quad (T, S \in A).$$

If there is an $S \in A$ for which $\delta(T) = TS - ST$ ($T \in A$), then we say that $\delta$ is an inner derivation.

There are two different types of operator algebras on which derivations are extensively studied. Namely, standard operator algebras, which are subalgebras of $B(X)$ containing the ideal $\mathcal{F}(X)$ of finite rank operators, and von Neumann algebras. From their vast literature we refer only to [1, 2, 5-9] which are in closer relation to our present considerations. The main result concerning the structure of derivations on a symmetric norm ideal is the following theorem:

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standard operator algebra $\mathcal{A}$ states that every derivation $\delta: \mathcal{A} \to \mathcal{B}(X)$ is spatially implemented, i.e., it is of the form $\delta(T) = TA - AT$ with a fixed $A \in \mathcal{B}(X)$ [2, 9]. Moreover, there is a classical theorem due to Sakai [7] stating that every derivation on a von Neumann algebra into itself is inner.

The important concept of local derivations was introduced by Larson, Sourour and Kadison [6, 5] as follows. A linear operator $\theta: \mathcal{A} \to \mathcal{B}(X)$ is said to be a local derivation if for every $T \in \mathcal{A}$ there exists a derivation $\delta_T: \mathcal{A} \to \mathcal{B}(X)$ such that

$$\theta(T) = \delta_T(T).$$

It was proved in [6, Theorem 1.2] that in the case $\mathcal{A} = \mathcal{B}(X)$, every local derivation is a derivation (cf. [1]). This statement can be easily extended to the case when $\mathcal{A}$ is an arbitrary standard operator algebra. Indeed, if $I \in \mathcal{A}$, then this follows from the proof of [6, Theorem 1.2]. Otherwise consider the local derivation $T + \lambda I \mapsto \theta(T)$ on the algebra $\mathcal{A} \oplus \mathbb{C}I$.

In [5] Kadison studied similar questions for von Neumann algebras. He proved that every norm-continuous local derivation on a von Neumann algebra into itself is a derivation. Since the derivations of a von Neumann algebra are inner, one can reformulate this theorem in the following way. Every norm-continuous locally inner derivation on a von Neumann algebra into itself is an inner derivation. Here, a linear mapping $\theta: \mathcal{A} \to \mathcal{B}(X)$ defined on the subalgebra $\mathcal{A} \subset \mathcal{B}(X)$ is called a locally inner derivation if for every $T \in \mathcal{A}$ there exists $S \in \mathcal{A}$ such that

$$\theta(T) = TS - ST.$$
If $s$ is a Calkin space, then the family $\mathcal{I}(s)$ of compact operators whose decreasing sequences of singular values (extended with 0's in the finite case) belong to $s$ is a proper ideal of $B(H)$.

Moreover, we have $\mathcal{I}(s(\mathcal{I})) = \mathcal{I}$ and $s(\mathcal{I}(s)) = s$.

Let $f$ denote the subspace of $c_0$ consisting of all sequences with finitely many non-zero elements. A norm $\Phi$ on $f$ is called symmetric if $\Phi(a) = \Phi(a^*)$ for every $a \in f$.

Let $s_\Phi = \{ a \in c_0: \lim_{n \to \infty} \Phi(a_1, \ldots, a_n, 0, 0, \ldots) \text{ exists and is finite} \}$.

Then $s_\Phi$ is a Calkin space onto which $\Phi$ can be extended as a Banach space norm in a natural way. The property of $s_\Phi$ that is most important for us can be found in [10, Theorem 1.16].

If $(\sigma_{nm})_{n,m \in \mathbb{N}}$ is a doubly substochastic matrix (i.e. $\sum_{n=1}^\infty |\sigma_{nk}| \leq 1$, $\sum_{m=1}^\infty |\sigma_{lm}| \leq 1$ (k, l $\in \mathbb{N}$)), then $(\sigma_{nm})^T \in s_\Phi$ for every $a \in s_\Phi$.

The ideal $\mathcal{I}(s_\Phi)$ is called a symmetric norm ideal on which there is a Banach space norm that is in a natural relation with $\Phi$. Note that $\mathcal{I}(f) = \mathcal{I}(s_\Phi)$ cannot occur, which follows from the fact that there is no Banach space norm on $f$. The standard examples of symmetric norm ideals are the Schatten-von Neumann $p$-classes of compact operators.

**Main results**

In the proofs of our results we shall need the following observation. Let $(e_n)_{n \in \mathbb{N}}$ be a c.o.n.s. (complete orthonormal system) in $H$. Suppose that $(\mu_n)_{n \in \mathbb{N}}$ is a bounded sequence of non-zero complex numbers and $(\lambda_n)_{n \in \mathbb{N}} \in \ell_\infty$. If

$$A = \sum_{n=1}^\infty \lambda_n e_n \otimes e_n \quad \text{and} \quad T = \sum_{n=1}^\infty \mu_n e_n \otimes e_{n+1},$$

then for arbitrary $S \in B(H)$ the equation

$$TA - AT = TS - ST$$

implies

$$\sum_{n=1}^\infty (\lambda_{n+1} - \lambda_n) \mu_n e_n \otimes e_{n+1} = \sum_{n=1}^\infty \mu_n e_n \otimes S^* e_{n+1} - \sum_{n=1}^\infty \mu_n S e_n \otimes e_{n+1}.$$
Considering the operators on both sides at $e^{n+1}$ and then taking inner products with $e_n$, we obtain that

$$(\lambda_{n+1} - \lambda_n)\mu_n = \mu_n\langle e_{n+1}, S^*e_{n+1} \rangle - \mu_n\langle Se_n, e_n \rangle,$$

which implies

$$\lambda_n - \lambda_1 = \langle Se_n, e_n \rangle - \langle S1, e_1 \rangle \quad (n \in \mathbb{N}).$$

Our first assertion will ensure the existence of a standard operator algebra on which not every locally inner derivation is inner.

**Theorem 1.** There are exactly three standard operator algebras on which every derivation is a locally inner derivation, namely $\mathcal{F}(H)$, $\mathcal{F}(H) \oplus CI$ and $B(H)$.

**Proof.** We know that every derivation on a standard operator algebra is of the form $T \mapsto TA - AT$ with an appropriate $A \in B(H)$. Hence, to prove that $\mathcal{F}(H)$, $\mathcal{F}(H) \oplus CI$ and $B(H)$ have the property in the statement, it is sufficient to consider only the case of $\mathcal{F}(H)$. But if $A \in B(H)$ is arbitrary and $T \in \mathcal{F}(H)$, then for the orthogonal projection $P$ onto the finite-dimensional subspace generated by the ranges of $T$ and $(TA)^*$ we have

$$TA - AT = T(AP) - (AP)T.$$ 

Now, let $\mathcal{A}$ be a standard operator algebra on which every derivation is locally inner. We infer

$$TA - AT \in \mathcal{A} \quad (T \in \mathcal{A}, \ A \in B(H)).$$

This means that $\mathcal{A}$ is a so-called Lie ideal. Then, by [3, Theorem 2] there is an (associative) ideal $\mathcal{I}$ in $B(H)$ for which

$$[\mathcal{I}, B(H)] \subset \mathcal{A} \subset \mathcal{I} + CI,$$

where $[\mathcal{I}, B(H)]$ denotes the subspace generated by the elements of $TA - AT$ ($T \in \mathcal{I}$, $A \in B(H)$). Since for arbitrary $A \in B(H)$ and $T \in \mathcal{I}$ we have $TA - AT \in \mathcal{A}$, thus there is an $S \in \mathcal{I}$ such that


We prove that $\mathcal{I} = \mathcal{F}(H)$ or $\mathcal{I} = B(H)$. To this end, suppose that there is an element in $\mathcal{I}$ with an infinite dimensional range. Then the set $\{\mu_n\}_{n \in \mathbb{N}}$ of its singular values
is certainly infinite. Let \((\lambda_n)_{n \in \mathbb{N}} \in \ell_\infty\) be a non-convergent sequence with pairwise different elements. Let \(A\) and \(T\) be just as in the first part of this section. We have 

\[ T \in \mathcal{I}. \text{ If } S \in \mathcal{I} \text{ is such that (2) holds, then just as above we can arrive at} \]

\[ \lambda_n - \lambda_1 = \langle S e_n, e_n \rangle = \langle S e_1, e_1 \rangle \quad (n \in \mathbb{N}). \]

However, in this case \((S e_n, e_n) \to 0\), i.e. \(S\) cannot be compact. Since \(\mathcal{I}\) contains a non-compact element, it follows that \(\mathcal{I} = B(H)\).

To complete the proof, observe that in the case \(\mathcal{I} = \mathcal{F}(H)\) we have \(A = \mathcal{F}(H)\) or \(A = \mathcal{F}(H) \oplus \mathcal{C}I\). Moreover, by \([B(H), B(H)] = B(H)\) (e.g. [3, Corollary 2, p. 187]), \(\mathcal{I} = B(H)\) implies \(A = B(H)\).  

Turning back to our remark right before Theorem 1, consider the standard operator algebra \(\mathcal{F}(H)\). If we show that there is a derivation on it which is not inner, then by the previous theorem this will mean the existence of a locally inner derivation which is not inner. In fact, since the commutant of \(\mathcal{F}(H)\) is \(\mathcal{C}I\), it is easy to see that a derivation \(T \mapsto TA - AT\) on this algebra is inner if and only if \(A\) is the sum of a finite rank operator and a scalar multiple of the identity. Now, take an \(A \in B(H)\) which cannot be written in such a form to obtain the desired statement.

**Theorem 2.** Let \(\mathcal{I}\) be a symmetric norm ideal in \(B(H)\). Then every locally inner derivation on \(\mathcal{I}\) is an inner derivation.

**Proof.** Suppose that \(\mathcal{I}\) is non-trivial. Then every locally inner derivation on \(\mathcal{I}\) is a derivation. Let \(A \in B(H)\) be such that for every \(T \in \mathcal{I}\) there is an \(S_T \in \mathcal{I}\) for which 

\[ TA - AT = T S_T - S_T T. \]

Considering this relation for \(T^*\) and taking adjoints we immediately have 

\[ TA^* - A^* T = T S_T^* - S_T^* T. \]

It follows from these equations that the self-adjoint operators \(Re\ A\) and \(Im\ A\) induce locally inner derivations on \(\mathcal{I}\). Consequently, we may suppose that \(A\) is self-adjoint.

By the well-known Weyl-von Neumann theorem there are a c.o.n.s. \(\{e_n\}_{n \in \mathbb{N}}\), a bounded sequence \((\nu_n)_{n \in \mathbb{N}}\) of real numbers and a compact self-adjoint operator \(K\) such that 

\[ A = \sum_{n=1}^{\infty} \nu_n e_n \otimes e_n + K. \]

Since \(\mathcal{I} \neq \mathcal{F}(H)\), \(\mathcal{I}\) must contain an operator with an infinite dimensional range. Let \(T\) be such as in the proof of Theorem 1. Since there is a compact operator \(S\) for which 

\[ T(A - K) - (A - K)T = TS - ST, \]
thus (1) implies that $(\nu_n)_{n \in \mathbb{N}}$ converges to a real $\nu$. Let

$$A' = A - \nu I,$$

which is a compact self-adjoint operator inducing the same locally inner derivation as $A$ does. Consequently, we may suppose even that $A$ is compact.

But such an $A$ can be written in the form

$$A = \sum_{n=1}^{\infty} \lambda_n e_n \otimes e_n,$$

where $\lambda_n \to 0$ and $(e_n)_{n \in \mathbb{N}}$ is a c.o.n.s. Let $T \in \mathcal{I}$ be just as above and let $S \in \mathcal{I}$ be such that

$$TA - AT = TS - ST.$$

From (1) we then have

$$\lambda_n = \langle Se_n, e_n \rangle \quad (n \in \mathbb{N}).$$

If $S = \sum a_k \psi_k \otimes \varphi_k$ is the canonical expansion of $S$, then

$$\langle Se_n, e_n \rangle = \sum_k a_k \langle e_n, \varphi_k \rangle \langle \psi_k, e_n \rangle.$$

Since

$$\sum_k |\langle e_n, \varphi_k \rangle| |\langle \psi_k, e_n \rangle| \leq \|e_n\|^2 = 1,$$

$$\sum_k |\langle e_n, \varphi_k \rangle| |\langle \psi_k, e_n \rangle| \leq \|\varphi_k\| \|\psi_k\| \leq 1,$$

by the introduction we have $A \in \mathcal{I}$. \hfill \Box

To conclude the paper we propose the following open question. Is there a non-trivial ideal of $\mathcal{B}(\mathcal{H})$ different from $\mathcal{F}(\mathcal{H})$ for which the conclusion of Theorem 2 fails?

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References


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