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OSCILLATION CRITERIA FOR SECOND ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. Our aim in this paper is to present sufficient conditions for the oscillation of the second order neutral differential equation

\[(x(t) - px(t - \tau))'' + q(t)x(\sigma(t)) = 0.\]

Keywords: neutral equation, delayed argument

MSC 1991: 34C10, 34K10

In this paper we deal with the oscillatory and asymptotic behavior of the solutions of the neutral differential equation. We consider the second order differential equation of the form

\[ \left( x(t) - px(t - \tau) \right)'' + q(t)x(\sigma(t)) = 0 \]

under the assumptions

(i) \( p \) and \( \tau \) are positive numbers;
(ii) \( q, \sigma \in C(\mathbb{R}_+, \mathbb{R}_+), \lim_{t \to \infty} \sigma(t) = \infty, \sigma(t) \leq t. \)

We put \( z(t) = x(t) - px(t - \tau). \) By a proper solution of Eq. (1) we mean a function \( x: [T_0, \infty) \to \mathbb{R} \) which satisfies (1) for all sufficiently large \( t \) and \( \sup\{ |x(t)| : t \geq T \} > 0 \) for any \( T \geq T_0 \) so that \( z(t) \) is twice continuously differentiable. Such a solution is called oscillatory if it has a sequence of zeros tending to infinity; otherwise it is called nonoscillatory. Eq. (1) is said to be oscillatory if all its solutions are oscillatory.

In the recent years there has been a growing interest in oscillation theory of functional differential equations of neutral type, see for example the papers [1–11] and...

The purpose of this paper is to establish criteria for oscillation problems of Eq. (1). We discuss Eq. (1) for the cases that $0 < p < 1$, $p = 1$ and $p > 1$, respectively.

In this paper we have been motivated by the fact that the existing criteria for oscillation problems of Eq. (1) in general do not contain the constant $p$ explicitly (see e.g. [3-5, 7, 8]). Further, we introduce some new techniques that yield a generalization of some criteria presented in the book by L. H. Erbe, Q. Kong and B. G. Zhang.

**Theorem 1.** Let $0 < p < 1$. Assume that

$$
\sigma \in C^1, \quad \sigma'(t) \geq 0.
$$

Further assume that there exists an integer $n \geq 0$ such that

$$
\int_0^\infty \left( q(s)\sigma(s) \frac{1 - p^{n+1}}{1 - p} - \frac{\sigma'(s)}{4\sigma(s)} \right) ds = \infty.
$$

Then the nonoscillatory solutions of Eq. (1) tend to zero as $t \to \infty$.

**Proof.** Without loss of generality let $x(t)$ be an eventually positive solution of Eq. (1) and define

$$
z(t) = x(t) - px(t - \tau).
$$

From Eq. (1) we have $z''(t) < 0$ for all large $t$, say $t \geq t_0$. If $z'(t) < 0$ eventually, then $\lim_{t \to \infty} z(t) = -\infty$. But $z(t) < 0$ eventually implies that

$$
x(t) < px(t - \tau) < p^2 x(t - 2\tau) < \ldots < p^n x(t - n\tau)
$$

for all large $t$, which implies $\lim_{t \to \infty} x(t) = 0$. Consequently, $\lim_{t \to \infty} z(t) = -\infty$, a contradiction.

Therefore, $z'(t) > 0$ for $t \geq t_0$. There are two possibilities for $z(t)$:

(a) $z(t) > 0$ for $t \geq t_1 \geq t_0$,

(b) $z(t) < 0$ for $t \geq t_1$.

For case (a), Eq. (1) can be written in the form

$$
z''(t) + q(t)x(\sigma(t)) = 0.
$$
Using (4) we have
\[ z''(t) + q(t)z(\sigma(t)) + pq(t)x(\sigma(t) - \tau) = 0. \]
Repeating this procedure we arrive at
\[ (5) \quad z''(t) + q(t) \sum_{i=0}^{n} p^i z(\sigma(t) - i\tau) + p^{n+1} q(t)z(\sigma(t) - (n + 1)\tau) = 0. \]
Denote \( a_n(t) = \sum_{i=0}^{n} p^i z(\sigma(t) - i\tau) \). Then
\[ (6) \quad z''(t) + a_n(t)q(t) \leq 0. \]
Define
\[ v(t) = \frac{\sigma(t) \sum_{i=0}^{n} p^i}{a_n(t)} - z'(t), \quad t \geq t_1. \]
Then \( v(t) > 0 \). Observe that
\[ v'(t) = \frac{\sigma'(t) \sum_{i=0}^{n} p^i}{a_n(t)} z'(t) + \frac{\sigma(t) \sum_{i=0}^{n} p^i}{a_n(t)} z''(t) - \frac{\sigma'(t) \sum_{i=0}^{n} p^i}{a_n(t)} z'(t) - \frac{\sigma(t) \sum_{i=0}^{n} p^i}{a_n(t)} z'(t). \]
Since \( z'(t) \) is decreasing, one gets that \( z'(\sigma(t) - i\tau) \geq z'(\sigma(t)) \) and therefore in view of (6) and (7)
\[ v'(t) \leq \frac{\sigma'(t)}{\sigma(t)} (v(t) - v_2) - q(t)z(\sigma(t)) \sum_{i=0}^{n} p^i. \]
It is easy to see that the polynomial \( P \) satisfies \( P(v) = v - v^2 \leq \frac{1}{2} \). Thus
\[ v'(t) \leq \frac{\sigma'(t)}{\sigma(t)} - q(t)z(\sigma(t)) \sum_{i=0}^{n} p^i. \]
Then integrating the last inequality from \( t_1 \) to \( t \), we are led to
\[ v(t) \leq v(t_1) - \int_{t_1}^{t} \left( q(s)z(\sigma(s)) \frac{1 - p^{n+1}}{1 - p} + \frac{\sigma'(s)}{4\sigma(s)} \right) ds. \]
Letting \( t \to \infty \) we have in view of (3) that \( v(t) \to -\infty \), a contradiction.
For case (b), as mentioned before, we are led to \( \lim_{t \to \infty} x(t) = 0. \)
Corollary 1. Assume that $0 < p < 1$ and (2) holds. Let

$$\liminf_{t \to \infty} \frac{q(t)\sigma^2(t)}{\sigma(t)} > \frac{1-p}{4},$$

Then the nonoscillatory solutions of Eq. (1) tend to zero as $t \to \infty$.

Proof. Denote $a = \liminf_{t \to \infty} \frac{q(t)\sigma^2(t)}{\sigma(t)}$. Let an integer $n$ be chosen such that

$$a - \varepsilon > \frac{1-p}{4(1-p^{n+1})},$$

where $\varepsilon > 0$ is small enough. Then there exists a $t_1$ (large enough) that

$$\frac{q(t)\sigma^2(t)}{\sigma(t)} > \frac{1-p}{4(1-p^{n+1})} > \varepsilon, \quad t \geq t_1.$$

Noting that (9) implies (3) we complete the proof. \(\Box\)

Example 1. Consider the neutral equation

$$\left(x(t) - px(t-r)\right)^{\gamma} + \frac{1}{t^{\frac{1}{\gamma}}}x^{\frac{1}{\gamma}} = 0, \quad p \in (0, 1), \quad \tau > 0.$$

Condition (8) for Eq. (10) reduces to $8 \geq (1-p)$ and therefore the nonoscillatory solutions of Eq. (1) tend to zero as $t \to \infty$. On the other hand Lemma 4.4.2 and Theorem 4.4.1 in [2] fail for Eq. (10).

The conclusion of Theorem 1 can be strengthened as follows.

Theorem 2. In addition to the condition of Theorem 1, assume that there exists an integer $k \geq 1$ such that $t - \sigma(t) > kr$ and

$$\limsup_{t \to \infty} \int_{\sigma(t)+kr}^{t} \left(s - \sigma(s) - kr\right)q(s)ds > \frac{(1-p)p^{k}}{1-p^{k}}.$$

Then every solution of Eq. (1) is oscillatory.

Proof. Taking the proof of Theorem 1 into account, it is sufficient to show that $z(t) < 0$ is impossible for $t \geq t_1$ under the assumptions. On the contrary, suppose that $z(t) > 0$, $z''(t) \leq 0$, $z'(t) > 0$ and $z(t) < 0$ eventually. We rewrite Eq. (1) as

$$z''(t) - \frac{1}{p}q(t)z'(t) + \frac{q(t)}{p}x(\sigma(t)+\tau) = 0.$$
Reiterating this process we are led to
\[ z''(t) - q(t)f(t)t + q(t)f(t)z(t + kT) = 0 \]
for all large \( t \). Then using the monotonicity of \( z(t) \) one gets
\[ z''(t) - q(t)f(t)t + q(t)f(t)z(t + kT) \leq 0. \]

Integrating (12) from \( s \) to \( t \) for \( t > s \) we have
\[ z'(t) - z'(s) - \int_s^t q(u)z(u + kT) du \leq 0. \]

Integrating (13) in \( s \) from \( \sigma(t) + kT \) to \( t \), we have
\[ z'(t) - z'(\sigma(t) + kT) \leq z(t) - z(\sigma(t) + kT) \]
\[ + \frac{1 - p}{(1 - p)p^k} \int_{\sigma(t) + kT}^t (s - \sigma(s) - kT)q(s)z(s + kT) ds. \]

Hence using the monotonicity of \( z(t) \) one gets
\[ z'(t) - z'(\sigma(t) + kT) \leq z(\sigma(t) + kT) \left\{ \frac{1 - p}{1 - p} \int_{\sigma(t) + kT}^t (s - \sigma(s) - kT)q(s) ds - 1 \right\}. \]

Therefore if (11) holds, then we arrive at a contradiction with the sign properties of \( z(t) \) and \( z'(t) \). The proof is complete.

**Remark 1.** For the special case when \( k = 1 \) in (10) we obtain the results presented in [2, Theorem 4.4.1].

Now we give an analogue of Theorems 1 and 2 for the case \( p = 1 \).

**Theorem 3.** Assume that \( p = 1 \) and (2) holds. Then the nonoscillatory solutions of Eq. (1) are bounded provided there exists an integer \( n \geq 1 \) such that
\[ \int_0^\infty \left( n\sigma(s)q(s) - \frac{\sigma'(s)}{4\sigma(s)} \right) ds = \infty. \]
Proof. Let \( x(t) \) be an eventually positive solution of Eq. (1) and \( z(t) = x(t) - x(t - \tau) \). Then \( z''(t) < 0 \) for \( t \geq t_0 \). If \( z'(t) < 0 \) eventually, then we have \( \lim_{t \to \infty} z(t) = -\infty \). Therefore

\[
x(t) \leq x(t - \tau) \quad \text{for all large } t,
\]

which implies that \( x(t) \) is bounded, a contradiction. Therefore, \( z'(t) > 0 \) for \( t \geq t_1 \). Then either \( z(t) > 0 \) or \( z(t) < 0 \). Assume that \( z(t) > 0 \). We rewrite Eq. (1) as

\[
z''(t) + q(t) \sum_{i=0}^{n} z(\sigma_i(t) - i\tau) + g(t)z(\sigma_i(t) - (n + 1)\tau) = 0.
\]

Thus

\[
z''(t) + q(t) \sum_{i=0}^{n} z(\sigma_i(t) - i\tau) \leq 0.
\]

Define

\[
v(t) = \frac{n \sigma(t)}{\sum_{i=0}^{n} z(\sigma_i(t) - i\tau)} z'(t).
\]

Then \( v(t) > 0 \) and

\[
v'(t) = \frac{n \sigma'(t)}{\sum_{i=0}^{n} z(\sigma_i(t) - i\tau)} z'(t) + \frac{n \sigma(t)}{\sum_{i=0}^{n} z(\sigma_i(t) - i\tau)} z''(t)
- \frac{n \sigma(t)}{\sum_{i=0}^{n} z(\sigma_i(t) - i\tau)} z'(t) \frac{\sigma'(t) \sum_{i=0}^{n} z(\sigma_i(t) - i\tau)}{\sum_{i=0}^{n} z(\sigma_i(t) - i\tau)}.
\]

Since \( z'(\sigma_i(t) - i\tau) \geq z'(\sigma(t)) \) then in view of (15)

\[
v'(t) \leq \frac{\sigma'(t)}{\sigma(t)} (v(t) - v^*(t)) - n q(t) \sigma(t) \leq \frac{\sigma'(t)}{4 \sigma(t)} - n q(t) \sigma(t).
\]

Integrating the last inequality from \( t_1 \) to \( t \), we have

\[
v(t) \leq v(t_1) - \int_{t_1}^{t} \left( n q(s) \sigma(s) - \frac{\sigma'(s)}{4 \sigma(s)} \right) ds,
\]

which implies \( v(t) \to -\infty \). This is a contradiction.

Assume that \( z(t) < 0 \) for \( t \geq t_1 \). Then \( x(t) < x(t - \tau) \), which implies that \( x(t) \) is bounded. \( \square \)
Corollary 2. Assume that $p = 1$. Let

$$
\liminf_{t \to \infty} \frac{q(t)\sigma^2(t)}{\sigma'(t)} > 0.
$$

Then the nonoscillatory solutions of Eq. (1) are bounded.

Proof. Denote $a = \liminf_{t \to \infty} \frac{4q(t)\sigma^2(t)}{\sigma'(t)}$. Let an integer $n$ be chosen such that $a - \varepsilon > \frac{1}{n}$, where $\varepsilon > 0$ is small enough. Then there exists a $t_1$ (large enough) such that

$$
4q(t)\sigma^2(t) \left( \frac{1}{n} - \varepsilon \right) > 0, \quad t \geq t_1.
$$

Note that (17) implies (16). The proof is complete.

Theorem 4. In addition to the condition of Theorem 3, assume that there exists an integer $k \geq 1$ such that $t - \sigma(t) > kr$ and

$$
\limsup_{t \to \infty} \int_{\sigma(t)+kr}^{t-kr} (s - \sigma(s) - kr)q(s) \, ds > \frac{1}{k}.
$$

Then every solution of Eq. (1) is oscillatory.

Proof. Let $x(t)$ be an eventually positive solution of Eq. (1) and $z(t) = x(t) - x(t - \tau)$. Taking the proof of Theorem 3 into account, it is sufficient to show that $z(t) < 0$ is impossible. On the contrary, suppose that $z(t) > 0$, $z''(t) \leq 0$, $z'(t) > 0$ and $z(t) < 0$ eventually. For (1) one gets

$$
z''(t) - q(t) \sum_{i=1}^{k} z(\sigma(t) + ir) + q(t)z(\sigma(t) + kr) = 0.
$$

Thus

$$
z''(t) - kq(t)z(\sigma(t) + kr) \leq 0.
$$

Exactly as in the proof of Theorem 2 we integrate twice the last inequality to have

$$
z'(t) \left( t - \sigma(t) - kr \right) \leq z(t) - z(\sigma(t) + kr)
$$

$$
+ k \int_{\sigma(t)+kr}^{t} (s - \sigma(s) - kr)q(s)z(\sigma(s) + kr) \, ds.
$$
Hence using monotonicity $z(t)$ we obtain

$$z'(t)(t - \sigma(t) - kr) \leq z(\sigma(t) + kr) \left\{ k \int_{\sigma(t)+kr}^{t} (s - \sigma(s) - kr)q(s) \, ds - 1 \right\},$$

which contradicts the sign properties of $z(t)$ and $z'(t)$. The proof is complete.

\[ \square \]

**Theorem 5.** Assume that $p > 1$ and (2) holds. Further assume that there exists an integer $n \geq 0$ such that

$$\int_{0}^{\infty} \left( q(s)\sigma(s) \frac{p^\alpha - 1}{p - 1} \frac{\sigma'(s)}{4\sigma'(s)} \right) \, ds = \infty.$$

Then every nonoscillatory solution $x(t)$ of Eq. (1) satisfies $x(t) < px(t - \tau)$.

**Proof.** Let $x(t)$ be an eventually positive solution of Eq. (1) and set $z(t) = x(t) - px(t - \tau)$. Then $z''(t) < 0$ eventually. There are three possibilities:

(i) $z'(t) > 0$, $z(t) > 0$,
(ii) $z'(t) > 0$, $z(t) < 0$,
(iii) $z'(t) < 0$, $z(t) < 0$.

For case (i), the proof runs exactly as in the proof Theorem 1 and so it can be omitted. For cases (ii) and (iii) we have assumed that $z(t) < 0$, then $x(t) < px(t - \tau)$ is obvious. \[ \square \]

**Corollary 3.** Assume that $p > 1$ and (2) holds. Let

$$\lim_{t \to \infty} q(t)\sigma(t) > 0.$$

Then every nonoscillatory solution $x(t)$ of Eq. (1) satisfies $x(t) < px(t - \tau)$.

**Remark 2.** When considering more general neutral differential equations with function $p(t)$ instead of a constant $p$,

$$\left( x(t) - p(t)x(t - \tau) \right)'' + q(t)x(\sigma(t)) = 0,$$

then it is usual to impose the condition $p_1 < p(t) < p_2$ on the function $p(t)$. From the proofs of the abovementioned results one can see that the technique presented in this paper can be applied to Eq. (19).
References


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