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THE DIRECTED DISTANCE DIMENSION OF ORIENTED GRAPHS

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Abstract. For a vertex \( v \) of a connected oriented graph \( D \) and an ordered set \( W = \{w_1, w_2, \ldots, w_k\} \) of vertices of \( D \), the (directed distance) representation of \( v \) with respect to \( W \) is the ordered \( k \)-tuple \( r(v \mid W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k)) \), where \( d(v, w_i) \) is the directed distance from \( v \) to \( w_i \). The set \( W \) is a resolving set for \( D \) if every two distinct vertices of \( D \) have distinct representations. The minimum cardinality of a resolving set for \( D \) is the (directed distance) dimension \( \dim(D) \) of \( D \). The dimension of a connected oriented graph need not be defined. Those oriented graphs with dimension 1 are characterized. We discuss the problem of determining the largest dimension of an oriented graph with a fixed order. It is shown that if the outdegree of every vertex of a connected oriented graph \( D \) of order \( n \) is at least 2 and \( \dim(D) \) is defined, then \( \dim(D) \leq n - 3 \) and this bound is sharp.

Keywords: oriented graphs, directed distance, resolving sets, dimension

MSC 1991: 05C12, 05C20

1. Introduction

For an oriented graph \( D \) of order \( n \), an ordered set \( W = \{w_1, w_2, \ldots, w_k\} \) of vertices of \( D \), and a vertex \( v \) of \( D \), the \( k \)-vector (ordered \( k \)-tuple)

\[ r(v \mid W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k)) \]

is referred to as the (directed distance) representation of \( v \) with respect to \( W \), where \( d(x, y) \) denotes the directed distance from \( x \) to \( y \), that is, the length of a shortest directed \( x - y \) path in \( D \). Since directed \( x - y \) paths need not exist in \( D \), even if \( D \) is connected (its underlying graph is connected); the vector \( r(v \mid W) \) need not exist as well. If \( r(v \mid W) \) exists for every vertex \( v \) of \( D \), then the set \( W \) is called a resolving set for \( D \) if every two distinct vertices of \( D \) have distinct representations. A resolving set of minimum cardinality is called a basis for \( D \) and this cardinality is
the (directed distance) dimension $\dim(D)$ of $D$. Of course, not every oriented graph has a dimension. An oriented graph of dimension $k$ is also called $k$-dimensional.

To determine whether an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices in an oriented graph $D$ is a resolving set, we need only show that the representations of the vertices of $V(D) - W$ are distinct since $r(w_i \mid W)$ is the only representation whose $i$th coordinate is 0.

The directed distance dimension of an oriented graph is a natural analogue of the metric dimension of a graph that was introduced independently by Harary and Melter [2] and Slater [3], [4]. This concept was also investigated in [1] as a result of studying a problem in pharmaceutical chemistry.

In the oriented graph $D$ of Figure 1, let $W_1 = \{u, v\}$. The five representations of the vertices of $D$ with respect to $W_1$ are $r(u \mid W_1) = (0, 2), r(v \mid W_1) = (1, 0), r(w \mid W_1) = (2, 1), r(x \mid W_1) = (2, 1), \text{ and } r(y \mid W_1) = (1, 3).$ Since $x$ and $w$ have the same representation, $W_1$ is not a resolving set for $D$.

The five representations of the vertices of $D$ with respect to $W_2 = \{u, v, w\}$ are

\[
\begin{align*}
    r(u \mid W_2) &= (0, 2, 2), &
    r(v \mid W_2) &= (1, 0, 3), &
    r(w \mid W_2) &= (2, 1, 0), \\
    r(x \mid W_2) &= (2, 1, 1), &
    r(y \mid W_2) &= (1, 3, 3)
\end{align*}
\]

Since these five 3-vectors are distinct, $W_2$ is a resolving set for $D$. However, $W_2$ is not a basis for $D$. To see this, let $W_3 = \{x, y\}$. Then $r(u \mid W_3) = (1, 3), r(v \mid W_3) = (2, 1), r(w \mid W_3) = (3, 1), r(x \mid W_3) = (0, 2), \text{ and } r(y \mid W_3) = (2, 0), \text{ which are distinct as well. So } W_3 \text{ is a resolving set for } D. \text{ Since there is no 1-element resolving set for } D, \text{ it follows that } W_3 \text{ is a basis and } \dim(D) = 2.$

Now let $T$ be the tournament shown in Figure 2. Table 1 gives all 2-element choices for $W$ and shows that for each such choice, there exist two equal 2-vectors, thus showing that $\dim(T) \geq 3$. However, $\dim(T) = 3$ since $\{v_1, v_3, v_6\}$ is a basis for $T$. Figure 3 shows an oriented graph $D$ containing $T$ as an induced subdigraph. The set $W = \{x, y\}$ is a basis of $D$, so $\dim(D) = 2$. Hence we have the possibly
unexpected property that the 3-dimensional tournament $T$ is an induced subdigraph of the 2-dimensional oriented graph $D$.

Figure 2. The tournament $T$

Figure 3. The digraph $D$

There is a fundamental question here—one whose answer is not known to us, but one which deserves further study. What is a necessary and sufficient condition for the dimension of a digraph $D$ to be defined? Certainly, if $D$ is strong, then $\dim(D)$ is defined. Also, if $D$ is connected and contains a vertex such that $D - v$ is strong, then $\dim(D)$ is defined. This last statement follows because if $\text{od}(v) > 0$, then $V(D) - \{v\}$ is a resolving set; while if $\text{id}(v) > 0$, then $V(D)$ is a resolving set. There are numerous other sufficient conditions for $\dim(D)$ to be defined.
Table 1.

2. 1-DIMENSIONAL ORIENTED GRAPHS

In this section we characterize those oriented graphs having dimension 1. We also describe some properties of bases for 1-dimensional oriented graphs.

Theorem 2.1. Let $D$ be a nontrivial oriented graph of order $n$. Then $\dim(D) = 1$ if and only if there exists a vertex $v$ in $D$ such that

(i) $D$ contains a hamiltonian path $P$ with terminal vertex $v$ such that $id_D v = 1$; and

(ii) if the hamiltonian path $P$ in (i) is of the form

$V_n - 1, V_{n-2}, \ldots, V_1, V$,

then, for each pair $i, j$ of integers with $1 \leq i < j \leq n - 1$, the digraph $D - E(P)$ contains no arc of the form $(v_i, v_j)$. 

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Proof. Assume that \( \dim(D) = 1 \). Let \( W = \{v\}, v \in V(D) \), be a basis of \( D \). Then the distance \( d(u,v) \) from \( u \) to \( v \) is defined for each vertex \( u \) in \( D \) and the set \( \{d(u,v); u \in V(D)\} = \{0,1,\ldots,n-1\} \). Thus, we may assume that \( V(D) = \{v,v_1,v_2,\ldots,v_{n-1}\} \) where \( d(v_i,v) = i \) (1 \( \leq i \leq n-1 \)). Clearly, \( id(v) = 1 \). Since \( d(v_{n-1},v) = n-1 \), there exists a hamiltonian path in \( D \), namely \( P: v_{n-1},v_{n-2},\ldots,v_1,v \), so (i) holds. Furthermore, if there exists a pair \( i,j \) of integers (1 \( \leq i < j \leq n-1 \)) such that the arc \( (v_i,v_j) \) is in \( D - E(P) \), then \( j \neq i+1 \) and \( d(v_j,v) = d(v_{i+1},v) \) (shown in Figure 4). This contradicts the fact that \( \{d(u,v); u \in V(D)\} \) consists of \( n \) distinct integers, so (ii) holds.

Conversely, assume that there is a vertex \( v \) in \( D \) such that (i) and (ii) hold.

We show that \( W = \{v\} \) is a resolving set of \( D \). Since \( d(u,v) \) is defined for each \( u \in V(D) \), it suffices to show that the set \( \{d(v_i,v); 1 \leq i \leq n-1\} \) consists of \( n-1 \) distinct integers. Suppose that this is not the case. Then there exist integers \( i,j \) (1 \( \leq i < j \leq n-1 \)) such that \( d(v_j,v) = d(v_i,v) = \ell \). Let \( P_1 \) be a \( v_i - v \) path and \( P_2 \) a \( v_j - v \) path in \( D \) such that \( P_1 \) and \( P_2 \) have the same length \( \ell \). Since \( id(v) = 1 \), there exists a vertex \( v_k \neq v \) in \( D \) that belongs to both \( P_1 \) and \( P_2 \). Assume that \( v_k \) is the vertex with largest index \( k \) such that the path \( v_k,v_{k-1},\ldots,v_1,v \) is on both \( P_1 \) and \( P_2 \) (see Figure 5).

Let \( (v_{k_1},v_{k_2}) \in E(P_1) \) and \( (v_{k_3},v_{k_4}) \in E(P_2) \) where \( (v_{k_1},v_{k_2}) \neq (v_{k_3},v_{k_4}) \). Clearly, \( k_3 > k \) and \( k_4 > k \). It follows that at least one of these arcs is in \( D - E(P) \), but this is a contradiction to (ii).

We now present some facts concerning bases in 1-dimensional oriented graphs.

Theorem 2.2. Let \( D \) be a digraph of order \( n \) with \( \dim(D) = 1 \). Furthermore, let \( v_1 \) and \( v_2 \) be distinct vertices of \( D \) with \( d(v_1,v_2) = 2 \) such that both \( \{v_1\} \) and
are bases of D. If v is a vertex of D such that \((v_1, v), (v, v_2) \in E(D)\), then \(\{v\}\) is also a basis of D.

**Proof.** To show that \(\{v\}\) is a basis of D, we show that for each \(u \in V(D)\), the distance \(d(u, v)\) is defined and the set \(\{d(u, v) ; u \in V(D)\}\) consists of \(n\) distinct integers.

First notice that \(idv = 1\), for otherwise there exist distinct vertices \(x\) and \(y\) of D such that \(d(x, v) = d(y, v) = 1\). Since \(idv = 1\), by Theorem 2.1, we have

\[
d(x, v_2) = d(y, v_2) = d(x, v) + 1 = 2
\]

This contradicts the fact that \(\{v_2\}\) is a basis of D.

Furthermore, suppose that there exist vertices \(u, w\) in D such that \(d(u, v) = d(w, v)\). Since \(idv = 1\), each \(u - v\) path contains the arc \((v_1, v)\) as its terminal arc, as does each \(w - v\) path, so

\[
d(u, v_1) = d(w, v_2) = d(u, v) - 1
\]

Again, this contradicts the fact that \(\{v_1\}\) is a basis of D.

We now have an immediate consequence of Theorem 2.2.

**Corollary 2.3.** If D is a 1-dimensional oriented graph of order \(n \geq 3\) such that \(\{v\}\) is a basis of D for every vertex v in D, then D is a directed cycle.

**Proof.** Let \(V(D) = \{v_1, v_2, \ldots, v_n\}\). By Theorem 2.2, \(idv = 1\) for every vertex \(v\) of D. Moreover, D contains a hamiltonian path \(P\). We can assume that

\[
P : v_n, v_{n-1}, \ldots, v_2, v_1
\]

Next, we show that D contains the cycle

\[
C_n : v_n, v_{n-1}, \ldots, v_2, v_1, v_n
\]

Since \(idv_n = 1\), there exists a unique vertex \(v\) such that \((v, v_n) \in E(D)\). If \(v \neq v_1\), then \((v, v_n) \in E(D)\) for some \(i (2 \leq i \leq n - 1)\). Since \(\{v_n\}\) is a basis of D, there exists a hamiltonian path in D with terminal vertex \(v_n\). However, since every vertex has indegree 1, the only possible path in D with \(v_n\) as its terminal vertex is

\[
P' : v_{n-1}, v_{n-2}, \ldots, v_{i+1}, v_i, v_n
\]

Since \(P'\) has length \(n - i\), it is not a hamiltonian path. This contradicts the fact that \(\{v_n\}\) is a basis. So D contains the cycle \(C_n\). Furthermore, since \(idv = 1\), D cannot contain any arc except those in \(C_n\). So \(D = C_n\).

\[\square\]
We can improve Corollary 2.3 slightly.

**Corollary 2.4.** If $D$ is a 1-dimensional oriented graph of order $n \geq 3$ such that

$$|\{v; \{v\} \text{ is a basis of } D\}| \geq n - 1$$

then $D$ is a directed cycle.

**Proof.** Let $V(D) = \{v, v_1, v_2, \ldots, v_{n-1}\}$. Without loss of generality, we assume that $\{v_i\}$ is a basis of $D$ for $1 \leq i \leq n - 1$. By Corollary 2.3, it suffices to show that $\{v\}$ is a basis as well.

We claim that $\od v > 0$. Suppose that this is not the case. Then for each vertex $u (\neq v)$, the distance $d(v, u)$ is not defined, which contradicts the fact that $\{u\}$ is a basis of $D$. Hence, there is a vertex $x (\neq v)$ such that $(v, x) \in E(D)$. Since $\{x\}$ is also a basis of $D$, then by Theorem 2.1(i), $D$ contains a hamiltonian path with terminal vertex $x$ and $d(x) = 1$. This implies that there exists a vertex $y$ distinct from $x$ and $v$ such that $(y, v) \in E(D)$. It follows that $d(y, x) = 2$ and by Theorem 2.2, $\{v\}$ is also a basis of $D$. $\square$

The bound in Corollary 2.4 cannot be improved in general. For example, consider the oriented graph $D$ of order $n$ in Figure 6. Since $\{v_1\}$ is a basis for $D$ for $1 \leq i \leq n - 2$, $\dim(D) = 1$. However, neither $\{v_{n-1}\}$ nor $\{v_n\}$ is a basis of $D$. So $|\{v; \{v\} \text{ is a basis of } D\}| = n - 2$ and $D$ is not a directed cycle.

![Figure 6. An oriented graph with $(n - 2)$ 1-element bases](image)

There is only one 1-dimensional oriented tree of every order.

**Theorem 2.5.** For every oriented tree $T$, $\dim(T) = 1$ or $\dim(T)$ is undefined. Furthermore, if $\dim(T) = 1$, then $T$ is a directed hamiltonian path.
Proof. There are certainly oriented trees whose dimension is undefined, for example, any orientation of a star $K_{1,t}$, where $t \geq 3$. Now let $T$ be an oriented tree whose dimension is defined. Since $T$ contains no cycles, for every pair $x, y$ of vertices, whenever $d(x, y)$ is defined, $d(y, x)$ is undefined. Thus $\dim(T) = 1$.

If $\dim(T) = 1$, then, by Theorem 2.1, $T$ contains a hamiltonian path $P$ and so $T = P$. 

3. ON ORIENTED GRAPHS WITH LARGE DIMENSION

We have characterized those oriented graphs with dimension 1. But how large can the dimension of an oriented graph of order $n$ be? In this section, we describe upper bounds for the dimension of a connected oriented graph in terms of lower bounds for the outdegrees of its vertices. The outdegree of every vertex in the oriented graph $D$ of Figure 7 is 2, yet $\dim(D)$ is undefined. Such examples exist regardless of the outdegrees.

![Figure 7. The oriented graph $D$](image)

**Theorem 3.1.** If $D$ is a connected oriented graph of order $n \geq 3$ with $od_v \geq 1$ for all $v \in V(D)$ such that $\dim(D)$ is defined, then $\dim(D) \leq n - 2$.

**Proof.** Let $D$ be an oriented graph satisfying the hypothesis of the theorem. Certainly $\dim(D) \leq n - 1$. Assume, to the contrary, that $\dim(D) = n - 1$. Let $W = \{v_1, v_2, \ldots, v_{n-1}\}$ be a basis for $D$ and let $V(D) - W = \{x\}$. Since $od_x \geq 1$, assume, without loss of generality, that $x$ is adjacent to $v_1$. Also, since $od_{v_1} \geq 1$, we may assume that $v_1$ is adjacent to $v_2$. Since $\dim(D) = n - 1$, $r(v_i | W - \{v_1\}) = r(x | W - \{v_1\})$ for $1 \leq i \leq n - 1$. Since $x$ is adjacent to $v_1$, it follows that $v_2$ is adjacent to $v_1$, but this contradicts the fact that $D$ is an oriented graph. 

We now describe a class of oriented graphs. For $k \geq 2$, let $D_k$ be an oriented graph with vertex set

$$V(D_k) = \{u, v, w_1, w_2, \ldots, w_k\}$$

and let $E(D_k)$ consist of the arc $(u, v)$ and the arcs $(v, w_j)$ and $(w_j, u)$ for $1 \leq j \leq k$.

The oriented graph $D_k$ is shown in Figure 8. Then $D_k$ has order $n = k + 2$ and $od_v \geq 1$ for all $v \in V(D_k)$. We claim that $\dim(D_k) = n - 3$. 
First we show that $\dim(D_k) \leq n - 3$. Let $W = \{w_2, w_3, \ldots, w_k\}$, where then $|W| = n - 3$. The distances $d(u, w_2) = 2$, $d(v, w_2) = 1$, and $d(w_1, w_2) = 3$ show that $W$ is a resolving set for $D_k$ and so $\dim(D_k) \leq n - 3$. On the other hand, at least $k - 1$ of the vertices $w_1, w_2, \ldots, w_k$ must belong to every resolving set of $D_k$ since the distance from any two of these vertices to every other vertex of $D_k$ is the same. Hence $\dim(D_k) \geq n - 3$ and so $\dim(D) = n - 3$. Of course, this does not show that sharpness of the bound in Theorem 3.1, except that if $D_1$ is the directed 3-cycle, then $\dim(D_1) = 1 = n - 2$.

We can, however, improve the bound in Theorem 3.1 if we require that the outdegree of every vertex is at least 2.

**Theorem 3.2.** If $D$ is a connected oriented graph of order $n \geq 5$ with $\text{od}(v) \geq 2$ for all $v \in V(D)$ such that $\dim(D)$ is defined, then $\dim(D) \leq n - 3$.

**Proof.** Suppose, to the contrary, that $D$ contains a basis $\mathcal{B}$ of cardinality $n - 2$. Let $\mathcal{B} = \{v_1, v_2, \ldots, v_{n-2}\}$, and $V(D) - \mathcal{B} = \{x, y\}$. For each $i (1 \leq i \leq n - 2)$, $\mathcal{B} - \{v_i\}$ is not a resolving set. Hence for each such $i$, some two of the three vertices $x, y, v_i$ have the same representations with respect to $\mathcal{B}$. We consider two cases.

**Case 1:** For some $i (1 \leq i \leq n-2)$, $x$ and $y$ have the same representations with respect to $\mathcal{B} - \{v_i\}$. Assume, without loss of generality, that $x$ and $y$ have the same representations with respect to $W = \mathcal{B} - \{v_{n-2}\}$. Then $x$ and $y$ have the same out-neighbors in $W$. Since $x$ and $y$ have distinct representations with respect to $\mathcal{B}$, exactly one of $x$ and $y$ is adjacent to $v_{n-2}$; for if neither $x$ nor $y$ is adjacent to $v_{n-2}$, then $d(x, v_{n-2}) = d(y, v_{n-2})$. Therefore, we may assume that $y$ is adjacent to $v_{n-2}$.

Let $W' = \{v_1, v_2, \ldots, v_{n-4}, v_{n-2}\}$. Two of $x, y$, and $v_{n-3}$ have the same representations with respect to $W'$. However, $y$ is adjacent to $v_{n-2}$ and $x$ is not, so $x$ and $y$ do not have the same representations with respect to $W'$. Thus there are two possibilities.
Subcase 1.1: $r(x \mid W) = r(v_{n-3} \mid W')$. We claim that $x$ is adjacent to at most one of $v_1, v_2, \ldots, v_{n-2}$. Suppose that this is not the case. Then we can assume without loss of generality that $x$ is adjacent to $v_1$ and $v_2$. Then $r(v_1 \mid B - \{v_1\}) = r(x \mid B - \{v_1\})$ or $r(v_1 \mid B - \{v_1\}) = r(y \mid B - \{v_1\})$. Similarly, $r(v_2 \mid B - \{v_2\}) = r(x \mid B - \{v_2\})$ or $r(v_2 \mid B - \{v_2\}) = r(y \mid B - \{v_2\})$. Since the out-neighbors of $y$ in $W$ are the same as the out-neighbors of $x$ in $W$, we have that $v_2$ is an out-neighbor of $v_1$, and that $v_1$ is an out-neighbor of $v_2$. Since $D$ is an oriented graph, this is impossible, so, as claimed, $x$ is adjacent to at most one of $v_1, v_2, \ldots, v_{n-2}$.

Subcase 1.2: $r(y \mid W') = r(v_{n-3} \mid W')$. We first suppose that $x$ is adjacent to some vertex in $W$, say $v_1$. Because of the assumptions in Case 1 and Subcase 1.2, it follows that $y$ and $v_{n-3}$ are also adjacent to $v_1$. However, since for $2 \leq i \leq n-3$, $r(v_i \mid B - \{v_1\}) = r(x \mid B - \{v_i\})$ or $r(v_i \mid B - \{v_1\}) = r(y \mid B - \{v_i\})$, it follows that $v_1$ is an out-neighbor of every vertex in the set $\{x, y, v_2, v_3, \ldots, v_{n-3}\}$, so $\od v_1 = 0$, which is a contradiction. Therefore, $x$ is not adjacent to any of $v_1, v_2, \ldots, v_{n-4}, v_{n-3}$. Thus, since $\od x \geq 2$, it follows that $x$ must be adjacent to both $y$ and $v_{n-3}$. But $y$ is adjacent to $v_{n-3}$ as well, because $x$ and $y$ have the same representations with respect to $W$. Since $x$ is not adjacent to any of $v_1, v_2, \ldots, v_{n-4}$, it follows that $y$ is not adjacent to any of $v_1, v_2, \ldots, v_{n-4}$. Now $r(y \mid W') = r(v_{n-3} \mid W')$, so it follows that $v_{n-3}$ is not adjacent to any of $v_1, v_2, \ldots, v_{n-4}$. All of this implies that $\od v_{n-3} = 1$, which is a contradiction.

Case 2: For every $i$ ($1 \leq i \leq n-2$), $x$ and $y$ have distinct representations with respect to $B - \{v_i\}$. We next prove that every vertex of $B$ is an out-neighbor of $x$ or $y$, but at most one vertex of $B$ is an out-neighbor of both $x$ and $y$. To prove this, we first show that among the out-neighbors $y_1, y_2, \ldots, y_n$ of $y$ in $B$, at most one $y_i$ has the same representation as $y$ with respect to $B - \{y_i\}$. Suppose that this is not the case. Then we may assume that $r(y_1 \mid B - \{y_1\}) = r(y \mid B - \{y_1\})$ and that $r(y_2 \mid B - \{y_2\}) = r(y \mid B - \{y_2\})$. The first equality tells us that $y_1$ is an out-neighbor of $y$, and the second equality tells us that $y_1$ is an out-neighbor of $y_2$, contradicting the fact that $D$ is an oriented graph. Similarly, among the out-neighbors $x_1, x_2, \ldots, x_r$ of $x$ in $B$, at most one $x_j$ has the same representation as $x$ with respect to $B - \{x_j\}$.

Next, we show that for each $i$ ($1 \leq i \leq n-2$), at least one of $x$ and $y$ is adjacent to $v_i$. This follows from the fact that if neither $x$ nor $y$ is adjacent to $v_i$, then no other
vertex \( v_j \) from \( B - \{v_i\} \) can be adjacent to \( v_i \) since \( r(v_j | B - \{v_i\}) = r(x | B - \{v_i\}) \) or \( r(v_j | B - \{v_i\}) = r(y | B - \{v_i\}) \). Thus \( \text{id}_{v_i} = 0 \), which is impossible since \( d(\sigma, v_i) \) must be defined for all \( z \in V(D) \). Finally, \( x \) and \( y \) are simultaneously adjacent to at most one vertex \( v_i \) \((1 \leq i \leq n - 2)\), for if \( v_i \) and \( v_k \) are distinct out-neighbors of both \( x \) and \( y \), then \( v_i \) and \( v_k \) are out-neighbors of each other, which is impossible.

This creates a natural partition of the vertices of \( B \) into either two or three subsets, depending on whether there exists a vertex to which \( x \) and \( y \) are simultaneously adjacent. We now consider these two subcases.

**Subcase 2.1:** There exists a unique common out-neighbor of \( x \) and \( y \).

We assume, without loss of generality, that \( v_{n-2} \) is an out-neighbor of both \( x \) and \( y \). Furthermore, we can assume, without loss of generality, that the set \( X = \{v_1, v_2, \ldots, v_k\} \) consists of the out-neighbors of \( x \) and not \( y \), and that the set \( Y = \{v_{k+1}, v_{k+2}, \ldots, v_{n-2}\} \) consists of the out-neighbors of \( y \) and not \( x \). We further assume, without loss of generality, that the representations of \( y \) and \( v_{n-2} \) with respect to \( B - \{v_i\} \) are the same. Therefore, there is no vertex in \( v_j \in Y \) for which the representations of \( y \) and \( v_j \) with respect to \( B - \{v_i\} \) are the same. Therefore, for every \( v_j \in Y \), the representations of \( x \) and \( v_j \) with respect to \( B - \{v_i\} \) are the same. Since \( x \) is adjacent to every vertex in \( X \), every vertex in \( Y \) is adjacent to every vertex in \( X \cup \{v_{n-2}\} \). Now, there is at most one \( v_i \in X \) for which the representations of \( x \) and \( v_i \) are the same. Therefore, if \( |X| \geq 2 \), there exists at least one vertex \( v_i \in X \) for which the representations of \( y \) and \( v_i \) with respect to \( B - \{v_i\} \) are the same. Hence, such a vertex \( v_i \) is adjacent to every vertex in \( Y \), but this implies that \( D \) is not an oriented graph since for any \( v_j \in Y \), there is an arc from \( v_i \) to \( v_j \) and an arc from \( v_j \) to \( v_i \). Therefore, \( |Y| \leq 1 \). But if \( |X| = 1 \), then \( v_1 \) is the only vertex that could possibly be an out-neighbor of \( v_{n-2} \). This contradicts the assumption that the out-degree of every vertex in \( D \) is at least 2, so \( |X| = 0 \). We have already seen that every vertex in \( Y \cup \{x\} \) is adjacent to vertex \( v_{n-2} \), so even if \( |X| = 0 \), we have that \( \text{od}_{v_{n-2}} = 0 \), which cannot occur.

**Subcase 2.2:** No vertex is a common out-neighbor of \( x \) and \( y \).

We assume, without loss of generality, that the set \( X = \{v_1, v_2, \ldots, v_k\} \) consists of the out-neighbors of \( x \) and not \( y \), and that the set \( Y = \{v_{k+1}, v_{k+2}, \ldots, v_{n-2}\} \) consists of the out-neighbors of \( y \) and not \( x \). Recall that there is at most one \( v_i \in X \) such that the representations of \( v_i \) and \( x \) with respect to \( B - \{v_i\} \) are equal and at most one \( v_j \in Y \) such that the representations of \( v_j \) and \( y \) with respect to \( B - \{v_i\} \) are equal. This produces three possibilities to consider.
Subcase 2.2.2: There is exactly one \( v_i \in X \) for which the representations of \( v_i \) and \( x \) with respect to \( B - \{ v_i \} \) are equal and there is no \( v_j \in Y \) for which \( v_j \) and \( y \) have the same representations with respect to \( B - \{ v_i \} \). (Note that this subcase is symmetric to the case when there is exactly one \( v_j \in Y \) for which the representations of \( v_j \) and \( y \) with respect to \( B - \{ v_j \} \) are equal and for which there is no \( v_i \in X \) such that \( v_i \) and \( x \) have the same representations with respect to \( B - \{ v_i \} \).) Now every vertex in \( Y \) has the same out-neighbors as \( x \), namely the vertices in the set \( X \). So if \( Y \neq \emptyset \), then every vertex in \( Y \) is adjacent to every vertex in \( A' \). Furthermore, every vertex in \( X - \{ v_i \} \) has the same out-neighbors as \( y \). So if \( |X| > 2 \), then there is at least one vertex in \( X \) which is adjacent to every vertex in \( Y \). But this produces a contradiction since \( D \) is an oriented graph. Note that if \( Y = \emptyset \), then \( y \) is adjacent to at most one vertex, namely \( x \), and this is a contradiction.

Assume now that \( |X| \leq 1 \) (so \( |Y| \geq 2 \)). If \( |X| = 1 \), then \( v_1 = v_1 \) and since every vertex in \( Y \) is adjacent to \( v_1 \), the vertex \( v_1 \) is adjacent to no vertex except possibly \( y \). Hence, od \( v_1 \leq 1 \), which is a contradiction. If \( X = \emptyset \), then \( x \) has no out-neighbors except possibly for \( y \), but this contradicts the assumption that the out-degree of \( x \) is at least 2.

Subcase 2.2.3: There exists exactly one \( v_i \in X \) for which the representations of \( v_i \) and \( x \) with respect to \( B - \{ v_i \} \) are the same and exactly one \( v_j \in Y \) for which the representations of \( v_j \) and \( y \) with respect to \( B - \{ v_j \} \) are the same. First, suppose that \( |X| \geq 2 \) and \( |Y| \geq 2 \). Then there exists at least one vertex \( v \in X \) for which the representations of \( v \) and \( y \) with respect to \( B - \{ v \} \) are the same. Therefore, \( v \) is adjacent to every vertex in \( Y \). Similarly, there is at least one vertex \( w \in Y \) for which the representations of \( w \) and \( x \) with respect to \( B - \{ w \} \) are the same. Therefore, \( w \) is adjacent to every vertex in \( X \). However, since \( v \in X \) and \( w \in Y \), it follows that \( v \) is adjacent to \( w \) and \( w \) is adjacent to \( v \). This contradicts the fact that \( D \) is an oriented graph.

Next suppose that \( |X| = 1 \), that \( |Y| \geq 2 \), and that \( X = \{ v_1 \} \). Then the out-neighbors of \( x \) are \( y \) and \( v_1 \). Furthermore, \( v_1 \) is an out-neighbor of every vertex in \( Y - \{ v_j \} \). The only possible out-neighbors of \( v_1 \) are \( y \) and \( v_j \). However, if \( v_j \) is adjacent to \( v_1 \), then \( x \) is adjacent to \( v_1 \), which contradicts the fact that \( v_1 \notin X \). Therefore, od \( v_1 \leq 1 \), contradicting the fact that every vertex in \( D \) has out-degree at least 2. The case where \( |Y| = 1 \) and \( |X| \geq 2 \) is similar. \( \square \)
The sharpness of the bound in Theorem 3.1 is not illustrated by the digraph $D_k$ shown in Figure 8 since the outdegrees of most vertices of $D_k$ are 1. We can, however, show that the upper bound in Theorem 3.2 is sharp. Let $F_k$ be an oriented graph with vertex set $V(F_k) = \{u_1, u_2, v_1, v_2, w_1, w_2, \ldots, w_k\}$ and let $E(F_k)$ consist of (1) the arcs $(u_i, v_j)$ for $1 \leq i, j \leq 2$ and (2) the arcs $(v_i, w_j)$ and $(w_i, u_j)$ for $1 \leq i \leq 2$ and $1 \leq j \leq k$. The oriented graph $F_k$ is shown in Figure 9. Then $F_k$ has order $n = k + 4$ and the property that $\text{od}(v) \geq 2$ for all $v \in V(F_k)$. We claim that $\text{dim}(F_k) = n - 3$.

First we show that $\text{dim}(F_k) \leq n - 3$. Let $W = \{u_1, v_1, w_1, w_2, \ldots, w_k\}$, where then $|W| = n - 3$. The distances $d(u_2, v_2) = 2$, $d(v_2, w_2) = 1$, and $d(w_1, v_2) = 3$ show that $W$ is a resolving set for $F_k$ and so $\text{dim}(F_k) \leq n - 3$. Next we show that $\text{dim}(F_k) \geq n - 3$. Let $W$ be a resolving set for $F_k$. Certainly at least $k - 1$ of the vertices $v_1, v_2, \ldots, v_k$ must belong to $W$ since the distance from any two of these vertices to every other vertex of $F_k$ is the same. Moreover, at least one of $v_1$ and $v_2$ must belong to $W$ since the distance from $v_1$ and $v_2$ to every other vertex of $F_k$ is the same. For the same reason, at least one of $v_1$ and $v_2$ must belong to $W$. Hence $\text{dim}(F_k) \geq n - 3$ and so $\text{dim}(F_k) = n - 3$.

No additional restriction on the outdegrees of the vertices of an oriented graph yields an improved bound, however. Let $r \geq 2$ be an integer. In the oriented graph of Figure 8, replace $u_1, u_2$ by the $r$ vertices $u_1, u_2, \ldots, u_r$ and $v_1, v_2$ by the $r$ vertices $v_1, v_2, \ldots, v_r$ and add the appropriate arcs. The resulting oriented graph $H_k$ has $\text{od}(v) \geq r$ for all $v \in V(H_k)$, but $\text{dim}(H_k) = n - 3$. 

![Figure 9. The oriented graph $F_k$ with minimum outdegree 2](image-url)
References


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