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DIGRAPHS CONTRACTIBLE ONTO *K₃.

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Abstract. We show that any digraph on n ≥ 3 vertices and with not less than 3n - 3 arcs is contractible onto *K₃.

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INTRODUCTION

The notation of contraction is well known for non-oriented graphs (cf. [4]). In this paper, Hadwiger gave the following conjecture: If χ(G) = p then G is contractible onto the complete graph on p vertices Kₚ. Here χ(G) denotes the chromatic number of G. Dirac [2] showed this conjecture to be true for p ≤ 4. Wagner [11] showed that the four color theorem implies the case p = 5. Robertson, Seymour and Thomas [9] proved the case p = 6. For a good survey on the relationship between the minor’s existence in G and the generalization of the coloring notion to the digraphs we refer the reader to [5], where an oriented version of Hadwiger’s conjecture is given, too.

Recently, Jagger [6] has shown that if p is large enough, then any digraph on n vertices having at least $10^p \sqrt{\log 2} \cdot n$ arcs is contractible onto *Kₚ. Nevertheless, this nice asymptotical result does not give a right information about the “little” cases. In this direction, Duchet and Kaneti [3] proved that any digraph on n vertices with not less than 5n - 8 arcs is contractible onto *K₄. We give a short proof of the following result discovered by Meyniel [8].

Theorem. Any digraph on n ≥ 3 vertices and with not less than 3n - 3 arcs is contractible onto *K₃, and this bound is attained for any n.
We consider only finite digraphs without loops and parallel arcs. An arc of a digraph \( G = (V(G), A(G)) \) from \( x \) to \( y \) is the couple \((x, y)\). We say that \((x, y)\) is incident to \( x \) and \( y \). The couple of arcs \((x, y)\) and \((y, x)\) is called a symmetrical arc and is denoted by \( xy \). We will say edge instead of arc whenever the orientation is insignificant. The set of out-neighbours (in-neighbours) of \( x \) is \( A^+(x) = \{ y \in V(G) : (x, y) \in A(G) \} \) \( A^-(x) = \{ y \in V(G) : (y, x) \in A(G) \} \). \( A(x) = A^+(x) \cup A^-(x) \) is the set of neighbours of \( x \). We denote by \( d^+(x) \) \( d^-(x) \) the in-degree (out-degree) and by \( d(x) = d^+(x) + d^-(x) \) the degree of \( x \). By contracting one arc we mean identifying its extremities and omitting the loop(s) created. We say that the digraph \( G \) is contractible onto \( G' \) (or \( G' \) is a minor of \( G \)) and we denote \( G \leq G' \) if \( G' \) can be obtained from \( G \) by a sequence (possibly empty) of contractions of arcs or removing of arcs or removing of vertices. Clearly, this relation is transitive. The digraph \( *K_p \) contains \( p \) vertices and a symmetrical arc between any pair of vertices. The digraph \( *K_3 \) is given in Fig. 1:

Fig. 1. The digraph \( *K_3 \).

**Proof and Remarks**

**Proof.** Let \( G = (V(G), A(G)) \) be a digraph with \( n = |V(G)| \) and \( m = |A(G)| \). The proof is done by induction on \( n + m \). The result is clearly true for \( n = 3 \), so we suppose that \( G \) has at least 4 vertices. If \( G \) contains a vertex \( x \) with \( d(x) \leq 3 \), then it is easy to see that \( G' = G - x \) verifies the induction hypothesis. This means that \( G' \geq *K_3 \) and by the transitivity of \( \geq \), we have \( G \geq *K_3 \). We can assume, in the following, that \( d(x) \geq 4 \) for every vertex \( x \) of \( G \). If all vertices of \( G \) have a degree \( \geq 6 \), then \( G \) has at least 3n arcs and the induction hypothesis applies to the graph \( G' \) obtained from \( G \) by removing one arc. So, we can suppose that \( G \) contains at least one vertex \( u \) such that \( d(u) \in \{4,5\} \). We can also assume that the following condition is verified:

\[(*) \quad \text{If } G' \text{ is obtained from } G \text{ by contraction of one arc with both its end-vertices in } A(u), \text{ then } |A(G')| \leq |A(G)| - 4.\]
Otherwise, the induction hypothesis applies to $G'$.

Let us suppose that $d^+(u) \geq d^-(u)$. We can ensure this to be always the case by changing the orientation of all arcs.

Now, let $d(u) = 4$. If $|A(u)| = 2$, let $x$ and $y$ be the neighbours of $u$. $G$ contains the symmetrical arcs $ux$ and $uy$ because $d(u) = 4$. The condition $(\ast)$ implies the existence of the symmetrical arc $xy$.

If $|A(u)| = 3$, let $A(u) = \{x, y, z\}$ and suppose that there is a symmetrical arc $ux$. By condition $(\ast)$, there must be at least one symmetrical arc either between $x$ and $y$ or $x$ and $z$ (see Fig. 2).

![Fig. 2](image)

If this is not the case, then it is easy to see that for any orientation of the arcs between vertices of $A(u)$, the contraction of any arc incident to $u$ decreases the number of arcs by at most 3. So, let $xz$ be a symmetrical arc. By the same argument, we can see that there is at least one arc between $y$ and $z$ (drawn as a segment because we don't know its orientation). The graph obtained by the contraction of the edge $uz$ must verify $(\ast)$.

This implies that either the two arcs incident to $y$ go to $y$, or they come out from $y$. But then, for any orientation of $uz$, by contracting either the edge $uy$ or the edge $yz$, we obtain the desired $K_3$.

Let $|A(u)| = 4$. The condition $(\ast)$ implies that there is at least one arc between any pair of vertices of $A(u)$. If $A(u) = A^+(u)$, then it is clear by $(\ast)$ that the graph induced by $A(u)$ is $K_4$. If $A^+(u) = \{x, y\}$ and $A^-(u) = \{z, v\}$ then we obtain a $K_3$ by identifying $x$ with $z$ and $y$ with $v$. If $A^+(u) = \{x, y, z\}$ and $A^-(u) = \{v\}$, then the graph induced by $A^+(u)$ is a $K_3$ and this completes the case $d(u) = 4$.

Suppose now that $u$ has the degree $d(u) = 5$. Let $|A(u)| = 3$ and let $x, y$ and $z$ be the neighbours of $u$. Suppose $(u, x)$ is not a symmetrical arc. The condition $(\ast)$ implies that there is at least one symmetrical arc either between $x$ and $y$ or between $x$ and $z$, for example between $x$ and $y$, and at least one of the edges $xz$ or $xy$, say $xz$. Then we obtain a $K_3$ by contracting $xz$. 

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If \(|A(u)| = 4\) and \(A^+(u) = \{x, y, z, v\}\) and \(A^-(u) = \{x\}\) then the graph \(G_i\) induced by \(u \cup A(u)\) contains at least 14 arcs and if \(n = 5\) we have \(14 > 3n - 3\). So, we can remove one of these arcs and apply the induction hypothesis to \(G_i\).

Suppose now that \(A^+(u) = \{x, y, z\}\) and \(A^-(u) = \{x, v\}\). If there is at least one symmetrical arc between \(x\) and one of \(y, z\) or \(v\), say between \(x\) and \(y\), then \((*)\) implies the existence of at least one of edges \(vy\) or \(vz\) and at least one edge \(yz\). We obtain a \({^*K}_3\) by contracting all thee edges. If there is only one edge between \(x\) and all the other vertices of \(A(u)\), then there is (by \((*)\)) a symmetrical arc \(yz\) and the edge \(vz\). We obtain a \({^*K}_3\) by contracting \(xy\) and \(zy\).

If \(|A(u)| = 5\) and \(|A^+(u)| = 4\) or 5 then the graph \(G_i\) induced by \(u \cup A(u)\) contains at least 16 arcs and for \(n = 6\) we have \(16 > 3n - 3\), so the induction hypothesis applies to \(G_i\). Suppose that \(A^+(u) = \{x, y, z\}\) and \(A^-(u) = \{v, w\}\). If there is one arc \((a, b)\) for any \(a \in A^-(u)\) and for any \(b \in A^+(u)\) then, by contracting \((w, x)\) and \((v, y)\), we obtain a \({^*K}_3\). On the contrary, if \((w, x)\) is not an arc of \(G\) then \(G\) contains (by \((*)\)) the arcs \((x, y), (z, x), (y, z), (w, y), (w, z)\) and \((v, w)\). We obtain a \({^*K}_3\) by identifying the vertices of the sets \(\{y, w\}\) and \(\{x, z, v\}\) and this completes the proof of the inequality in the theorem. The graph \(G\) drawn in Fig. 3 is an example showing that the bound of the theorem is attained for any \(n\).

![Fig. 3. The bound is attained for any \(n\).](image)

In this graph there is a symmetrical arc between \(1\) and \(i\) for \(i = 2, \ldots, n\) and an arc \((i, i + 1)\) for \(i = 2, \ldots, n - 1\). Thus, \(G\) has \(3n - 4\) arcs and it is not contractible onto \({^*K}_3\). This completes the proof of the theorem.

We conclude this paper with some remarks. First, by our Theorem and the result of Duchet and Kaneti [3] the following intuitive conjecture is suggested:

**Conjecture.** If a digraph \(G\) on \(p\) vertices has at least \((2h - 3)p - h(h - 2)\) arcs, then \(G\) is contractible onto \({^*K}_h\) for any integer \(h \geqslant 3\).

We remark that this conjecture is not true for “great” valued of \(p\). This fact is a consequence of a result from Bollobas, Catlin and Erdős [1], by taking a “great” a non-oriented graph and by replacing any edge by a symmetrical arc.
Concerning the contraction of non-oriented graphs onto cliques, Kostochka and then Thomason [10] have shown that if there is a constant $c_p$ that any graph on $n$ vertices having $c_p n$ edges is contractible onto $K_p$, then $c_p = o(p(\log p))$. Unfortunately, this bound does not apply to the digraphs.

References


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