Bohdan Zelinka
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A REMARK ON GRAPH OPERATORS

BOHDAN ZELINKA, Liberec

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Abstract. A theorem is proved which implies affirmative answers to the problems of E. Prisner. One problem is whether there are cycles of the line graph operator $L$ with period other than 1, the other whether there are cycles of the 4-edge graph operator $V_4$ with period greater than 2. Then a similar theorem follows.

Keywords: graph operator, line graph, $k$-edge graph

MSC 1991: 05C99

In [1], page 71, E. Prisner suggests the problem whether there are $L$-cycles with period greater than 1. In the same book on page 131 the problem whether $V_4$-periods greater than 2 are possible is given. We shall prove a general theorem which implies affirmative answers to both these questions.

Let $\Gamma$ be a class of graphs. A graph operator on $\Gamma$ is a mapping $\Phi$ which assigns to every graph $G \in \Gamma$ a graph $\Phi(G) \in \Gamma$.

We consider the class $\Gamma$ of all undirected graphs (finite and infinite) without loops and multiple edges. We denote by $K_0$ the empty graph, i.e. the graph in which both the vertex set and the edge set are empty.

The operator $L$ is the line graph operator which to every graph $G \in \Gamma$ assigns its line graph $L(G)$, i.e. the graph whose vertex set is the edge set of $G$ and in which two vertices are adjacent if and only if there exists a vertex in $G$ incident to both of them (as edges).

The operator $V_4$ is the $k$-edge graph operator. For an integer $k \geq 2$, a $k$-edge of a graph $G$ is either a clique (i.e. a maximal complete subgraph) in $G$ with at most $k$ vertices, or a complete subgraph of $G$ with $k$ vertices. The $k$-edge graph $V_k(G)$ of a graph $G$ is the graph whose vertex set is the set of all $k$-edges of $G$ and in which two vertices are adjacent if and only if they have at least one common vertex (as subgraphs).
Note that if a graph $G$ has no triangles and no isolated vertices, then $\nabla_k(G) = L(G)$ for any $k$. A different situation occurs for graphs with isolated vertices; namely an isolated vertex is not an edge, but it is a clique. We have $\nabla_k(K_1) = K_1$ while $L(K_1) = K_0$.

A graph operator $\Phi$ on $\Gamma$ will be called additive, if $\Phi(K_0) = K_0$ and for every graph $G \in \Gamma$ the image $\Phi(G)$ is the disjoint union of graphs $\Phi(C_i)$, where $C_i$ for $i$ from some index set $I$ are connected components of $G$. (Most of commonly used graph operators have this property.)

If $\Phi$ is a graph operator on $\Gamma$, then we define $\Phi^0$ to be the identical mapping on $\Gamma$ and $\Phi^n$ for a positive integer $n$ to be the operator such that $\Phi^n(G) = \Phi(\Phi^{n-1}(G))$ for every graph $G \in \Gamma$.

By $P_n$ we denote the path of length $n$, i.e. with $n+1$ vertices. In particular, $P_0 = K_1$.

**Theorem 1.** Let $\Phi$ be an additive graph operator on $\Gamma$ and let $r$ be a positive integer. If there is an infinite sequence $(H_n)_{n=0}^\infty$ of pairwise non-isomorphic graphs such that $\Phi(H_0) = H_0$ and $\Phi(H_n) = H_{n-1}$ for any $n \geq 1$, then there are $r$ pairwise non-isomorphic graphs $G_i$, $0 \leq i \leq r-1$, such that the sequence $(\Phi^n(G_i))_{n=0}^\infty$ is periodic with period $r$.

**Proof.** The graph $G_i$ for $0 \leq i \leq r-1$ will be defined as the disjoint union of all graphs $H_j$ such that $j \equiv i \pmod{r}$ and of infinitely many disjoint copies of $H_0$. Evidently the graphs $G_0, \ldots, G_{r-1}$ are pairwise non-isomorphic. If $i, p$ are positive integers and $p \leq i$, then $\Phi^i(H_p) = H_{i-p}$; if $p > i$, then $\Phi^i(H_p) = H_0$. This implies that for $0 \leq i \leq r-1$ we have $\Phi(G_i) = G_i$, where $0 \leq q \leq r-1$ and $q \equiv i - p \pmod{r}$. This implies the assertion. \qed

**Corollary 1.** Let $L$ be the line graph operator and let $r$ be a positive integer. Then there exist at least $r$ graphs $G_i$, $0 \leq i \leq r-1$, such that the sequence $(L^n(G_i))_{n=0}^\infty$ is periodic with period $r$.

**Proof.** The assertion follows from Theorem 1 if we put $H_0 = K_0$ and $H_i = P_{i-1}$ for every positive integer $i$. \qed

**Corollary 2.** Let $\nabla_k$ be the $k$-edge graph operator for an integer $k \geq 2$, let $r$ be a positive integer. Then there exist at least $r$ graphs $G_i$, $0 \leq i \leq r-1$, such that the sequence $(\nabla_k(G_i))_{n=0}^\infty$ is periodic with period $r$.

**Proof.** This again follows from Theorem 1 if we put $H_i = P_i$ for every non-negative integer $i$. \qed
We will prove another theorem similar to the preceding one.

**Theorem 2.** Let $\Phi$ be an additive operator on $\Gamma$, let $H$ be a graph such that $\Phi^n(H)$ is a proper subgraph of $\Phi^{n+1}(H)$ for each non-negative integer $n$. Then there exists a graph $G$ such that $\Phi^{n+1}(G)$ is a proper subgraph of $\Phi^n(G)$ for each non-negative integer $n$.

**Proof.** The graph $G$ is the disjoint union of all graphs $\Phi^n(H)$ for non-negative integers $n$. The graph $\Phi^{n+1}(G)$ is obtained from $\Phi^n(G)$ by deleting the subgraph $\Phi^n(H)$ for any $n$.

At the end we remark that in the proof of Corollary 2 the paths need not necessarily occur.

We may define graphs $P_0^{k}$ analogous to the paths $P_n$. Let $k \geq 2$. We have $P_0^{k} = K_1$ and $P_1^{k} = K_k$. The graph $P_2^{k}$ has $k$ blocks which are complete graphs with $k$ vertices each and a unique articulation common to all of them. If the graph $P_{n-2}^{k}$ is constructed for an integer $n \geq 3$, then to each vertex $v$ of $P_{n-2}^{k}$ which belongs to only one block we assign $k - 1$ new copies of $K_k$, choose one vertex in each of them and identify it with $v$. (In the case $k = 2$ we have $P_n^{2} = P_n$ for any $n$.) We have $\nabla_k(P_0^{k}) = P_0^{k}$, $\nabla_k(P_1^{k}) = P_{n-1}^{k}$ for any positive integer $n$.

**References**


**Author’s address:** Bohdan Zelinka, Katedra aplikovane matematiky Technicke university, Voronská 13, 461 17 Liberec 1, Czech Republic.