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HAMILTONIAN CONNECTEDNESS AND A MATCHING IN POWERS OF CONNECTED GRAPHS

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Summary. In this paper the following results are proved:
1. Let $P_n$ be a path with $n$ vertices, where $n \geq 5$ and $n \neq 7, 8$. Let $M$ be a matching in $P_n$. Then $(P_n)^4 - M$ is hamiltonian-connected.
2. Let $G$ be a connected graph of order $p \geq 5$, and let $M$ be a matching in $G$. Then $G^5 - M$ is hamiltonian-connected.

Keywords: power of a graph, matching, hamiltonian connectedness

AMS classification: 05C70, 05C45

1. INTRODUCTION

By a graph we mean a finite undirected graph with no loops or multiple edges (a graph in the sense of [1] and [2]). If $G$ is a graph, then we denote by $V(G)$, $E(G)$ and $\delta(G)$ the vertex set, the edge set and the diameter of $G$, respectively. The number $|V(G)|$ is called the order of $G$. If $u, v, w \in V(G)$, then the degree of $u$ in $G$ and the distance between $v$ and $w$ in $G$ will be denoted by $\deg(u)$ and $d_G(v, w)$, respectively. If $W \subseteq V(G)$, then we denote by $(W)_G$ the subgraph of $G$ induced by $W$.

A path connecting vertices $u$ and $v$ in $G$ is referred to as a $u - v$ path in $G$. We say that a graph $G$ is hamiltonian-connected if for every pair of distinct vertices $u$ and $v$ of $G$, there exists a hamiltonian $u - v$ path in $G$.

If a spanning subgraph $F$ of $G$ is a regular graph of degree one, then we say that $F$ is a 1-factor of $G$. A set $M \subseteq E(G)$ is called a matching in $G$ if no two edges in $M$ are incident with the same vertex. We denote by $M(G)$ and $H(G)$ the set of matchings in $G$ and the set of hamiltonian paths of $G$, respectively.
For every integer $n \geq 1$, by the $n$-th power $G^n$ of $G$ we mean the graph with $V(G^n) = V(G)$ and

$$E(G^n) = \{uv; u, v \in V(G) \text{ and } 1 \leq d_G(u, v) \leq n\}.$$ 

We now mention some results concerning hamiltonian properties of powers of connected graphs.

**Theorem A.** [5] If $G$ is a nontrivial connected graph, then $G^3$ is Hamilton-connected.

**Theorem B.** [6] Let $G$ be a connected graph of order $p \geq 4$ and let $M$ be a matching in $G$. Then there exists a Hamiltonian cycle $C$ of $G^4$ such that $E(C) \cap M = \emptyset$.

**Theorem C.** [3] Let $G$ be a connected graph of order $p \geq 4$. Then for every matching $M$ in $G^4$ there exists a Hamiltonian cycle $C$ of $G^4$ such that $E(C) \cap M = \emptyset$.

2. RESULTS

In the present paper we prove the following two theorems:

**Theorem 1.** Let $P_n$ be a path with $n$ vertices, where $n \geq 5$ and $n \neq 7, 8$. Let $M$ be a matching in $P_n$. Then $(P_n)^4 - M$ is Hamilton-connected.

**Theorem 2.** Let $G$ be a connected graph of order $p \geq 5$ and let $M$ be a matching in $G$. Then $G^5 - M$ is Hamilton-connected.

To prove Theorem 1 we will use two lemmas and five remarks. The following lemma immediately follows from Theorem B.

**Lemma 1.** Let $M$ be a matching in a complete graph $K_n$, where $n \geq 5$. Then $K_n - M$ is Hamilton-connected.

The following notation will be useful for us.

Let $n \geq 1$ be an integer, and let $w_1, \ldots, w_n$ be mutually distinct vertices. We denote by $A_n$ the path with

$$V(A_n) = \{w_1, \ldots, w_n\} \quad \text{and} \quad E(A_n) = \{w_iw_{i+1}; 1 \leq i \leq n-1\}.$$
A permutation \((k_1, k_2, \ldots, k_n)\) of the set \(\{1, 2, \ldots, n\}\) with the property that 
\(|k_i - k_{i+1}| \leq k\) for every \(i \in \{1, 2, \ldots, n-1\}\) determines the hamiltonian path 
\(P \in \mathcal{H}(\mathcal{A}_n^k)\) with \(E(P) = \{w_1, w_2, w_3, \ldots, w_k, \ldots, w_{k-1}, w_k\}\). The path \(P\) is a 
\(w_{k+1} - w_{k+1}\) path of \((\mathcal{A}_n)^k\) and also a \(w_{k+1} - w_{k+1}\) path of \((\mathcal{A}_n)^k\).

Finally, we define 
\[A_n = w_{n-1}w_n + w_{n-2}w_n\] for \(n \geq 3\).

**Remark 1.** Let \(M\) be a matching in \(\mathcal{A}_n\). Then there exist hamiltonian \(w_1 - w_3\), 
\(w_2 - w_3\) and \(w_2 - w_4\) paths of \((\mathcal{A}_n)^3\) - \(M\).

Let \(T\) be a tree of order \(p = 4\) which is not isomorphic to \(\mathcal{A}_4\). Then \(T\) is isomorphic 
to \(\mathcal{A}_4\). For the sake of simplicity we will assume that \(T = \mathcal{A}_4\). Let \(M\) be a matching 
in \(T\). For every \(j, j \in \{1, 3, 4\}\), there exists a hamiltonian \(w_2 - w_3\) path of \(T^3 - M\).

**Remark 2.** Let \(M\) be a matching in \(\mathcal{A}_n\). Clearly, \((\mathcal{A}_n)^4\) is the complete graph. 
It follows from Lemma 1 that \((\mathcal{A}_n)^4 - M\) is hamiltonian-connected.

We define the following matchings in \(\mathcal{A}_5\):

\[M_1 = \{w_1w_2, w_3w_4\},\ M_2 = \{w_1w_2, w_4w_5\}, \ M_3 = \{w_2w_3, w_4w_5\}\]

For every matching \(M' \in \mathcal{M}(\mathcal{A}_5)\) there exists \(k \in \{1, 2, 3\}\) such that \(M' \subseteq M_k\).

The permutations

\[
(1, 3, 5, 4, 2), (1, 4, 5, 2, 3), (1, 3, 2, 5, 4), (1, 4, 2, 3, 5), (2, 4, 1, 3, 5), \\
(3, 1, 4, 2, 5), (4, 1, 3, 2, 5) \quad \text{for } k = 1,
\]

\[
(1, 4, 3, 5, 2), (1, 4, 2, 5, 3), (1, 3, 5, 2, 4), (1, 4, 3, 2, 5), (2, 4, 1, 3, 5), \\
(3, 1, 4, 2, 5), (4, 1, 3, 2, 5) \quad \text{for } k = 2,
\]

\[
(1, 4, 3, 5, 2), (1, 4, 2, 5, 3), (1, 3, 5, 2, 4), (1, 3, 4, 2, 5), (2, 1, 4, 3, 5), \\
(3, 4, 1, 2, 5), (4, 2, 1, 3, 5) \quad \text{for } k = 3
\]

of the set \(\{1, 2, 3, 4, 5\}\) determine in \((\mathcal{A}_5)^3 - M_k\) the hamiltonian \(w_1 - w_j\) and \(w_1 - w_i\) 
paths, where \(1 \leq i < j \leq 5\).

Hence for every \(i, j, i \in \{1, 2, 3, 4\}\) and \(j \in \{2, 3, 4, 5\}\) there exist hamiltonian 
\(w_i - w_5\) and \(w_1 - w_j\) paths of \((\mathcal{A}_5)^3 - M\).

**Remark 3.** Let \(M\) be a matching in \(\mathcal{A}_n\). The permutations
of the set \{1, \ldots, 6\} determine the hamiltonian \(w_i - w_j\) paths of \((A_6)^4 - M\), where \(1 \leq i < j \leq 6\).

This means that \((A_6)^4 - M\) is hamiltonian-connected.

Remark 4. Let \(M\) be a matching in \(A_7\). The permutations

\[(1,4,6,3,5,2), (1,4,6,2,5,3), (1,3,5,2,6,4), (1,3,6,2,5,4), (3,6,2,4,1,5), (3,5,1,4,2,6), (4,1,3,6,2,5), (4,1,3,5,2,6), (5,2,4,1,3,6)\]

of the set \{1, \ldots, 7\} determine the hamiltonian \(w_i - w_j\) paths of \((A_7)^4 - M\), where \(i \in \{1,2,6,7\}\), \(j \in \{1,2,\ldots,7\}\) and \(i \neq j\).

The permutations

\[(3,6,2,7,5,1,4), (3,6,2,7,4,1,5), (4,1,3,6,2,7,5)\]

of the set \{1, \ldots, 7\} determine the hamiltonian \(w_i - w_j\) paths of \((A_7)^4 - M\), where \(i \in \{1,2,6,7\}\), \(j \in \{1,2,\ldots,7\}\) and \(i \neq j\).

If \(M = \{w_1w_2, w_6w_7\}\), then there exist no hamiltonian \(w_3 - w_4, w_3 - w_5, w_4 - w_5\) paths of \((A_7)^4 - M\).

This means that \((A_7)^5 - M\) is hamiltonian-connected and for every \(j, j \in \{4,6,7,8\}\), there exists a hamiltonian \(w_i - w_j\) path \(P \in \mathcal{H}(A_7)^4 - M\).

Remark 5. Let \(M\) be a matching in \(A_8\).

1. We denote

\[M_1 = E(A_8 - w_1) \cap M.\]

Then \(M_1 \in \mathcal{M}(A_8 - w_1)\). It follows from Remark 4 that for every \(j, j \in \{2,4,5,6,7,8\}\), there exists a hamiltonian \(w_3 - w_j\) path \(P_1 \in \mathcal{H}(A_8 - w_1)^4 - M_1\).

Then

\[P = P_1 + w_1w_3\] is a hamiltonian \(w_3 - w_j\) path of \((A_8)^4 - M_1\).

2. We denote

\[M_1 = E(A_8 - w_2 - w_3) \cap M.\]

Then \(M_1 \in \mathcal{M}(A_8 - w_2 - w_3)\). It follows from Remark 2 that for every \(j, j \in \{4,6,7,8\}\), there exists a hamiltonian \(w_5 - w_j\) path \(P_1 \in \mathcal{H}(A_8 - w_2 - w_3)^4 - M_1\).
We put

$$P = P_1 + w_3 w_3 + w_3 w_1 + w_1 w_2 \quad \text{if} \quad w_1 w_2 \notin M,$$

$$P = P_1 + w_3 w_1 + w_1 w_3 + w_3 w_2 \quad \text{if} \quad w_1 w_2 \in M.$$  

Then $P$ is a hamiltonian $w_2 - w_j$ path of $(A_8)^4 - M$.

Further, we put

$$P = P_1 + w_j w_1 + w_1 w_3 + w_3 w_2 \quad \text{if} \quad j = 4 \quad \text{and} \quad w_2 w_3 \notin M,$$

$$P = P_1 + w_j w_1 + w_1 w_3 + w_2 w_2 \quad \text{if} \quad j = 4 \quad \text{and} \quad w_2 w_3 \in M.$$  

Then $P$ is a hamiltonian $w_2 - w_3$ path of $(A_8)^4 - M$.

The path

$$P = P_1 + w_3 w_1 + w_1 w_2 + w_2 w_j \quad \text{if} \quad j = 4$$

is a hamiltonian $w_2 - w_3$ path of $(A_8)^4 - M$.

Analogously we can show that for every $j$, $j \in \{1, 2, \ldots, 6, 8\}$, there exists a hamiltonian $w_i - w_j$ path of $(A_8)^4 - M$.

3. The permutations

$$(3, 8, 6, 2, 7, 5, 1, 4), (3, 8, 6, 2, 7, 4, 1, 5), (3, 8, 5, 2, 7, 4, 1, 6),$$

$$(4, 1, 3, 8, 6, 2, 7, 5), (4, 1, 3, 8, 5, 7, 2, 6), (5, 1, 3, 8, 4, 7, 2, 6)$$

of the set $\{1, \ldots, 8\}$ determine the hamiltonian $w_i - w_j$ paths of $(A_8)^5 - M$, where $3 \leq i < j \leq 6$.

4. If $M = \{w_1 w_2, w_3 w_4, w_5 w_6, w_7 w_8\}$, then for $i, j, 3 \leq i < j \leq 6$ there exists no hamiltonian $w_i - w_j$ path of $(A_8)^6 - M$.

This means that $(A_8)^6 - M$ is hamiltonian-connected and for $i \in \{1, 2, 7, 8\}$, $j \in \{1, 2, \ldots, 8\}$, $i \neq j$ there exists a hamiltonian $w_i - w_j$ path of $(A_8)^7 - M$.

**Lemma 2.** Let $n \geq 9$, and let $M$ be a matching in $A_n$. Then $(A_n)^4 - M$ is hamiltonian-connected.

**Proof.** We distinguish the following cases and subcases:

1. Let $n = 9$. In $(A_9)^4 - M$ we shall construct hamiltonian $w_i - w_j$ paths, where $1 \leq i < j \leq 9$. Denote

$$W_1 = \{w_1, \ldots, w_9\}, \quad W_2 = \{w_9, \ldots, w_1\},$$

$$G_1 = (W_1)_{A_9} \quad \text{and} \quad G_2 = (W_2)_{A_9}.$$
Moreover, denote by $M_1$ and $M_2$ the matchings with the properties

$$M_1 \in \mathcal{M}(G_1), \quad M_2 \in \mathcal{M}(G_2) \quad \text{and} \quad M_1 \cup M_2 = M.$$  

1.1. $1 \leq i < j \leq 5$ or $5 \leq i < j \leq 9$.

We prove the proposition of Lemma 2 for the case $1 \leq i < j \leq 5$.

If $5 \leq i < j \leq 9$, then the proof is analogous.

It follows from Remark 2 that there exists a hamiltonian $w_i - w_j$ path $P_1 \in \mathcal{H}((G_1)^4 - M_1)$ and a hamiltonian $w_5 - w_6$ path $P_2 \in \mathcal{H}((G_2)^4 - M_2)$. If $w_j = w_6$, then according to Remark 2 there exists a hamiltonian $w_i - w_5$ path $P_1 \in \mathcal{H}((G_1)^4 - M_1)$. This implies that there exists $x \in V(G_1)$ such that $xw_5 \in E(P_1)$ and $x \neq w_1$.

Then $d_{A_4}(x, w_6) \leq 4$. We put

$$P = (P_1 \cup P_2) - xw_5 + xw_6.$$

Then $P$ is a hamiltonian $w_1 - w_j$ path of $(A_9)^4 - M$.

1.2. $1 \leq i < 5$ and $5 < j \leq 9$.

According to Lemma 1 there exists a hamiltonian $w_i - w_5$ path $P_1 \in \mathcal{H}((G_1)^4 - M_1)$ and a hamiltonian $w_5 - w_j$ path $P_2 \in \mathcal{H}((G_2)^4 - M_2)$. We put

$$P = P_1 \cup P_2.$$

Then $P$ is a hamiltonian $w_i - w_j$ path of $(A_9)^4 - M$.

From these two subcases it follows that $(A_9)^4 - M$ is hamiltonian-connected.

2. Let $n > 10$. Assume that for every tree $A_m$, where $9 \leq m < n$, it is proved that $(A_m)^4 - M^*$ is hamiltonian-connected for any matching $M^* \in \mathcal{M}(A_m)$.

In $(A_n)^4 - M$ we shall construct hamiltonian $w_i - w_j$ paths, where $1 \leq i < j \leq n$.

2.1. $1 \leq i < j \leq 5$ or $(n - 4) \leq i < j \leq n$.

We prove the proposition of Lemma 2 for the case $1 \leq i < j \leq 5$. If $(n - 4) \leq i < j \leq n$, then the proof is analogous.

Denote

$$W_1 = \{w_1, \ldots, w_5\}, \quad W_2 = \{w_5, \ldots, w_n\},$$

$$G_1 = \langle W_1 \rangle_{A_4} \quad \text{and} \quad G_2 = \langle W_2 \rangle_{A_4}.$$

Moreover, denote by $M_1$ and $M_2$ the matchings with the properties

$$M_1 \in \mathcal{M}(G_1), \quad M_2 \in \mathcal{M}(G_2) \quad \text{and} \quad M_1 \cup M_2 = M.$$  

It follows from the induction hypothesis and Remarks 3, 4, 5 that there exists a hamiltonian $w_5 - w_6$ path $P_2 \in \mathcal{H}((G_2)^4 - M_2)$. It follows from Remark 2 that $310$
there exists a hamiltonian $w_i - w_j$ path $P_1 \in \mathcal{H}((G_1)^k - M_1)$ and if $w_j = w_k$, then $P_1 \in \mathcal{H}((G_j)^k - M_1)$. This implies that there exists $x \in V(G_1)$ such that $xw_3 \in E(P_1)$ and $x \neq w_1$. Then $d_{A_n}(x, w_3) \leq 4$ and

$$P = (P_1 \cup P_2) - xw_3 + xw_6$$

is a hamiltonian $w_i - w_j$ path of $(A_n)^k - M$.

2.2. 1 $\leq i \leq 4$ and 6 $\leq j \leq n$ or 5 $\leq i < j \leq n - 4$ or 5 $\leq i \leq n - 5$ and $n - 3 < j \leq n$.

2.2.1. There exists $w_k \in V(A_n)$ with the property

$$i < k < j \quad \text{and} \quad 5 \leq k \leq n - 4.$$

Denote

$$W_1 = \{w_1, \ldots, w_k\}, \quad W_2 = \{w_k, w_{k+1}, \ldots, w_n\},$$

$$G_1 = (W_1)_{A_n} \quad \text{and} \quad G_2 = (W_2)_{A_n}.$$

Further, denote by $M_1$ and $M_2$ the matchings with the properties

$$M_1 \in \mathcal{M}(G_1), \quad M_2 \in \mathcal{M}(G_2) \quad \text{and} \quad M_1 \cup M_2 = M.$$

According to the induction hypothesis and Remarks 2, 3, 4, 5 there exists a hamiltonian $w_i - w_j$ path $P_1 \in \mathcal{H}((G_1)^k - M_1)$ and a hamiltonian $w_k - w_j$ path $P_2 \in \mathcal{H}((G_2)^k - M_2)$. Then

$$P = P_1 \cup P_2$$

is a hamiltonian $w_i - w_j$ path of $(A_n)^k - M$.

2.2.2. There exists no $w_k \in V(A_n)$ with the property (1). Then $w_iw_j \in E(A_n)$ and $5 \leq i < j \leq n - 4$. Hence $w_j = w_{i+1}$.

We denote by $G_1$ or $G_2$ the component of $A_n - w_iw_{i+1}$ which contains $w_i$ or $w_{i+1}$, respectively. Further, we denote by $M_1$ and $M_2$ the matchings with the properties

$$M_1 \in \mathcal{M}(G_1), \quad M_2 \in \mathcal{M}(G_2), \quad M_1 = M \cap E(G_1) \quad \text{and} \quad M_2 = M \cap E(G_2).$$

It follows from the induction hypothesis and Remarks 2, 3, 4, 5 that there exists a hamiltonian $w_{i-1} - w_i$ path $P_1 \in \mathcal{H}((G_1)^k - M_1)$ and a hamiltonian $w_{i+1} - w_{i+2}$ path $P_2 \in \mathcal{H}((G_2)^k - M_2)$. Then

$$P = P_1 \cup P_2 + w_{i+1}w_{i+2}$$

is a hamiltonian $w_i - w_j$ path of $(A_n)^k - M$.

From this subcases it follows that $(A_n)^k - M$ is hamiltonian-connected. Thus the proof of Lemma 2 is complete.

\[\square\]

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Theorem 1 immediately follows from Lemma 2 and Remarks 2 and 3.

To prove Theorem 2 we will use the previous lemmas and remarks as well as the two following lemmas.

**Lemma 3.** Let $T$ be a tree of order $p \geq 5$ and let $M$ be a matching in $T$. Then $T^5 - M$ is hamiltonian-connected.

**Proof.** The cases when $p \in \{5,6,7\}$ follows immediately from Lemma 1 and Remark 4.

Let $p = 8$. If $T$ is isomorphic to $A_8$, or $\delta(T) \leq 5$, then the proposition of Lemma 3 follows from Remark 5 and Lemma 1.

Denote

$T_1 = A_8,$

$T_2 = A_8 - w_2w_9 + w_5w_8,$

$T_3 = A_8 - w_7w_9 + w_4w_8,$

$T = \{T_1, T_2, T_3\}.$

If $T$ is not isomorphic to $A_8$ and $\delta(T) > 5$, then $T$ is isomorphic to one of the elements of $T$. For the sake of simplicity we shall assume that $T \in T$. Further, we denote

$M_0 = E(T - w_8) \cap M.$

Then $T - w_8 = A_7$ and $M_0 \in M(A_7)$. It follows from Remark 4 that there exists a hamiltonian $w_i - w_j$ path $P_0 \in H((A_7)^5 - M_0)$, where $i, j \in \{1,\ldots,7\}, i \neq j$. Since $|E(P_0)| = 6$, there exist integers $k, l \in \{1,\ldots,7\}, k \neq l$, such that $w_kw_l \in E(P_0)$ and

$k, l \notin \{1,6\} \text{ if } T = T_1,$

$k, l \neq 5 \text{ if } T = T_2,$

$k, l \neq 4 \text{ if } T = T_3.$

Then

$P = P_0 - w_kw_l + w_kw_9 + w_lw_8$ is a hamiltonian $w_i - w_j$ path of $T^5 - M$, where $i, j \in \{1,\ldots,7\},$

$P = P_0 + w_jw_9$ is a hamiltonian $w_i - w_j$ path of $T^5 - M$ if $j = 3$ and $i \in \{1,2,4,5,6,7\},$

$P = P_0 + w_kw_9$ is a hamiltonian $w_j - w_k$ path of $T^5 - M$ if $i = 2$ and $j = 3$.

This means that for $p = 8$ the statement of Lemma 3 is correct.

Let $p \geq 9$. Assume that for every tree $T^*$ of order $p^*$, where $5 \leq p^* < p$, it is proved that $(T^*)^5 - M^*$ is hamiltonian-connected for any matching $M^* \in M(T^*)$. 

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If $T$ is isomorphic to $A_p$, or if $\delta(T) \leq 5$, then the result follows from Lemma 2 or Lemma 1. We shall assume that $T$ is not isomorphic to $A_p$ and $\delta(T) > 5$.

Let $x$ and $y$ be arbitrary distinct vertices of $T$. We shall construct a hamiltonian $x - y$ path $P$ of $T^5 - M$.

We denote by $t_x$, $t_y$ the vertices of $T$ with the following properties:

1. $t_x, t_y \in E(T)$,
2. $t_x, t_y$ belong to the $x - y$ path in $T$,
3. $0 \leq d_T(t_x, x) < d_T(t_y, x)$.

Then $T - t_x t_y$ has two components. We denote by $T_x$ or $T_y$ the component of $T - t_x t_y$ which contains $x$, $t_x$ or $y$, $t_y$, respectively. Further, we denote by $M_x$ and $M_y$ the matching with the properties

$$M_x \in \mathcal{M}(T_x), \quad M_y \in \mathcal{M}(T_y), \quad M_x = M \cap E(T_x) \quad \text{and} \quad M_y = M \cap E(T_y).$$

We define graphs $T_1$ and $T_2$:

$$T_1 = T_x \quad \text{and} \quad V(T_2) = V(T_y) \cup \{t_x\}, \quad E(T_2) = E(T_y) \cup \{t_x t_y\}.$$ 

Finally, we denote by $M_1$ and $M_2$ the matchings with the properties

$$M_1 \in \mathcal{M}(T_1), \quad M_2 \in \mathcal{M}(T_2), \quad M_1 = M_x \quad \text{and} \quad M_2 = M \cap E(T_2).$$

We distinguish the following cases and subcases:

1. There exist $t_x, t_y \in V(T)$ with the properties (1)–(3) such that $|V(T_x)| \geq 5$ and $|V(T_y)| \geq 5$. Then $|V(T_1)| \geq 5$ and $|V(T_2)| \geq 5$.

   1.1. Let $t_x \neq x$. According to the induction hypothesis there exists a hamiltonian $x - t_x$ path $P_1 \in \mathcal{H}((T_x)^5 - M_1)$ and a hamiltonian $t_x - y$ path $P_2 \in \mathcal{H}((T_y)^5 - M_2)$. We put

   $$P = P_1 \cup P_2.$$ 

   1.2. Let $t_x = x$. We denote by $x_1$ the vertex of $T_x$ with the property that $xx_1 \in E(T_x)$. If $t_y = y$, then we denote by $y_1$ the vertex of $T_y$ with the property that $yy_1 \in E(T_y)$. Then $d_T(x_1, t_x) = 2$ and $d_T(x_1, y_1) = 3$. It follows from the induction hypothesis that there exists a hamiltonian $x - x_1$ path $P_1 \in \mathcal{H}((T_x)^5 - M_1)$ and a hamiltonian path $P_2 \in \mathcal{H}((T_y)^5 - M_2)$. Let us suppose that

   $$P_1 \text{ is a hamiltonian } t_x - y \text{ path if } t_x \neq y \quad \text{and} \quad P_2 \text{ is a hamiltonian } y_1 - y \text{ path if } t_x = y.$$
We put

\[ P = P_1 \cup P_2 + x_1 t_y \quad \text{if} \quad t_y \neq y \]
\[ P = P_1 \cup P_2 + x_1 y_1 \quad \text{if} \quad t_y = y. \]

2. For every two vertices \( t_x, t_y \) with the properties (1)-(3) we have \( |V(T_x)| < 5 \) or \( |V(T_y)| < 5 \). We put \( t_y = y \). Without loss of generality we assume that \( |V(T_y)| < 5 \).

2.1. Let \( |V(T_y)| = 1 \). Then \( V(T_y) = \{y\} \) and \( |V(T_x)| \geq 3 \). There exists \( u \in V(T_x) \) such that \( u \neq x, u \neq t_y \) and \( 1 \leq d_T(u, t_y) < 2 \). Then \( 2 \leq d_T(u, y) \leq 3 \). It follows from the induction hypothesis that there exists a hamiltonian \( x - u \) path \( P_1 \in \mathcal{H}(T_x)^5 - M_x \). We put

\[ P = P_1 + uy. \]

2.2. Let \( |V(T_y)| = 4 \). According to Remark 1 there exists a hamiltonian \( y - v \) path \( P_2 \in \mathcal{H}(T_y)^5 - M_y \), where \( v \in V(T_y) \) and

\[ d_T(v, y) = 1 \quad \text{if} \quad T_y \text{ is not isomorphic to } A_4, \]
\[ d_T(v, y) = 2 \quad \text{if} \quad T_y \text{ is isomorphic to } A_4. \]

Since \( |V(T_y)| = 4 \) and \( p \geq 9 \), we have \( |V(T_x)| \geq 5 \). We denote by \( u \) the vertex with the properties
\[ u \in V(T_x), \quad u \neq x \quad \text{and} \quad d_T(u, y) \leq 2. \]
Then \( d_T(u, v) \leq 4 \). It follows from the induction hypothesis that there exists a hamiltonian \( x - u \) path \( P_1 \in \mathcal{H}(T_x)^5 - M_x \). We put

\[ P = P_1 + vu. \]

2.3. Let \( 1 < |V(T_y)| < 4 \). Let \( S_1, \ldots, S_m \) be all components of \( T - t_x \) which are different from \( T_y \). We denote by \( L_1, \ldots, L_m \) the matchings in \( S_1, \ldots, S_m \) such that \( L_j = M \cap E(S_j) \) for \( j = 1, \ldots, m \).

2.3.1. There exists \( i, i \in \{1, \ldots, m\} \) such that \( |V(S_i)| \geq 5 \). Then there exist \( u_1, u_2 \in V(S_i) \) such that \( u_1 \neq u_2 \neq x, d_T(u_1, t_y) \leq 2, 1 < d_T(u_2, t_y) \leq 3 \), and if \( x \notin V(S_i) \), then \( d_T(u_1, t_y) = 1 \). According to the induction hypothesis there exists a hamiltonian path \( P_1 \in \mathcal{H}(S_i)^5 - L_i \). Let us suppose that

\[ P_1 \text{ is a hamiltonian } u_1 - u_2 \text{ path if } x \notin V(S_i), \]
\[ P_1 \text{ is a hamiltonian } u_2 - x \text{ path if } x \in V(S_i). \]

Denote

\[ T_0 = T - V(S_i). \]
Then $T_0$ is a tree, $|V(T_0)| \geq 3$ and $y \in V(T_0)$. Further we denote by $M_0$ the matching in $T_0$ such that $M_0 = M \cap E(T_0)$.

2.3.1.1. Let $|V(T_0)| = 3$. Then $m = i = 1$ and there exists $v \in V(T_0)$ such that $V(T_0) = \{i, y, v\}$ and $E(T_0) = \{i y, v y\}$. If $x \notin V(S_i)$, then $x = t_x$. We put

$P = P_1 + u_1 v + v x + u_2 y$ if $x \notin V(S_i)$,
$P = P_1 + u_2 v + v x + t_x y$ if $x \in V(S_i)$ and $t_x y \notin M$,
$P = P_1 + u_2 t_x + t_x v + v y$ if $x \in V(S_i)$ and $t_x y \in M$.

2.3.1.2. Let $|V(T_0)| = 4$. Assume that $x \in V(S_i)$. Then according to Remark 1 there exists a hamiltonian $y - v$ path $P_2 \in \mathcal{H}((T_0)^3 - M_0)$, where $v \in V(T_0)$, $v \neq y$ and

$$d_T(t_x, v) = 2 \text{ if } \deg_T t_x = 1,$$
$$d_T(t_x, v) = 1 \text{ if } \deg_T t_x = 2.$$

Then $d_T(v, u_2) \leq 5$. We put

$P = P_1 \cup P_2 + u_2 v$.

Let $x \notin V(S_i)$. There exist $v_1, v_2 \in V(T_0)$ such that $v_1 \neq v_2 \neq t_x \neq y$. Then $V(T_0) = \{t_x, y, v_1, v_2\}$. We put

$P = P_1 + u_1 v_2 + v_2 y + u_3 v_1 + v_1 x$ if $x = t_x$ and $E(T_0) = \{x y, y v_1, v_1 v_2\}$,
$P = P_1 + u_1 v_2 + v_2 y + v_1 x + u_2 y$ if $x = t_x$ and $E(T_0) = \{x y, y v_1, v_2 x\}$

or if $x = t_x$ and $E(T_0) = \{x y, y v_1, x v_2\}$,
$P = P_1 + u_3 y + u_2 t_x + t_x v_1 + v_1 x$ if $x = v_2$ and $E(T_0) = \{x t_x, t_x y, v_2 x\}$.

2.3.1.3. Let $|V(T_0)| \geq 5$. Since $|V(T_0)| < 5$ or $|V(T_y)| < 5$ for every two vertices $t_x, t_y$ of $T$ with the properties (1)–(3), we have $x \notin V(S_i)$. It follows from the induction hypothesis that there exists a hamiltonian $x - y$ path $P_2 \in \mathcal{H}((T_0)^3 - M_0)$. Since $|V(T_0)| \geq 4$, there exists $v \in V(T_0)$ such that $v y \in E(P_2)$ and $d_T(v, t_x) \leq 4$. We put

$P = P_1 \cup P_2 - v y + u_1 v + u_2 y$ if $v \neq t_x$,
$P = P_1 \cup P_2 - v y + u_2 v + u_1 y$ if $v = t_x$.

2.3.2. For every $i, i \in \{1, \ldots, m\}$ we have $|V(S_i)| < 5$. Denote

$$T_0 = T - V(T_y), \quad M_0 = M \cap E(T_0).$$

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Then $|V(T_0)| > 5$, $M_0 \in \mathcal{M}(T_0)$, $x \in V(T_0)$ and for every $i, i \in \{1, \ldots, m\}$, we have $V(S_i) \subset V(T_0)$. There exists $v \in V(T_0)$ such that $v \neq x$ and $1 \leq d_T(v, t_x) \leq 2$. It follows from the induction hypothesis that there exists a hamiltonian $x - v$ path $P_0 \in H((T_0)^5 - M_0)$. Since $|V(T_0)| \in \{2, 3\}$ and $\delta(T) > 5$, there exists $k, k \in \{1, \ldots, m\}$, such that $S_k$ is isomorphic to one of the elements of $A$, where

$$A = \{A_2, A_3, A_4\}.$$ 

For the sake of simplicity we shall assume that $S_k \in A$. Then

$$V(S_k) = \{w_1, \ldots, w_n\}, \text{ where } n \in \{3, 4\},$$

$$d_T(w_j, t_x) = j \quad \text{for every } j, j \in \{1, 2, 3\},$$

$$d_T(w_4, t_x) = 4 \quad \text{if } S_k = A_4 \text{ and } d_T(w_4, t_x) = 3 \quad \text{if } S_k = A_4.$$ 

Let $a_2$ and $a_3$ be distinct vertices of $T_0$ such that $a_2w_2, a_3w_3 \in E(P_0)$. If $S_k = A_4$, then there exists $h, h \in \{2, 3\}$, such that $a_h \neq w_4$. Then $d_T(a_h, t_x) \leq 3$. The component $T_y$ is isomorphic to one of the elements of $B$, where

$$B = \{A_2, A_3, A_3\}.$$ 

We denote the vertices of $T_y$ by $t_1, \ldots, t_n \ (n \in \{2, 3\})$ so that

$$d_T(t_j, t_x) = j \quad \text{if } j \in \{1, 2\},$$

$$d_T(t_3, t_x) = 3 \quad \text{if } T_y \text{ is isomorphic to } A_2,$$

$$d_T(t_3, t_x) = 2 \quad \text{if } T_y \text{ is isomorphic to } A_3.$$ 

Then $t_1 = y, d_T(a_h, t_2) \leq 5, d_T(w_2, t_2) = 4, d_T(w_3, t_2) = 5$ and $d_T(v, t_2) \leq 5$. We put

$$P = P_0 - a_hw_n + vy + a_h t_2 + w_h t_2 \quad \text{if } T_y \text{ is isomorphic to } A_2,$$

$$P = P_0 - a_hw_n + v3y + a_h t_2 + w_h t_2 \quad \text{if } T_y \text{ is isomorphic to } A_3,$$

$$P = P_0 - a_hw_n + vy + a_h t_2 + t_2t_3 + t_3w_n \quad \text{if } T_y \text{ is isomorphic to } A_3.$$ 

We can see that in each subcase $P$ is the hamiltonian $x - y$ path of $T^5 - M$. Thus the proof of Lemma 3 is complete.

**Lemma 4.** ([4] p.63) Let $G$ be a connected graph and let $L$ be a subgraph of $G$ which contains no cycle. Then there exists a spanning tree $T$ of $G$ such that $L$ is a subgraph of $T$.

**Proof of Theorem 2.** Let $G$ be a graph satisfying the conditions of Theorem 2 and let $M$ be an arbitrary matching in $G$. As follows from Lemma 4, there exists a spanning tree $T$ of $G$ such that $M$ is a matching in $T$. According to Lemma 3, $T^5 - M$ is hamiltonian-connected. Thus $G^5 - M$ is also hamiltonian-connected. □
Remark 6. Let $n \geq 1$ be an integer, and let $G$ be the tree of order $p = 4n + 4$ which is given in Fig. 1. Let

$$M = \{u_{i1}u_{i2}, u_{i3}u_{i4} ; 1 \leq i \leq n\} \cup \{xy, w_3w_4\}$$

be a matching in $G$. Then there exists no hamiltonian $x - y$ path of $G^4 - M$.

This means that the value 5 of the power in Theorem 2 is the best possible.

References


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