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A MATCHING AND A HAMILTONIAN CYCLE
OF THE FOURTH POWER OF A CONNECTED GRAPH

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Summary. The following result is proved: Let $G$ be a connected graph of order $\geq 4$. Then for every matching $M$ in $G^4$ there exists a hamiltonian cycle $C$ of $G^4$ such that $E(C) \cap M = \emptyset$.

Keywords: power of a graph, matching, hamiltonian cycle

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Let $G$ be a graph (in the sense of the book [1], for example) with a vertex set $V(G)$ and an edge set $E(G)$; note that the number $|V(G)|$ is referred to as the order of $G$. If $n$ is a positive integer, then by the $n$-th power $G^n$ of $G$ we mean the graph $G'$ such that $V(G') = V(G)$ and vertices $u$ and $v$ are adjacent in $G'$ if and only if $1 \leq d_G(u,v) \leq n$, where $d_G$ denotes the distance in $G$.

Chartrand, Polimeni and Stewart [2] and Sumner [6] have proved that if $G$ is a connected graph of an even order, then $G^2$ has a 1-factor. As follows from Sekanina's paper [5], if $G$ is a connected graph of order $\geq 3$, then $G^3$ has a hamiltonian cycle. The existence of 1-factors and/or a hamiltonian cycle of the fourth power of a connected graph was investigated in [3], [7], [4] and [8].

Let $G$ be a connected graph of an even order $\geq 4$. The present author [3] proved that $G^4$ has a 3-factor each component of which is $K_4$ or $K_2 \times K_3$, where $\times$ denotes the cartesian product of graphs. Consequently, $G^4$ has tree mutually edge-disjoint 1-factors. Wisztová [7] proved that there exist a hamiltonian cycle $C$ of $G^3$ and a 1-factor $F$ of $G^4$ such that $E(F) \cap E(C) = \emptyset$. This result was improved by the present author [4] as follows: for any factor $H$ of $G^3$ such that $H$ contains no triangle and the maximum degree of $H$ does not exceed 2, there exists a 1-factor $F$ of $G^4$ such that $E(F) \cap E(H) = \emptyset$. Consequently, for every hamiltonian cycle $C$ of $G^3$ there exists a 1-factor $F$ of $G^4$ such that $E(F) \cap E(C) = \emptyset$. 

43
Recently, Wisztová [8] has proved that if $G$ is a connected graph of an order $\geq 4$ and $M$ is a matching in $G$, then there exists a hamiltonian cycle $C$ of $G^4$ such that $E(C) \cap M = \emptyset$. In the present paper the result obtained in [8] will be improved as follows: if $G$ is a connected graph of an order $\geq 4$ and $M$ is a matching in $G^4$, then there exists a hamiltonian cycle $C$ of $G^4$ such that $E(C) \cap M = \emptyset$.

Before proving the main result of the paper we shall introduce some auxiliary notions and prove three lemmas.

If $F_1$ and $F_2$ are graphs, then we denote by $F_1 \cup F_2$ the graph $F'$ with $V(F') = V(F_1) \cup V(F_2)$ and $E(F') = E(F_1) \cup E(F_2)$. If $F$ is a graph and $u$ and $v$ are distinct vertices, then we denote by $F + uv$ the graph $F''$ with $V(F'') = V(F) \cup \{u, v\}$ and $E(F'') = E(F) \cup \{uv\}$. If $H$ is a graph and $W$ is a nonempty subset of $V(H)$, then we denote by $(W)_H$ the subgraph of $H$ induced by $W$.

An ordered pair $(T, v)$, where $T$ is a tree and $v \in V(T)$ will be referred to as a rooted tree. We say that rooted trees $(T_1, v_1)$ and $(T_2, v_2)$ are isomorphic if there exists an isomorphism $f$ of $T_1$ onto $T_2$ such that $f(v_1) = v_2$.

Now, let $k \geq 1$ and $m \geq 1$ be integers, and let $u_0, \ldots, u_k, w_1, \ldots, w_m$ be mutually distinct vertices. We shall generalize some constructions used in [8]. By a $Y_m$-tree ($m \geq 5$) we mean a tree $T$ such that

$$V(T) = \{w_1, \ldots, w_m\},$$
$$\{w_jw_{j+1}; 1 \leq j \leq m-2\} \subseteq E(T),$$
and
either $w_mw_1 \in E(T)$ or $w_1w_m \in E(T)$.

By a $Y^*_m$-tree ($m \geq 5$) we mean a tree isomorphic to a $Y_m$-tree. By an $X_m$-tree ($m \geq 5$) we mean a tree $T'$ such that

$$V(T') = \{w_1, \ldots, w_m\},$$
$$\{w_jw_{j+1}; 2 \leq j \leq m-2\} \subseteq E(T'),$$
and
either $w_1w_2 \in E(T')$ or $w_1w_3 \in E(T')$, and
either $w_mw_2 \in E(T')$ or $w_mw_3 \in E(T')$.

By an $X^*_m$-tree ($m \geq 5$) we mean a tree isomorphic to $X_m$-tree. By a $U_{k,m}$-tree we mean a rooted tree $(T'', u_0)$ such that

$$V(T'') = \{u_k, \ldots, u_0, w_1, \ldots, w_m\},$$
$$\{u_{i+1}u_i; 1 \leq i \leq k-2\} \cup \{u_1u_0, u_0w_1\} \cup \{w_jw_{j+1}; 1 \leq j \leq m-2\} \subseteq E(T'');$$
if $k = 2$, then $u_2u_1 \in E(T'')$,
if $k \geq 3$, then either $u_ku_{k-1} \in E(T'')$ or $u_ku_{k-2} \in E(T'')$,
if $m = 2$, then $w_1w_2 \in E(T'')$, and
if $m \geq 3$, then either $w_mw_{m-2} \in E(T'')$ or $w_{m-1}w_m \in E(T'')$. 

44
Finally, by a $U_{k,m}$-tree we mean a rooted tree isomorphic to $U_{k,m}$.

**Lemma 1.** Let $m \geq 5$ be an integer, let $T$ be a $Y_m$-tree, and let $M$ be a matching in $T^3$. Then there exists a hamiltonian $w_1-w_2$ path $P$ of $T^3$ such that $E(P) \cap M = \emptyset$.

**Proof.** We shall construct a hamiltonian $w_1-w_2$ path $P$ of $T^3$ such that $E(P) \cap M = \emptyset$.

First, let $m = 5$. We put

$$E(P) = \{w_1w_3, w_3w_4, w_4w_5, w_5w_2\} \text{ if } w_3w_5 \in M,$$

$$E(P) = \{w_1w_4, w_4w_3, w_3w_5, w_5w_2\} \text{ if } w_4w_5 \in M,$$

$$E(P) = \{w_1w_3, w_3w_5, w_5w_4, w_4w_2\} \text{ if } (w_3w_5, w_4w_5 \notin M, w_2w_3 \in M)$$

$$\text{or } (w_2w_3, w_3w_5, w_5w_6 \notin M, w_1w_4 \in M),$$

$$E(P) = \{w_1w_4, w_4w_5, w_5w_3, w_3w_2\} \text{ if } w_1w_4, w_2w_3, w_3w_5, w_4w_5 \notin M.$$

Now let $m = 6$. We put

$$E(P) = \{w_1w_3, w_3w_6, w_6w_5, w_5w_4, w_4w_2\} \text{ if } w_2w_3, w_4w_6 \in M,$$

$$E(P) = \{w_1w_4, w_4w_5, w_5w_6, w_6w_3, w_3w_2\} \text{ if } w_2w_3 \notin M, w_4w_6 \in M,$$

$$E(P) = \{w_1w_3, w_3w_5, w_5w_6, w_6w_4, w_4w_5, w_5w_2\} \text{ if } (w_2w_3 \in M, w_4w_6 \notin M)$$

$$\text{or } (w_2w_3, w_4w_6 \notin M, w_1w_4, w_3w_5 \in M) \text{ or } (w_2w_3, w_4w_6 \notin M, w_1w_4 \in M, w_3w_5 \notin M, w_5w_6 \in M),$$

$$E(P) = \{w_1w_3, w_3w_4, w_4w_6, w_6w_5, w_5w_2\} \text{ if } \begin{cases} w_2w_3 \in M, & w_4w_6 \notin M, \\
   w_5w_6 \notin M, & w_1w_4 \in M \text{ or } (w_2w_3, w_4w_6 \notin M, w_1w_4 \notin M, w_3w_5 \in M), \\
   w_1w_4 \notin M, & w_3w_5 \in M, \end{cases}$$

$$E(P) = \{w_1w_4, w_4w_3, w_3w_6, w_6w_5, w_5w_2\} \text{ if } w_2w_3, w_4w_6 \notin M,$$

$$w_5w_6 \notin M, w_1w_4 \notin M,$$

$$E(P) = \{w_1w_3, w_3w_5, w_5w_6, w_6w_4, w_4w_2\} \text{ if } w_2w_3, w_4w_6 \notin M,$$

$$w_1w_4 \in M, w_3w_5 \notin M, w_5w_6 \notin M,$$

$$E(P) = \{w_1w_4, w_4w_6, w_6w_3, w_3w_5, w_5w_2\} \text{ if } w_2w_3, w_4w_6 \notin M,$$

$$w_1w_4 \notin M, w_3w_5 \notin M, w_5w_6 \in M,$$

$$E(P) = \{w_1w_4, w_4w_6, w_6w_5, w_5w_3, w_3w_2\} \text{ if } w_2w_3, w_4w_6 \notin M,$$

$$w_1w_4 \in M, w_3w_5 \notin M, w_5w_6 \notin M.$$

Finally, let $m \geq 7$. We assume that for $m-2$ the statement of the lemma is proved. Denote $T_0 = T - w_1 - w_2$ and $M_0 = M \cap E((T_0)^3)$. According to our assumption, there exists a hamiltonian $w_3-w_4$ path $P_0$ of $(T_0)^3$ such that $E(P_0) \cap M_0 = \emptyset$. We
Thus, the proof of the lemma is complete. \(\square\)

As immediately follows from Lemma 1, if \(m \geq 5\) is an integer, \(T\) is a \(Y_m\)-tree, and \(M\) is a matching in \(T^4\), then there exists a hamiltonian \(w_1 - w_2\) path \(P\) of \(T^4\) such that \(E(P) \cap M = \emptyset\).

In the proof of the next lemma an idea from the proof of Lemma 3 in [8] will be used.

**Lemma 2.** Let \(m \geq 5\) be an integer, let \(T\) be an \(X_m\)-tree, and let \(M\) be a matching in \(T^4\). Then there exists a hamiltonian cycle \(C\) of \(T^4\) such that \(E(C) \cap M = \emptyset\).

**Proof.** Obviously, if \(m = 5\) then \(T^4 = K_5\), and if \(m = 6\) then \(T^4 = K_6 - e\) or \(K_6\). Thus, we can see that if \(m = 5\) or \(6\), the statement of the lemma holds.

Let \(m \geq 7\). Denote \(T_0 = T - w_1 - w_2\). Clearly, \(T_0\) is a \(Y_{m-2}\)-tree. According to Lemma 1, there exists a hamiltonian \(w_3 - w_4\) path \(P_0\) of \((T_0)^3\) such that \(E(P_0) \cap m = \emptyset\).

First, let \(w_1w_2 \in M\). Obviously, there exists \(w \in V(T_0 - w_3)\) such that \(w_3w \in E(P_0)\). We put

\[ C = P_0 - w_3w + w_2w_3 + w_3w_1 + w_1w_4. \]

Now let \(w_1w_2 \notin M\). We put

\[ C = P_0 + w_3w_1 + w_1w_2 + w_2w_4 \quad \text{if} \quad w_1w_4 \in M \quad \text{or} \quad w_2w_3 \in M, \quad \text{and} \]

\[ C = P_0 + w_3w_2 + w_2w_1 + w_1w_4 \quad \text{if} \quad w_1w_4, \ w_2w_3 \notin M. \]

We can see that \(C\) is a hamiltonian cycle of \(T^4\) such that \(E(C) \cap M = \emptyset\). Thus, the proof of the lemma is complete. \(\square\)

**Lemma 3.** Let \(T\) be a tree of an order \(n \geq 4\), and let \(M\) be a matching in \(T^4\). Then there exists a hamiltonian cycle \(C\) of \(T^4\) such that \(E(C) \cap M = \emptyset\).

**Proof.** We proceed by induction on \(n\). If the diameter of \(T\) does not exceed four, then \(T^4\) is a complete graph and thus the statement of the lemma holds. If \(T\) is an \(X_n^*\)-tree, then—according to Lemma 2—the statement of the lemma holds, too.

We shall assume that the diameter of \(T\) is at least five and \(T\) is not a \(X_n^*\)-tree. This implies that \(n \geq 7\). We distinguish the following cases and subcases:

1. Assume that there exist mutually distinct vertices \(v, v_1, v_2, v_3\) such that \(vv_1, vv_2, vv_3 \in E(T)\) and \(v_1, v_2, v_3\) are vertices of degree one in \(T\). Obviously, there
exist distinct \( g, h \in \{1, 2, 3\} \) such that \( v_g v_h \notin M \). Without loss of generality, let 
\( v_2 v_3 \notin M \). Denote \( T_0 = T - v_2 - v_3 \). Since \( |V(T_0)| = n - 2 \geq 5 \), it follows from 
the induction hypothesis that there exists a hamiltonian cycle \( C_0 \) of \( (T_0)^4 \) such that 
\( E(C_0) \cap (M - \{v_{v_2}, v_{v_3}\}) = \emptyset \). Since \( v_1 \) is a vertex of degree one in \( T_0 \), there exists 
\( v_0 \in V(T_0 - v_1) \) such that \( v_0 v_1 \in E(C_0) \) and \( d_T(v, v_0) \leq 3 \). We put 
\[
C = C_0 - v_0 v_1 + v_0 v_2 + v_2 v_3 + v_3 v_1 \quad \text{if } v_1 v_2 \in M \text{ or } v_0 v_3 \in M,
C = C_0 - v_0 v_1 + v_0 v_3 + v_2 v_3 + v_3 v_1 \quad \text{if } v_1 v_2, v_0 v_3 \notin M.
\]
Obviously, \( C \) is a hamiltonian cycle of \( T^4 \) and \( E(C) \cap M = \emptyset \).

2. Assume that for every vertex \( v \) of \( T \), at most two vertices adjacent to \( v \) have 
degree one. It is not difficult to see that there exist positive integers \( k \) and \( m \), 
a vertex \( u \) of a degree \( \geq 3 \) in \( T \) and a subtree \( T' \) of \( T \) with the properties that 
\( 3 \leq k + m \leq n - 4 \), \( u \in V(T') \), the degree of \( u' \) in \( T' \) is equal to the degree of \( u' \) in 
\( T \) for each \( u' \in V(T' - u) \), and \( (T', u) \) is a \( U_{k,m} \)-tree.

For the sake of simplicity we shall assume that \( (T', u) \) is a \( U_{k,m} \)-tree. Thus \( u = u_0 \) 
and \( V(T_0) = \{u_k, \ldots, u_0, w_1, \ldots, w_m\} \). Without loss of generality we assume that
(1) \[ k \geq 2; \text{ if } m = 2, \text{ then } k \leq 3; \text{ if } m = 3, \text{ then } k = 3; \]
if \( m = 4 \), then \( k \leq 4 \).

Denote \( T_0 = T - w_1 - \ldots - w_m \) and \( M_0 = M \cap E((T_0)^4) \). Since \( 5 \leq |V(T_0)| \leq n - 1 \), 
it follows from the induction hypothesis that there exists a hamiltonian cycle \( C_0 \) of 
\( (T_0)^4 \) such that \( E(C_0) \cap M_0 = \emptyset \). We shall construct a hamiltonian cycle \( C \) of \( T^4 \) 
such that \( E(C) \cap M = \emptyset \).

2.1. Let \( m \neq 2, 3, 4 \).

2.1.1. Assume that 
(2) there exist mutually distinct \( v_{i1}, v_{i2}, v_{i1}, v_{i2} \in V(T) \)
such that \( v_{i1} v_{i2} \in E(C_0), d_T(u_0, v_{i1}) \leq d_T(u_0, v_{i2}) \leq 3 \)
and \( d_T(u_0, v_{i1}) + d_T(u_0, v_{i2}) \leq 4 \) for \( i = 1 \) and \( 2 \).

Without loss of generality we assume that \( v_{i2} w_1, v_{i2} w_1 \notin M \).

2.1.1.1. Let \( m = 1 \). We put 
\[
C = C_0 - v_{i1} v_{i2} + v_{i1} w_1 + w_1 v_{i2}.
\]

2.1.1.2. Let \( m \geq 5 \). Obviously, \( v_{i1} w_2, v_{i2} w_1 \in E(T^4) \) and if \( d_T(v_{i1}, w_2) = 4 \), then 
\( d_T(v_{i2}, w_2) = 4 \).

2.1.1.2.1. Assume that \( v_{i1} w_2 \notin M \) or \( d_T(v_{i1}, w_2) = 4 \). According to Lemma 1 
there exists a hamiltonian \( w_1 - w_2 \) path \( P \) of \( ((\{w_1, \ldots, w_m\})^T)^4 \). We put 
\[
C = (C_0 - v_{i1} v_{i2}) \cup P + v_{i1} w_2 + w_1 v_{i2} \quad \text{if } v_{i1} w_2 \notin M, \text{ and}
C = (C_0 - v_{i1} v_{i2}) \cup P + v_{i1} w_1 + w_2 v_{i2} \quad \text{if } v_{i1} w_2 \in M \text{ and } d_T(v_{i1}, w_2) = 4.
\]
2.1.1.2.2. Assume that $v_{11}w_2 \in M$ and $d_T(v_{11}, w_2) \leq 3$. Then $v_{11}w_3 \in E(T^4) - M$. Moreover, $w_1w_2, w_2w_3 \notin M$.

First, let $m = 5$. We put

$$C = C_0 - v_{11}v_{12} + v_{11}w_3 + w_3w_4 + w_4w_2 + w_2w_5 + w_5w_1 + w_1v_{12}$$

if $w_4w_5 \in M$,

$$C = C_0 - v_{11}v_{12} + v_{11}w_3 + w_3w_2 + w_2w_5 + w_5w_4 + w_4w_1 + w_1v_{12}$$

if $w_4w_5 \notin M$, $w_1w_5 \in M$, and

$$C = C_0 - v_{11}v_{12} + v_{11}w_3 + w_3w_2 + w_2w_4 + w_4w_5 + w_5w_1 + w_1v_{12}$$

if $w_4w_5$, $w_1w_5 \notin M$.

Now let $m \geq 6$. According to Lemma 1 there exists a hamiltonian $w_2 - w_3$ path $P'$ of $\left(\{w_2, \ldots, w_m\}\right)_T^4$. We put

$$C = (C_0 - v_{11}v_{12}) \cup P' + v_{11}w_3 + w_2w_1 + w_1v_{12}.$$

2.1.2. Assume that (2) does not hold. According to (1), $k \geq 2$. It is not difficult to see that $k \geq 4$ and there exists $v \in V(T_0 - u_0 - \ldots - u_k)$ such that $d_T(u_0, v) \leq 3$ and $C_0 - u_1 - \ldots - u_k$ is an $u_0 - v$ hamiltonian path of $(T_0 - u_1 - \ldots - u_k)^4$. Moreover, we can see that if $k = 4$, then $u_0u_4 \in E(C_0)$ and therefore $u_0u_4 \notin M$.

2.1.2.1. Let $vw_1 \in M$. First, let $k = 4$. Recall that $u_0u_4 \notin M$. We put

$$C = (C_0 - u_1 - u_2 - u_3 - u_4) + u_0u_2 + u_2u_4 + u_4u_3 + u_3w_1$$

+ $w_1u_1 + u_1v$ if $u_2u_3 \in M$,

$$C = (C_0 - u_1 - u_2 - u_3 - u_4) + u_0u_4 + u_4u_2 + u_2u_3 + u_3w_1$$

+ $w_1u_1 + u_1v$ if $u_3u_4 \in M$, and

$$C = (C_0 - u_1 - u_2 - u_3 - u_4) + u_0u_4 + u_4u_3 + u_3u_2 + u_2w_1$$

+ $w_1u_1 + u_1v$ if $u_2u_3, u_3u_4 \notin M$.

Now let $k \geq 5$. As follows from Lemma 1, there exists a hamiltonian $u_1 - u_2$ path $P$ of $\left(\{u_1, \ldots, u_k\}\right)_T^4$. We put

$$C = (C_0 - u_1 - \ldots - u_k) \cup P + u_0w_1 + w_1u_2 + u_1v.$$

2.1.2.1.2. Let $vw_1 \notin M$. According to Lemma 1, there exists a hamiltonian $w_1 - u_0$ path $P$ of $\left(\{w_1, u_0, \ldots, u_k\}\right)_T^4$. We put

$$C = (C_0 - u_1 - \ldots - u_k) \cup P + w_1v.$$

2.1.2.2. Assume that $m \geq 5$.  

48
2.1.2. Let \( k = 4 \). First, let \( v w_1 \in M \) or \( u_1 w_2 \in M \). Then \( v u_1 \notin M \). There exists a hamiltonian \( u_0 - w_1 \) path \( P \) of \( \langle \{u_0, w_1, \ldots, w_m\} \rangle_T^4 \). Clearly, \( u_1 u_4 \notin M \) or \( u_3 w_1 \notin M \). We put

\[
C = (C_0 - u_1 - u_2 - u_3 - u_4) + P + v u_1 + u_1 u_3 + u_3 u_4 + u_4 u_2 + u_2 w_1
\]

if \( u_2 u_3 \in M \),

\[
C = (C_0 - u_1 - u_2 - u_3 - u_4) + P + v u_1 + u_1 u_2 + u_2 u_4 + u_4 u_3 + u_3 w_1
\]

if \( u_2 u_3, u_3 w_1 \notin M, u_1 u_4 \in M \),

\[
C = (C_0 - u_1 - u_2 - u_3 - u_4) + P + v u_1 + u_1 u_4 + u_4 u_3 + u_3 u_2 + u_2 w_1
\]

if \( u_2 u_3, u_1 u_4 \notin M, u_3 w_1 \notin M, u_2 u_4 \in M \),

\[
C = (C_0 - u_1 - u_2 - u_3 - u_4) + P + v u_1 + u_1 u_4 + u_4 u_2 + u_2 u_3 + u_3 w_1
\]

if \( u_2 u_3, u_1 u_4, u_3 w_1, u_2 u_4 \notin M \).

Now let \( v w_1, u_1 w_2 \notin M \). According to Lemma 1 there exist a hamiltonian \( u_0 - u_1 \) path \( P' \) of \( \langle \{u_0, \ldots, u_4\} \rangle_T^4 \) and a hamiltonian \( w_1 - w_2 \) path \( P'' \) of \( \langle \{w_1, \ldots, w_m\} \rangle_T^4 \). We put

\[
C = (C_0 - u_1 - u_2 - u_3 - u_4) \cup P' \cup P'' + v w_1 + w_2 u_1.
\]

2.1.2.2. Let \( k \geq 5 \). According to Lemma 1 there exist hamiltonian \( u_1 - u_2 \) path \( P \) of \( \langle \{u_1, \ldots, u_k\} \rangle_T^4 \) and a hamiltonian \( w_1 - w_2 \) path \( P' \) of \( \langle \{w_1, \ldots, w_m\} \rangle_T^4 \). Obviously, \( v w_1 \notin M \) or \( v u_1 \notin M \). Without loss of generality we assume that \( v w_1 \notin M \). We put

\[
C = (C_0 - u_1 - \ldots - u_k) \cup P \cup P' + u_0 u_1 + u_2 w_2 + w_1 v
\]

if \( u_0 u_2 \in M \) or \( u_1 w_2 \in M \), and

\[
C = (C_0 - u_1 - \ldots - u_k) \cup P \cup P' + u_0 u_2 + u_1 w_2 + w_1 v
\]

if \( u_0 u_2, u_1 w_2 \notin M \).

2.2. Let \( m = 2 \). According to (1), \( k = 2 \) or 3. It is easy to see that there exist \( u_1', u_2' \in V(T_0) \) with the properties that \( u_1' \neq u_1, u_2' \neq u_2, u_1 u_1', u_2 u_2' \in E(C_0), u_1 u_1' \neq u_2 u_2', d_T(u_0, u_1') \leq 3 \) and \( d_T(u_0, u_2') \leq 2 \). We put

\[
C = C_0 - u_1 u_1' - u_2 u_2' + u_1 w_1 + w_1 u_1' + u_2 w_2 + w_2 u_2'
\]

if \( w_1 w_2 \in M \),

\[
C = C_0 - u_2 u_2' + u_2 w_1 + w_1 w_2 + u_2' w_2'
\]

if \( w_1 w_2 \notin M \) and \( u_2' w_1 \in M \),

\[
C = C_0 - u_2 u_2' + u_2' w_1 + w_1 w_2 + w_2 u_2
\]

if \( w_1 w_2, u_2' w_1, w_2 u_2 \notin M \).

2.3. Let \( m = 3 \). According to (1), \( k = 3 \).
2.3.1. Assume that

(3) there exist \( u_1' \in V(T_0 - u_1) \) such that \( u_1u_1' \in E(C_0) \) and \( d_T(u_0, u_1') \leq 2 \).

We put

\[
C = C_0 - u_1u_1' + u_1w_3 + w_3w_2 + w_2w_1 + w_1u_1' \quad \text{if} \quad w_1w_3 \in M,
\]
\[
C = C_0 - u_1u_1' + u_1w_3 + w_3w_1 + w_1w_2 + w_2u_1' \quad \text{if} \quad w_2w_3 \in M,
\]
\[
C = C_0 - u_1u_1' + u_1w_1 + w_1w_3 + w_3w_2 + w_2u_1' \quad \text{if} \quad w_1w_3, w_2w_3 \notin M,
\]
\[
\text{and (}u_1w_2 \in M \text{ or } u_1u_1' \in M, \text{)} \quad \text{and}
\]
\[
C = C_0 - u_1u_1' + u_1w_2 + w_2w_3 + w_3w_1 + w_1u_1' \quad \text{if} \quad w_1w_3, w_2w_3, u_1w_2, u_1u_1' \notin M.
\]

2.3.2. Assume that (2) does not hold. Then there exist mutually distinct \( u_1', u_1'', u_2' \in V(T_0 - u_1 - u_2) \) such that \( u_1u_1', u_1u_1'', u_2u_2' \in E(C_0) \) and \( d_T(u_0, u_1') \leq 2 \). Clearly, \( d_T(u_0, u_1') = 3 = d_T(u_0, u_1'') \). Obviously, \( u_1'w_1 \notin M \) or \( u_1''w_1 \notin M \). Without loss of generality we assume that \( u_1'w_1 \notin M \). We put

\[
C = C_0 - u_1u_1' + u_1w_3 + w_3w_2 + w_2w_1 + w_1u_1' \quad \text{if} \quad w_1w_3 \in M,
\]
\[
C = C_0 - u_1u_1' - u_2u_2' + u_1w_3 + w_3w_1 + w_1u_1' + u_2w_2 + w_2u_2' \quad \text{if} \quad w_2w_3 \in M,
\]
\[
C = C_0 - u_2u_2' + u_2w_1 + w_1w_3 + w_3w_2 + w_2u_2' \quad \text{if} \quad w_1w_3, w_2w_3 \notin M
\]
\[
\text{and (}u_2w_2 \in M \text{ or } u_2u_2' \in M, \text{)} \quad \text{and}
\]
\[
C = C_0 - u_2u_2' + u_2w_2 + w_2w_3 + w_3w_1 + w_1u_2' \quad \text{if} \quad w_1w_3, w_2w_3, u_2w_2, u_2u_2' \notin M.
\]

2.4. Let \( m = 4 \). According to (1), \( 2 \leq k \leq 4 \). Without loss of generality we assume that

(4) if \( k = 4 \) and \( w_3w_4 \in M \), then \( u_3u_4 \in M \).

2.4.1. Assume that

(5) there exist \( v_{11}, v_{12}, v_{21}, v_{22} \in V(T_0) \) such that

\[
v_{12} \neq v_{22}, v_{11} \neq v_{12} \neq v_{21}, v_{11} \neq v_{22} \neq v_{21}, v_{11}v_{12},
\]
\[
v_{21}v_{22} \in E(C_0), d_T(u_0, v_{11}) \leq 1, d_T(u_0, v_{12}) \leq 3,
\]
\[
d_T(u_0, v_{21}) \leq 1 \text{ and } d_T(u_0, v_{22}) \leq 3.
\]

Obviously, \( v_{12}w_1 \notin M \) or \( v_{22}w_1 \notin M \). Without loss of generality we assume that \( v_{12}w_1 \notin M \). We put
\[ C = C_0 - v_{11}v_{12} + v_{11}w_2 + w_2w_3 + w_3w_4 + w_4w_1 + w_1v_{12} \]
\[
\text{if } w_2w_4 \in M, \\
C = C_0 - v_{11}v_{12} + v_{11}w_3 + w_3w_2 + w_2w_4 + w_4w_1 + w_1v_{12} \\
\text{if } w_3w_4 \in M, \\
C = C_0 - v_{11}v_{12} + v_{11}w_3 + w_3w_4 + w_4w_2 + w_2w_1 + w_1v_{12} \\
\text{if } (w_2w_4, w_3w_4 \notin M, v_{11}w_2 \in M) \\
\text{or } (v_{11}w_2, w_2w_4, w_3w_4 \notin M, w_1w_3 \in M), \text{ and} \\
C = C_0 - v_{11}v_{12} + v_{11}w_2 + w_2w_4 + w_4w_3 + w_3w_1 + w_1v_{12} \\
\text{if } v_{11}w_2, w_1w_3, w_2w_4, w_3w_4 \notin M. \\
\]

2.4.2. Assume that (5) does not hold. Then \( k = 4 \) and \( u_1u_4 \in E(C_0) \) and \( d_T(u_0, u_4) = 4 \).

We first assume that \( u_2u_3, u_2u_4 \in E(C_0) \). Then there exist \( u'_1, u'_3 \in V(T_0 - u_1 - u_3) \) such that \( u'_1 \neq u'_3, u_1u'_1, u_3u'_3 \in E(C_0), d_T(u_0, u'_1) \leq 3 \) and \( d_T(u_0, u'_3) \leq 1 \), which contradicts (5).

Now we assume that \( u_2u_3 \notin E(C_0) \) or \( u_2u_4 \notin E(C_0) \). Then there exists \( u'_2 \in V(T_0 - u_2) \) such that \( u_2u'_2 \in E(C_0) \) and \( d_T(u_0, u'_2) \leq 2 \).

2.4.2.1. Let \( w_3w_4 \notin M \). Obviously, \( u_2w_1 \notin M \) or \( u'_2w_1 \notin M \). Without loss of generality we assume that \( u_2w_1 \notin M \). We put

\[ C = C_0 - u_2u'_2 + u_2w_1 + w_1w_3 + w_3w_4 + w_4w_2 + w_2u'_2 \\
\text{if } w_2w_3 \in M \text{ or } (w_1w_4 \in M, u'_2w_2 \notin M), \\
C = C_0 - u_2u'_2 + u_2w_2 + w_2w_4 + w_4w_3 + w_3w_1 + w_1u'_2 \\
\text{if } u'_2w_2, w_1w_4 \in M, \\
C = C_0 - u_2u'_2 + u_2w_2 + w_2w_3 + w_3w_4 + w_4w_1 + w_1u'_2 \\
\text{if } u'_2w_2 \in M, w_1w_4 \notin M, \text{ and} \\
C = C_0 - u_2u'_2 + u_2w_1 + w_1w_4 + w_4w_3 + w_3w_2 + w_2u'_2 \\
\text{if } u'_2w_2, w_1w_4, w_3w_3 \notin M. \\
\]

2.4.2.2. Let \( w_3w_4 \in M \). According to (4), \( u_3u_4 \in M \). Therefore, \( u_3u_4 \notin E(C_0) \).

There exists \( u''_3 \in V(T_0 - u_2 - u_3) \) such that \( u'''_3u''_3 \in E(C_0) \). Since \( u_3u_4 \notin E(C_0) \) and \( d_T(u_0, u_3) = 3 \), we have \( d_T(u_0, u''_3) \leq 1 \). We put \( v_{11} = u''_3 \) and \( v_{12} = u_3 \). Since \( u_3u_4 \in M \), we have \( v_{12}w_1 \notin M \). Thus we can construct \( C \) in the same way as in 2.4.1.

The proof of the lemma is complete.

The following theorem is the main result of the present paper:
Theorem. Let $G$ be a connected graph of an order $\leq 4$. Then for every matching $M$ in $G^4$ there exists a hamiltonian cycle $C$ of $G^4$ such that $E(C) \cap M = \emptyset$.

Proof. Consider an arbitrary spanning tree $T$ of $G$. Denote $M_0 = M \cap E(T^4)$. Obviously, $M_0$ is a matching in $T^4$. According to Lemma 3, there exists a hamiltonian cycle $C$ of $T^4$ such that $E(C) \cap M_0 = \emptyset$. Clearly, $C$ is a hamiltonian cycle of $G^4$. Since $E(C) \subseteq E(T^4)$, we can see that $E(C) \cap M = \emptyset$, which completes the proof.

As follows from [2] and [5], if $G$ is a connected graph of an even order, then $G^2$ has a 1-factor. Combining this result with our theorem, we get the following corollary:

Corollary. Let $G$ be a connected graph of an even order $\geq 4$. Then there exist a 1-factor $F$ of $G^2$ and hamiltonian cycle $C$ of $G^4$ such that $E(C) \cap E(F) = \emptyset$.

References


Souhrn

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LADISLAV NEBESKÝ

Nechť $G$ je souvislý graf s alespoň čtyřmi uzly. V článku je dokázáno, že pro každé párování $M$ v grafu $G^4$ existuje hamiltonovská kružnice grafu $G^4$, jejíž žádná hrana do $M$ nepatří.