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CONVEX ISOMORPHISM OF Q-LATTICES

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Summary. V. I. Marmazejev introduced in [3] the following concept: two lattices are convex isomorphic if their lattices of all convex sublattices are isomorphic. He also gave a necessary and sufficient condition under which the lattices are convex isomorphic, in particular for modular, distributive and complemented lattices.

The aim of this paper is to generalize this concept to the q-lattices defined in [2] and to characterize the convex isomorphic q-lattices.

Keywords: q-lattice, convex sub$q$-lattice, convex isomorphism

AMS classification: 06A10, 06B15

1. THE LATTICE OF CONVEX SUBq-LATTICES

I. Chajda studied in [2] properties of the sets with a quasiorder (i.e. reflexive and transitive binary relation). If relation Q is a quasiorder over a set A then $E_Q = Q \cap Q^{-1}$ is an equivalence relation on A. Let us denote by A/E_Q the partition of A induced by E_Q. Then the relation Q/E_Q is an order over A/E_Q; this order will be denoted by $\leqslant_Q$. In [2] the author described especially the L-quasiordered set (i.e. a quasiordered set (A, Q) such that $(A/E_Q, \leqslant_Q)$ is a lattice). For $x \in A$ we will denote by $[x]$ the set $\{y \in A;\, yE_Qx\}$. We can choose a choice function κ over the power set $\operatorname{Exp}(A)$ of A such that $\kappa(B) \in B$ for each $b \in A/E_Q$. If A is the L-quasiordered set then we can define two binary operations $\vee$, $\wedge$ on A as follows: $x \vee y = \kappa\big(\sup_{\leqslant_Q}([x],[y])\big)$, $x \wedge y = \kappa\big(\inf_{\leqslant_Q}([x],[y])\big)$. The algebra $(A, \vee, \wedge)$ is then a q-lattice in the sense of the following definition: An algebra $(A, \vee, \wedge)$ is called a q-lattice if it satisfies the following identities in $(A, \vee, \wedge)$:

(c) :	$a \vee b = b \vee a$	$a \wedge b = b \wedge a$
(a) :	$a \vee (b \vee c) = (a \vee b) \vee c$	$a \wedge (b \wedge c) = (a \wedge b) \wedge c$
$(w - ab)$:	$a \vee (b \wedge a) = a \vee a$	$a \wedge (b \vee a) = a \wedge a$
$(w - id)$:	$a \vee b = a \vee (b \vee b)$	$a \wedge b = a \wedge (b \wedge b)$
(e) :	$a \vee a = a \wedge a$	

A q-lattice $(A, \vee, \wedge)$ induces the quasiorder Q as follows: aQb if and only if $a \vee b = b \vee b$.

Let $A = (A, Q)$ be a quasiordered set. We say that its subset $C = (C, Q)$ is convex if for each $x \in A$ the following implication holds: if $a, b \in C$, aQx and xQb, then $x \in C$. It is clear that $\emptyset, A$ are convex subsets of A. A one-element subset of A need not be convex, e.g. $\{a\}$ is not convex in $(\{a, b\}, Q)$, where aQb, bQa. Let $(A, \vee, \wedge)$ be a q-lattice and $S \subseteq A$ such that $x \vee y \in S$, $y \wedge y \in S$ for each $x, y \in S$. Then we say that $(S, \vee, \wedge)$ is the sub$q$-lattice of $(A, \vee, \wedge)$.

Let $(A, \vee, \wedge)$ be a q-lattice and (A, Q) its induced L-quasiordered set. We say that $(S, \vee, \wedge)$ is a convex sub$q$-lattice of $(A, \vee, \wedge)$ if $(S, \vee, \wedge)$ is a sub$q$-lattice of $(A, \vee, \wedge)$ and (S, Q) is a convex subset of (A, Q).

We will denote by $Cq(A)$ the set of all convex sub$q$-lattices of a q-lattice A together with $\emptyset$. Let $Cq_A(X_i, i \in I) = \{Z \in Cq(A);\, X_i \subseteq Z \quad \forall i \in I\}$.

Lemma 1. *Let $\{X_i\colon i \in I\}$ be an arbitrary system of convex subq-lattices of a q-lattice A. Then $\bigcap\{X_i, i \in I\} \in Cq(A)$.*

The proof is evident. □

Theorem 1. *Let $(A, \vee, \wedge)$ be a q-lattice and $\{X_i;\, i \in I\}$ an arbitrary system of convex subq-lattices of A. Then $(Cq(A), \subseteq)$ is a complete lattice, where $\bigwedge\{X_i;\, i \in I\} = \bigcap\{X_i;\, i \in I\}$ and $\bigvee\{X_i, i \in I\} = \bigcap Cq_A(Xi, i \in I)$ are the infimum and the supremum, respectively.*

The proof follows immediately from Lemma 1 and Theorem 17 in [4]. □

2. Convex isomorphic q-lattices

We say that q-lattices A, B are convex isomorphic if the lattices $(Cq(A), \subseteq)$ and $(Cq(B), \subseteq)$ are isomorphic.

For a quasiordered set (A, Q), $a, b \in A$, aQb we say that the set $[a, b] = \{x \in A;\, aQx, xQb\}$ is the segment of (A, Q) determined by the elements a, b (if Q is an order then $[a, b]$ is called an interval).

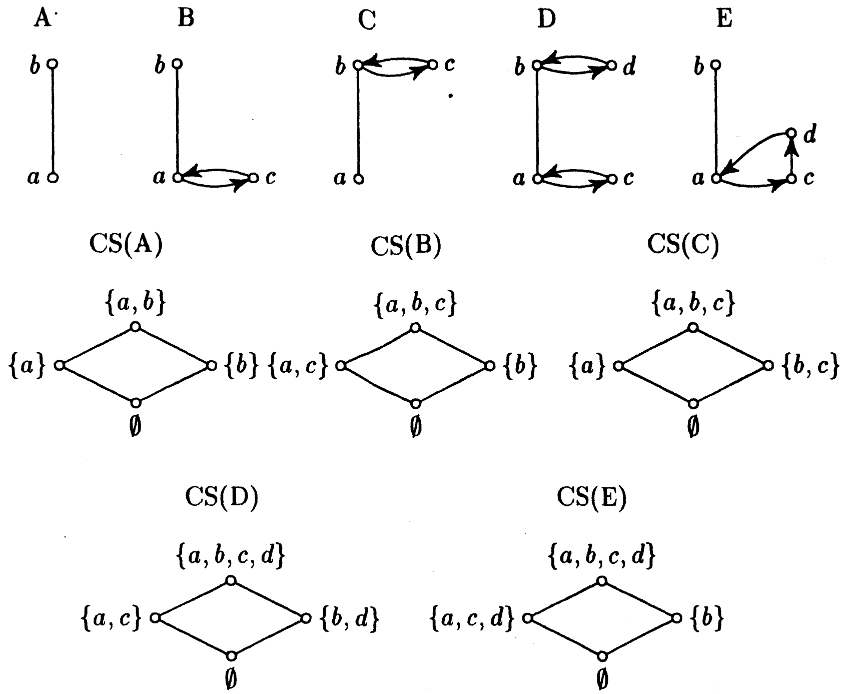


Fig. 1

E x a m p l e 1. The q-lattices A, B, C, D, E in Figure 1 are convex isomorphic.

E x a m p l e 2. Unlike the lattices and the ordered sets, a segment $[a, a]$ need not be a one-element set, e.g. for the segment $[a, a]$ of B in the foregoing example, we have $[a, a] = \{a, c\} = [c, c]$.

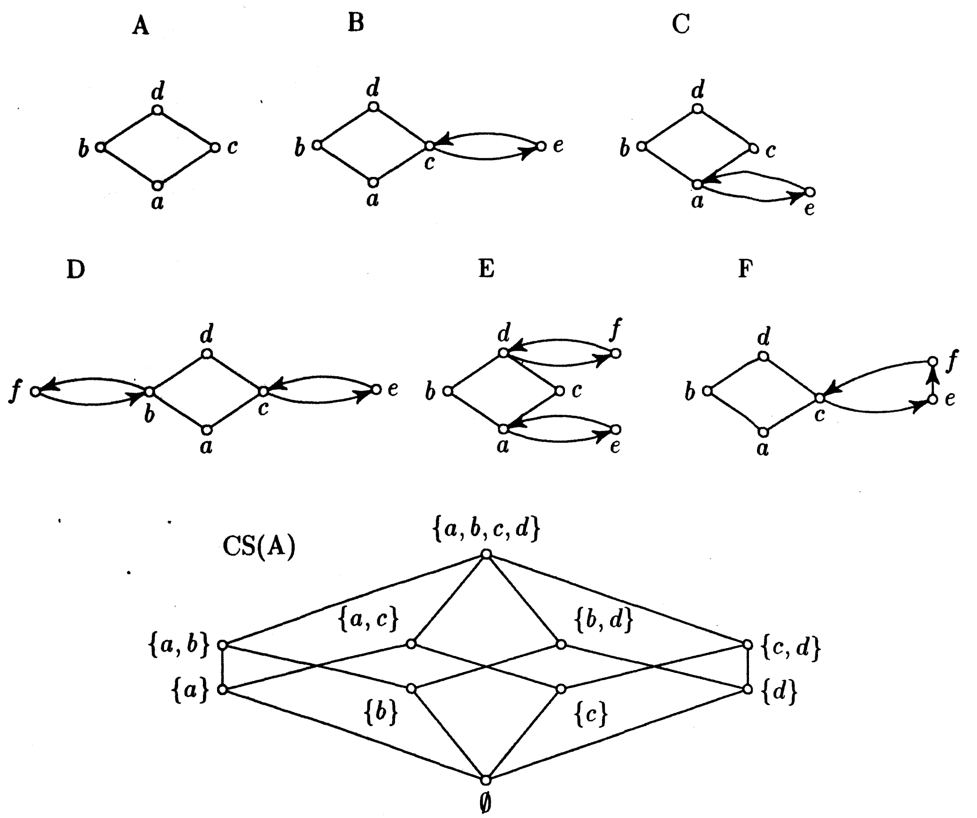
E x a m p l e 3. The q-lattices A, B, C, D, E, F in Figure 2 are convex isomorphic.

Theorem 2. *An arbitrary q-lattice $(A, \vee, \wedge)$ and its induced lattice $(A/E_Q, \leqslant_Q)$ are convex isomorphic.*

P r o o f. Since E_Q is a congruence on $(A, \vee, \wedge)$ (see [2]) we can write $X = \bigcup\{Y_i : i \in I\}$ uniquely for every set $\emptyset \neq X \in Cq(A)$, where $Y_i \in A/E_Q$ for each $i \in I$ and $Y_i \neq Y_j$ whenever $i \neq j$ for any $i, j \in I$. Therefore we can define a mapping F: $Cq(A) \to \text{Exp}(A/E_Q)$ as follows:

(i) $F(\emptyset) = \emptyset$,

(ii) $F(X) = \big\{Y_i ; Y_i \in A/E_Q ; X = \bigcup\{Y_i ; i \in I, Y_i \neq Y_j \text{ for any } i, j \in I, i \neq j\}\big\}$ for $\emptyset \neq X \in Cq(A)$.



$$CS(A) \simeq CS(B) \simeq CS(C) \simeq CS(D) \simeq CS(E) \simeq CS(F)$$

Fig. 2

We will show that F is an isomorphism of $Cq(A)$ onto $Cq(A/E_Q)$. We have $xQzQy$ for $x, y, z \in X \in Cq(A)$ if and only if $[x] \leqslant_Q [z] \leqslant_Q [y]$ for $[x], [y], [z] \in F(X)$, and also $z \in X$ if and only if $[z] \in F(X)$, according to the definition of the relation $\leqslant_Q$. Thus $F(X)$ is a convex subset of A/E_Q. Furthermore, $x, y, x \vee y, x \wedge y \in X \in Cq(A)$ if and only if $[x], [y] \in F(X), [x \vee y] = \sup\limits_{\leqslant_Q} ([x], [y]) \in F(X)$ and $[x \wedge y] = \inf\limits_{\leqslant_Q} ([x], [y]) \in F(X)$, so $F(X)$ is a sub$q$-lattice of A/E_Q. It is evident that F is a bijection of $Cq(A)$ onto $Cq(A/E_Q)$, and for arbitrary $X, Y \in Cq(A)$ we have $X \subseteq Y$ if and only if $F(X) \subseteq F(Y)$, according to the definition of the mapping F. So F is an isomorphism of the lattices $(Cq(A), \subseteq)$ and $(Cq(A/E_Q), \subseteq)$. □

The following theorem is a corollary of Theorem 2 and Theorem 1 in [3].

Theorem 3. *Let $(A, \vee, \wedge)$, $(A', \vee, \wedge)$ be q-lattices, Q, Q' their induced quasiorders. Then the following conditions are equivalent:*

a) $(A, \vee, \wedge)$, $(A', \vee, \wedge)$ *are convex isomorphic.*

b) $(A/E_Q, \leqslant_Q)$, $(A'/E_{Q'}, \leqslant_{Q'})$ *are convex isomorphic.*

c) *There exists a bijection* $f\colon A/E_Q \to A'/E_{Q'}$ *such that* $f([\inf_{\leqslant_Q}(X, Y)]) = [\inf_{\leqslant_{Q'}}(f(X), f(Y)), \sup_{\leqslant_{Q'}}(f(X), f(Y))]$ *for each* $X, Y \in A/E_Q$.

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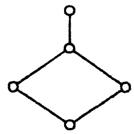

Fig. 3

E x a m p l e 4. Let L be the lattice in Figure 3 and let us denote its dual lattice by L^*. Then the lattices $S = L \times L$ and $S' = L \times L^*$ are neither isomorphic nor antiisomorphic but they are convex isomorphic (according to Lemma I in [3]). It is possible to construct q-lattices such that their induced lattices are isomorphic to S, S' by suitable adding some other elements to S, S'. Thus we get an example of convex isomorphic q-lattices (see Theorem 2) which are not lattices and which, as well as their induced lattices, are neither isomorphic nor antiisomorphic.

R e m a r k . Let $(A, \leqslant)$ be an ordered set. We say that its subset $S = (S, \leqslant)$ is convex if for each $x \in A$ the following implication holds: if $a, b \in S, a \leqslant x \leqslant b$ then $x \in S$. In [1] the author studied the lattice $(CS(A), \subseteq)$ of all convex subsets of ordered set A. There is also given the following necessary and sufficient condition under which $(CS(A), \subseteq) \simeq (CS(A'), \subseteq)$ for any ordered sets A, A' (i.e. A and A' are convex isomorphic) in [1]: The ordered ssets A, A' are convex isomorphic if and only if there exists a bijection $f\colon A \to A'$ such that $f(CS_A\langle a, b\rangle) = CS_{A'}\langle f(a), f(b)\rangle$ for each $a, b \in A$, where $CS_A\langle a, b\rangle = [a, b]$ for $a \leqslant b$ and $CS_A\langle a, b\rangle = \{a, b\}$ for $a \parallel b$.

Let us consider the more general class of quasiordered sets. Similarly, we can study the lattice $(CS(A), \subseteq)$ of all convex subsets of a quasiordered set A. We can see that $(CS(A), \subseteq) \simeq (CS(A/E_Q), \subseteq)$ for any quasiordered set A. Consequently, if (A, Q), (A', Q') are any quasiordered sets, we have $(CS(A), \subseteq) \simeq (CS(A'), \subseteq)$ if and only if there exists a bijection $f\colon A/E_Q \to A'/E_{Q'}$, such that $f(CS_{A/E_Q}\langle X, Y\rangle) = CS_{A'/E'_Q}\langle f(X), f(Y)\rangle$ for each $X, Y \in A/E_Q$.

The concepts of distributive and modular q-lattices were defined in [2]. We say that the q-lattice $(A, \wedge, \vee)$ is distributive if $a \vee (b \wedge c) = (a \vee b) \vee (a \wedge c)$ for each $a, b, c \in A$ and $(A, \wedge, \vee)$ is modular if $a \wedge ((a \wedge b) \vee c) = (a \wedge b) \vee (a \wedge c)$ for each a, b,

$c \in A$. We can also define the complemented q-lattices. A q-lattice A has zero 0 or unit 1 if $x \wedge 0 = 0$ and $x \vee 1 = 1$ for each $x \in A$. A q-lattice $(A, \wedge, \vee)$ with 0 and 1 is complemented if for each $a \in A$ there exists $b \in A$ such that $a \vee b = 1$ and $a \wedge b = 0$; b is called the complement of a. According to [2] a q-lattice $(A, \wedge, \vee)$ is distributive (modular, complemented) if and only if its induced lattice $(A/E_Q, \leqslant_Q)$ has the same property. Thus we have the following corollary with respect to Theorem 6 in [3] and Theorem 3:

Corollary. *Let* $(A, \wedge, \vee)$, $(A', \wedge, \vee)$ *be convex isomorphism q-lattices. Then* $(A, \wedge, \vee)$ *is distributive (modular, complemented) if and only if* $(A', \wedge, \vee)$ *has the same property.*

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