Mathematica Bohemica

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Mathematica Bohemica, Vol. 116 (1991), No. 4, 401–404

Persistent URL: http://dml.cz/dmlcz/126022

116 (1991) MATHEMATICA BOHEMICA No. 4, 401–404

ELEMENTARY EVALUATION OF FRESNEL'S INTEGRALS

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(Received May 7, 1990)

Summary. We evaluate the Fresnel integrals by using the Leibniz rule only on a finite interval.

Keywords: Fresnel's integrals.

INTRODUCTION

The method of complex variables is most often used to evaluate the integrals

$$G_0 = \int_0^\infty \cos x^2 \, \mathrm{d}x$$

and

$$F_0 = \int_0^\infty \sin x^2 \, \mathrm{d}x .$$

The similarity of G_0 and F_0 with

$$J_0 = \int_0^\infty \mathrm{e}^{-x^2} \, \mathrm{d}x$$

has been exploited in [FL] and [YR], real variables method is also used in [JA]. Although these evaluations are not particularly demanding they do use tools not always available in an undergraduate course like transformation to polar coordinates for an improper double integral, the Leibniz rule for an improper integral etc. Recently Weinstock [WE] obtained all three integrals by using the Leibniz rule on a finite interval only, however, the calculation for F_0 and G_0 is not as simple as for J_0. In this note we find all three integrals practically simultaneously almost as simply as Weinstock found J_0. The auxilary function h below is a slight modification of a function from [SW] where it is used for evaluation of J_0. In connection with this reference it should be mentioned that the use of the gauge integral (see also [ML] [Mac]) makes interchange of limit and integration, differentiation with respect to a parameter (even for an infinte interval) and similar tools far more accessible hence rendering elementary evaluation unnecessary. However, our method is not only elementary but also very simple.

We set $\gamma = \alpha + i\beta$ with $\alpha \leqq 0$, $\gamma \neq 0$ and define

$$J(t) = \int_0^t \exp \gamma x^2 \,\mathrm{d}x \,,$$

$$s(t) = [J(t)]^2 \,,$$

$$h(t) = \int_0^1 \frac{\exp \gamma(1 + x^2)\, t^2}{\gamma(1 + x^2)} \,\mathrm{d}x \,.$$

We first show by integration by parts that J has a limit as $t \to \infty$. As a consequence we obtain that J is bounded, say $|J(t)| \leqq K$, and that J_0 exists. Clearly

$$\int_1^t \exp \gamma x^2 \,\mathrm{d}x = \int_1^t 2\gamma x \exp \gamma x^2 \frac{1}{2\gamma x} \,\mathrm{d}x = \frac{\exp \gamma t^2}{2\gamma t} - \frac{\exp \gamma}{2\gamma} +$$

$$+ \frac{1}{2\gamma} \int_1^t \frac{\exp \gamma x^2}{x^2} \,\mathrm{d}x \,.$$

Since $|\exp \gamma x^2| \leqq 1$, the right hand side has a limit as $t \to \infty$ and so has J.

Differentiating s and applying the Leibniz rule to h shows that

$$s'(t) = 2 \exp \gamma t^2 \int_0^t \exp \gamma x^2 \,\mathrm{d}b$$

and

$$h'(t) = 2 \exp \gamma t^2 \int_0^1 t \exp \gamma x^2 t^2 \,\mathrm{d}x = 2 \exp \gamma t^2 \int_0^t \exp \gamma x^2 \,\mathrm{d}x.$$

Since s and h have the same derivative

$$(1) \qquad s(t) = h(t) - h(0) = h(t) - \frac{\pi}{4\gamma} \,.$$

Now we show that $h(t) \to 0$ as $t \to \infty$. We have

$$|\gamma\, h(t)| \leqq \left|\int_0^1 \frac{\exp \gamma x^2 t^2}{1 + x^2} \,\mathrm{d}x\right| \leqq \left|\int_0^t \frac{t \exp \gamma y^2}{t^2 + y^2} \,\mathrm{d}y\right|$$

and integration by parts shows that the last integtal equals

$$W = \frac{J(t)}{2t} + \int_0^t \frac{2ty\, J(y)}{(t^2 + y^2)^2} \,\mathrm{d}y \,.$$

Since J is bounded we obtain

$$|W| \leqq K \left(\frac{1}{2t} + \frac{1}{2t} + \frac{1}{t}\right).$$

Sending t to ∞ in (1) gives

$$\text{(2)} \qquad \lim_{t\to\infty} s(t) = -\frac{\pi}{4\gamma}.$$

Setting $\gamma = -1$ gives the value of $J_0 = \sqrt{(\pi)}/2$. Let now $\gamma = i$, it follows from (2) that

$$\text{(3)} \qquad (G_0 + iF_0)^2 = \frac{\pi i}{4}.$$

It will be shown below that $F_0 > 0$ and taking this into account it is easy to calculate from (3) that $G_0 = F_0 = \sqrt{(\pi)}/2\sqrt{2}$.

For the rest of this note we assume $\beta > 0$ and denote

$$\mathcal{G}_0 = \int_0^\infty \exp \alpha x^2 \cos \beta x^2 \, dx$$

and

$$\mathcal{F}_0 = \int_0^\infty \exp \alpha x^2 \sin \beta x^2 \, dx .$$

By separating real and imaginary parts we obtain from (2)

$$\text{(4)} \qquad \mathcal{G}_0^2 - \mathcal{F}_0^2 = -\frac{\alpha\pi}{4|\gamma|^2}$$

and

$$\text{(5)} \qquad 2\mathcal{G}_0\mathcal{F}_0 = \frac{\beta\pi}{4|\gamma|^2}$$

In order to solve (4) and (5) for $\mathcal{G}_0$ and $\mathcal{F}_0$ we show that $\mathcal{F}_0 \geqq 0$ (and therefore also F_0). Equation (5) then implies that $\mathcal{G}_0$ is also nonnegative. The substitution $y = \beta x^2$ brings $\mathcal{F}_0$ to the form

$$\int_0^\infty f(y) \sin y \, dy$$

with decreasing f. We show that for a non-negative integer k we have

$$\int_{2k\pi}^{(2k+2)\pi} f(x) \sin x \, dx \geqq 0 ,$$

This will establish the required inequality $\mathcal{F}_0 \geqq 0$. Clearly

$$\int_{2k\pi}^{(2k+1)\pi} f(x) \sin x \, dx \geqq f((2k+1)\pi)) \int_{2k\pi}^{(2k+1)\pi} \sin x \, dx + + f((2k+1)\pi)) \int_{(2k+1)\pi}^{(2k+2)\pi} \sin x \, dx = 0 .$$

Squaring (4) and (5), adding it together and taking square root gives

$$\text{(6)} \qquad \mathcal{G}_0^2 + \mathcal{F}_0^2 = \frac{\pi}{4|\gamma|}.$$

It is now easy to find $\mathscr{G}_0$ from (5) and (6)

$$\mathscr{G}_0 = \frac{\sqrt{(-\alpha + |\gamma|)}}{2\sqrt{(2)}|\gamma|}\sqrt{\pi}\,.$$

Using this and (5)

$$\mathscr{F}_0 = \frac{\sqrt{(\alpha + |\gamma|)}}{2\sqrt{(2)}|\gamma|}\sqrt{\pi}\,.$$

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Souhrn

JEDNODUCHÝ VÝPOČET FRESNELOVÝCH INTEGRÁLŮ

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V článku se vypočítávají Fresnelovy integrály, za použití Leibnizova pravidla, pouze na konečném intervalu.

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