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DISTANCES BETWEEN PARTIALLY ORDERED SETS

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Summary. A distance between finite partially ordered sets is studied. It is a certain measure of the difference of their structure.

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There exist various distances between isomorphism classes of graphs; see [1], [2], [3], [4], [5]. In a similar way as in [4] for graphs, the distance between isomorphism classes of partially ordered sets (shortly posets) may be introduced. For the sake of brevity, we will speak about the distance between posets instead of the distance between isomorphism classes of posets; we must bear in mind that then two posets having the zero distance need not be identical, but they are isomorphic.

If a poset \( P \) with an ordering \( \leq \) is given, then a subposet of \( P \) is a subset of \( P \) whose ordering is the restriction of \( \leq \) onto it. When it does not lead to a misunderstanding, we shall use the same symbol \( \leq \) in distinct posets.

Consider the class \( \mathcal{P}_n \) of all posets with \( n \) elements, where \( n \) is a positive integer. Let \( P_1, P_2 \) be two posets from \( \mathcal{P}_n \). The distance \( d(P_1, P_2) \) between the posets \( P_1, P_2 \) is equal to \( n \) minus the maximum number of elements of a poset which is isomorphic simultaneously to a subposet of \( P_1 \) and to a subposet of \( P_2 \).

We will prove a theorem.

Theorem 1. Let \( P_1, P_2 \) be two posets from \( \mathcal{P}_n \). Let \( P_0 \) be a poset containing subgraphs isomorphic to both \( P_1 \) and \( P_2 \), and let \( P_0 \) have the minimum number of elements among all posets with this property. Then the number of elements of \( P_0 \) is \( n + d(P_1, P_2) \).
Proof. Denote $d(P_1, P_2) = p$. Then there exists a subposet $P$ of $P_1$ which has $n - p$ elements and is isomorphic to a subposet $P'$ of $P_2$. Suppose that $P_1, P_2$ are disjoint and take their union $P_1 \cup P_2$; in this union $x \leq y$ if and only if $x \leq y$ in $P_1$ or in $P_2$. Now choose an isomorphism $\phi$ of $P$ onto $P'$ and for each $x \in P$ identify the vertices $x$ and $\phi(x)$. The poset thus obtained from $P_1 \cup P_2$ will be denoted by $P_0$. If $x \in P_1, y \in P_1$, then $x \leq y$ in $P_0$ if and only if $x \leq y$ in $P_1$. If $x \in P_2, y \in P_2$, then $x \leq y$ in $P_0$ if and only if $x \leq y$ in $P_2$. If $x \in P_1 - P, y \in P_2 - P$, then $x \leq y$ in $P_0$ if and only if there exists $z \in P$ such that $x \leq z$ in $P_1$ and $z \leq y$ in $P_2$. If $x \in P_2 - P, y \in P_1 - P$, then $x \leq y$ in $P_0$ if and only if there exists $z \in P$ such that $x \leq z$ in $P_2$ and $z \leq y$ in $P_1$. We shall prove that $P_0$ is really a poset. The transitivity of the ordering $\leq$ is clear. We shall only prove that $x \leq y$ and $y \leq x$ is not possible for $x \neq y$. It suffices to prove this for $x \in P_1 - P, y \in P_2 - P$; the proof for $x \in P_2 - P, y \in P_1 - P$ is analogous and for other pairs $x, y$ the assertion is clear. Suppose that $x \in P_1 - P, y \in P_2 - P$ and simultaneously $x \leq y$ and $y \leq x$. Then there exist elements $z_1, z_2$ of $P$ such that $x \leq z_1$ in $P_1$, $z_1 \leq y$ in $P_2$, $y \leq z_2$ in $P_2$, $z_2 \leq x$ in $P_1$. As $z_1 \leq y, y \leq z_2$ in $P_2$, we have $z_1 \leq z_2$ in $P_2$ and also in $P$. As $z_2 \leq x, x \leq z_1$ in $P_1$, we have $z_2 \leq z_1$ in $P_1$ and also in $P$. Therefore in $P$ we have simultaneously $z_1 \leq z_2$ and $z_2 \leq z_1$ and thus $z_1 = z_2$. But then $x \leq z_1$ and $z_1 \leq x$ in $P_1$; we have $x = z_1$, which is a contradiction with the assumption that $x \in P_1 - P, z_1 \in P$. The poset $P_0$ has $2n - (n - p) = n + p$ elements and has the required property.

Now suppose that $P_3$ is a poset containing subposets isomorphic to both $P_1$ and $P_2$. Let these subposets be $P'_1$ and $P'_2$. As $|P'_1| = |P'_2| = n$, the intersection $P'_1 \cap P'_2$ has at least $2n - |P_3|$ elements. This intersection is isomorphic to a subgraph of $P_1$ and to a subgraph of $P_2$ and therefore $|P'_1 \cap P'_2| \leq n - p$, which yields $2n - |P_3| \leq |P'_1 \cap P'_2| \leq n - p$ and therefore $|P_3| \geq n + p$.

By a chain we mean a totally ordered set, i.e. a set in which $x \leq y$ or $y \leq x$ for any two elements $x, y$. An antichain is a poset in which $x \leq y$ if and only if $x = y$. If $P$ is a poset, then by $c(P)$ (or $a(P)$) we denote the maximum number of elements of a subposet of $P$ which is a chain (or an antichain, respectively).

Theorem 2. Let $P_1, P_2$ be two posets from $\mathcal{P}_n$. Then $d(P_1, P_2) \leq n - \min \{c(P_1), c(P_2), a(P_1), a(P_2)\}$.

Proof. Both the posets $P_1, P_2$ have a subposet which is a chain with $\min \{c(P_1), c(P_2)\}$ elements, and hence $d(P_1, P_2) \leq n - \min \{c(P_1), c(P_2)\}$. They have also a subposet which is an antichain with $\min \{a(P_1), a(P_2)\}$ elements, and hence $d(P_1, P_2) \leq n - \min \{a(P_1), a(P_2)\}$. This yields the result.

Now we shall construct the distance graph $G(P_n)$ of $\mathcal{P}_n$. The vertex set of $G(\mathcal{P}_n)$ is the set of all isomorphism classes of posets from $\mathcal{P}_n$. (An isomorphism class of
posets if the class of all posets which are isomorphic to a given poset.) Two vertices of \(G(\mathcal{P})\) are adjacent if and only if the distance between posets from these classes is equal to 1.

**Theorem 3.** Let \(P_1, P_2\) be two posets from \(\mathcal{P}_n\). Then the distance between the isomorphism classes containing \(P_1\) and \(P_2\) in the graph \(G(\mathcal{P}_n)\) is equal to \(d(P_1, P_2)\).

**Proof.** If \(d(P_1, P_2) = 0\) or \(d(P_1, P_2) = 1\), then the assertion is clear. Suppose that \(d(P_1, P_2) = p > 1\). There exists a poset \(P_0\) having \(n-p\) elements and isomorphic to a subposet of \(P_1\) and to a subposet of \(P_2\). For the sake of simplicity we may suppose that \(P_0 = P_1 \cap P_2\). Let \(P_1 - P_0 = \{x_1, \ldots, x_p\}\), \(P_2 - P_0 = \{y_1, \ldots, y_p\}\). For \(i = 1, \ldots, p\) we define \(X_i = \{x_{i+1}, \ldots, x_p\}\), \(Y_i = \{y_{1}, \ldots, y_i\}\); further \(X_0 = \{x_1, \ldots, x_p\}\), \(Y_0 = X_p = \emptyset\), \(Y_p = \{y_{1}, \ldots, y_p\}\). For \(i = 0, \ldots, p\) then \(Q_i = P_0 \cup X_i \cup Y_i\). We determine the ordering \(\leq\) on \(Q_i\) for each \(i\). If both \(x\) and \(y\) are in \(P_0 \cup X_i\), then \(x \leq y\) if and only if \(x \leq y\) in \(P_1\). If both \(x\) and \(y\) are in \(P_0 \cup Y_i\), then \(x \leq y\) if and only if \(x \leq y\) in \(P_2\). If \(x \in X_i\), \(y \in Y_i\), then \(x \leq y\) if and only if there exists \(z \in P_0\) such that \(x \leq z\) in \(P_1\) and \(z \leq y\) in \(P_2\). If \(x \in X_i\), \(y \in Y_i\), then \(x \leq y\) if and only if there exists \(z \in P_0\) such that \(x \leq z\) in \(P_2\) and \(z \leq y\) in \(P_1\). Analogously as in the proof of Theorem 1 we can prove that this is really an ordering on \(Q_i\). Evidently \(Q_0 = P_1\), \(Q_p = P_2\) and \(d(Q_i, Q_{i+1}) = 1\) for \(i = 0, \ldots, p-1\) and therefore the isomorphism classes to which these posets belong form a path of length at most \(p\) in \(G(\mathcal{P}_n)\) connecting the isomorphism classes of \(P_1\) and \(P_2\). Now suppose that there exists a path in \(G(\mathcal{P}_n)\) connecting these classes and having the length \(q < p\). Let its vertices be the classes containing the posets \(P_1 = R_0, R_1, \ldots, R_q = P_2\). For \(i = 0, \ldots, p-1\) we have \(d(R_i, R_{i+1}) = 1\). Now we shall define the posets \(S_0, \ldots, S_q\). We put \(S_0 = P_1\). According to Theorem 1 there exists a poset \(S_1\) with \(n+1\) elements which has subposets \(R'_0 \cong R_0\) and \(R'_1 \cong R_1\). Now let \(2 \leq i \leq p; p\) suppose that we have constructed the set \(S_{i-1}\) which has at most \(n+i-1\) elements and contains the mentioned subposet \(R'_0 \cong R_0\) and subposet \(R'_{i-1} \cong R_{i-1}\). Again according to Theorem 1 there exists a poset having \(n+1\) elements and containing the above mentioned subposet \(R'_{i-1}\) and subposet \(R'_i \cong R_i;\) we put \(S_i = S_{i-1} \cup R'_i\) and determine the ordering in it analogously as above. As \(|R'_{i-1}| = n\) and \(R'_{i-1} \subseteq S_{i-1} \cap R'_i\), we have \(|S_i| \leq n+i\). Hence \(|S_q| \leq n+q\) and \(S_q\) contains subposets isomorphic to \(P_1\) and to \(P_2\). The intersection of these subposets has at least \(2n - (n+q) = n - q\) elements and \(d(P_1, P_2) \leq q < p\), which is a contradiction. This proves the assertion. \(\square\)

**Theorem 4.** The diameter of \(G(\mathcal{P}_n)\) is \(n-1\). The unique pair of vertices of \(G(\mathcal{P}_n)\) having the distance \(n-1\) consists of the class containing a chain and the class containing an antichain.
Proof. Evidently \( c(P) \geq 1 \), \( a(P) \geq 1 \) for every non-empty poset \( P \) and therefore \( d(P_1, P_2) \leq n - 1 \) for any two posets \( P_1, P_2 \) from \( \mathcal{P}_n \). According to Theorem 2, if \( C \) is a chain and \( A \) is an antichain with \( n \) vertices, then the maximum number of elements of a poset isomorphic simultaneously to a subposet of \( C \) and to a subposet of \( A \) is 1 and \( d(C, A) = n - 1 \). If \( P \) is a poset with \( n \) vertices being neither a chain, nor an antichain, then \( P \) contains a chain with two elements and an antichain with two elements as its subposet and hence \( d(P_1, P_2) \leq n - 2 \) whenever \( \{P_1, P_2\} \neq \{C, A\} \).

References


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