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DISTANCES BETWEEN PARTIALLY ORDERED SETS

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Summary. A distance between finite partially ordered sets is studied. It is a certain measure of the difference of their structure.

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There exist various distances between isomorphism classes of graphs; see [1], [2], [3], [4], [5]. In a similar way as in [4] for graphs, the distance between isomorphism classes of partially ordered sets (shortly posets) may be introduced. For the sake of brevity, we will speak about the distance between posets instead of the distance between isomorphism classes of posets; we must bear in mind that then two posets having the zero distance need not be identical, but they are isomorphic.

If a poset P with an ordering $\leqslant$ is given, then a subposet of P is a subset of P whose ordering is the restriction of $\leqslant$ onto it. When it does not lead to a misunderstanding, we shall use the same symbol $\leqslant$ in distinct posets.

Consider the class $\mathscr{P}_n$ of all posets with n elements, where n is a positive integer. Let P_1, P_2 be two posets from $\mathscr{P}_n$. The distance $d(P_1, P_2)$ between the posets P_1, P_2 is equal to n minus the maximum number of elements of a poset which is isomorphic simultaneously to a subposet of P_1 and to a subposet of P_2.

We will prove a theorem.

Theorem 1. *Let P_1, P_2 be two posets from $\mathscr{P}_n$. Let P_0 be a poset containing subgraphs isomorphic to both P_1 and P_2, and let P_0 have the minimum number of elements among all posets with this property. Then the number of elements of P_0 is $n + d(P_1, P_2)$.*

Proof. Denote $d(P_1, P_2) = p$. Then there exists a subposet P of P_1 which has $n - p$ elements and is isomorphic to a subposet P' of P_2. Suppose that P_1, P_2 are disjoint and take their union $P_1 \cup P_2$; in this union $x \leqslant y$ if and only if $x \leqslant y$ in P_1 or in P_2. Now choose an isomorphism φ of P onto P' and for each $x \in P$ identify the vertices x and $\varphi(x)$. The poset thus obtained from $P_1 \cup P_2$ will be denoted by P_0. If $x \in P_1$, $y \in P_1$, then $x \leqslant y$ in P_0 if and only if $x \leqslant y$ in P_1. If $x \in P_2$, $y \in P_2$, then $x \leqslant y$ in P_0 if and only if $x \leqslant y$ in P_2. If $x \in P_1 - P$, $y \in P_2 - P$, then $x \leqslant y$ in P_0 if and only if there exists $z \in P$ such that $x \leqslant z$ in P_1 and $z \leqslant y$ in P_2. If $x \in P_2 - P$, $y \in P_1 - P$, then $x \leqslant y$ in P_0 if and only if there exists $z \in P$ such that $x \leqslant z$ in P_2 and $z \leqslant y$ in P_1. We shall prove that P_0 is really a poset. The transitivity of the ordering $\leqslant$ is clear. We shall only prove that $x \leqslant y$ and $y \leqslant x$ is not possible for $x \neq y$. It suffices to prove this for $x \in P_1 - P$, $y \in P_2 - P$; the proof for $x \in P_2 - P$, $y \in P_1 - P$ is analogous and for other pairs x, y the assertion is clear. Suppose that $x \in P_1 - P$, $y \in P_2 - P$ and simultaneously $x \leqslant y$ and $y \leqslant x$. Then there exist elements z_1, z_2 of P such that $x \leqslant z_1$ in P_1, $z_1 \leqslant y$ in P_2, $y \leqslant z_2$ in P_2, $z_2 \leqslant x$ in P_1. As $z_1 \leqslant y$, $y \leqslant z_2$ in P_2, we have $z_1 \leqslant z_2$ in P_2 and also in P. As $z_2 \leqslant x$, $x \leqslant z_1$ in P_1, we have $z_2 \leqslant z_1$ in P_1 and also in P. Therefore in P we have simultaneously $z_1 \leqslant z_2$ and $z_2 \leqslant z_1$ and thus $z_1 = z_2$. But then $x \leqslant z_1$ and $z_1 \leqslant x$ in P_1; we have $x = z_1$, which is a contradiction with the assumption that $x \in P_1 - P$, $z_1 \in P$. The poset P_0 has $2n - (n - p) = n + p$ elements and has the required property.

Now suppose that P_3 is a poset containing subposets isomorphic to both P_1 and P_2. Let these subposets be P_1' and P_2'. As $|P_1'| = |P_2'| = n$, the intersection $P_1' \cap P_2'$ has at least $2n - |P_3|$ elements. This intersection is isomorphic to a subgraph of P_1 and to a subgraph of P_2 and therefore $|P_1' \cap P_2'| \leqslant n - p$, which yields $2n - |P_3| \leqslant |P_1' \cap P_2'| \leqslant n - p$ and therefore $|P_3| \geqslant n + p$. □

By a chain we mean a totally ordered set, i.e. a set in which $x \leqslant y$ or $y \leqslant x$ for any two elements x, y. An antichain is a poset in which $x \leqslant y$ if and only if $x = y$. If P is a poset, then by $c(P)$ (or $a(P)$) we denote the maximum number of elements of a subposet of P which is a chain (or an antichain, respectively).

Theorem 2. *Let P_1, P_2 be two posets from $\mathscr{P}_n$. Then $d(P_1, P_2) \leqslant n - \min\{c(P_1), c(P_2), a(P_1), a(P_2)\}$.*

Proof. Both the posets P_1, P_2 have a subposet which is a chain with $\min\{c(P_1), c(P_2)\}$ elements, and hence $d(P_1, P_2) \leqslant n - \min\{c(P_1), c(P_2)\}$. They have also a subposet which is an antichain with $\min\{a(P_1), a(P_2)\}$ elements, and hence $d(P_1, P_2) \leqslant n - \min\{a(P_1), a(P_2)\}$. This yields the result. □

Now we shall construct the distance graph $G(P_n)$ of $\mathscr{P}_n$. The vertex set of $G(\mathscr{P}_n)$ is the set of all isomorphism classes of posets from $\mathscr{P}_n$. (An isomorphism class of posets if the class of all posets which are isomorphic to a given poset.) Two vertices of $G(\mathscr{P})$ are adjacent if and only if the distance between posets from these classes is equal to 1.

Theorem 3. *Let P_1, P_2 be two posets from $\mathscr{P}_n$. Then the distance between the isomorphism classes containing P_1 and P_2 in the graph $G(\mathscr{P}_n)$ is equal to $d(P_1, P_2)$.*

Proof. If $d(P_1, P_2) = 0$ or $d(P_1, P_2) = 1$, then the assertion is clear. Suppose that $d(P_1, P_2) = p > 1$. There exists a poset P_0 having $n-p$ elements and isomorphic to a subposet of P_1 and to a subposet of P_2. For the sake of simplicity we may suppose that $P_0 = P_1 \cap P_2$. Let $P_1 - P_0 = \{x_1, \ldots, x_p\}$, $P_2 - P_0 = \{y_1, \ldots, y_p\}$. For $i = 1, \ldots, p-1$ we define $X_i = \{x_{i+1}, \ldots, x_p\}$, $Y_i = \{y_1, \ldots, y_i\}$; further $X_0 = \{x_1, \ldots, x_p\}$, $Y_0 = X_p = \emptyset$, $Y_p = \{y_1, \ldots, y_p\}$. For $i = 0, \ldots, p$ then $Q_i = P_0 \cup X_i \cup Y_i$. We determine the ordering $\leqslant$ on Q_i for each i. If both x and y are in $P_0 \cup X_i$, then $x \leqslant y$ if and only if $x \leqslant y$ in P_1. If both x and y are in $P_0 \cup Y_i$, then $x \leqslant y$ if and only if $x \leqslant y$ in P_2. If $x \in X_i$, $y \in Y_i$, then $x \leqslant y$ if and only if there exists $z \in P_0$ such that $x \leqslant z$ in P_1 and $z \leqslant y$ in P_2. If $x \in Y_i$, $y \in X_i$, then $x \leqslant y$ if and only if there exists $z \in P_0$ such that $x \leqslant z$ in P_2 and $z \leqslant y$ in P_1. Analogously as in the proof of Theorem 1 we can prove that this is really an ordering on Q_i. Evidently $Q_0 = P_1$, $Q_p = P_2$ and $d(Q_i, Q_{i+1}) = 1$ for $i = 0, \ldots, p-1$ and therefore the isomorphism classes to which these posets belong form a path of length at most p in $G(\mathscr{P}_n)$ connecting the isomorphism classes of P_1 and P_2. Now suppose that there exists a path in $G(\mathscr{P}_n)$ connecting these classes and having the length $q < p$. Let its vertices be the classes containing the posets $P_1 = R_0, R_1, \ldots, R_q = P_2$. For $i = 0, \ldots, p-1$ we have $d(R_i, R_{i+1}) = 1$. Now we shall define the posets $S_0, \ldots, S_q$. We put $S_0 = P_1$. According to Theorem 1 there exists a poset S_1 with $n+1$ elements which has subposets $R_0' \cong R_0$ and $R_1' \cong R_1$. Now let $2 \leqslant i \leqslant p$; suppose that we have constructed the set S_{i-1} which has at most $n+i-1$ elements and contains the mentioned subposet $R_0' \cong R_0$ and subposet $R_{i-1}' \cong R_{i-1}$. Again according to Theorem 1 there exists a poset having $n+1$ elements and containing the above mentioned subposet R_{i-1}' and subposet $R_i' \cong R_i$; we put $S_i = S_{i-1} \cup R_i'$ and determine the ordering in it analogously as above. As $|R_{i-1}'| = n$ and $R_{i-1}' \subseteq S_{i-1} \cap R_i'$, we have $|S_i| \leqslant n+i$. Hence $|S_q| \leqslant n+q$ and S_q contains subposets isomorphic to P_1 and to P_2. The intersection of these subposets has at least $2n - (n+q) = n - q$ elements and $d(P_1, P_2) \leqslant q < p$, which is a contradiction. This proves the assertion. □

Theorem 4. *The diameter of $G(\mathscr{P}_n)$ is $n-1$. The unique pair of vertices of $G(\mathscr{P}_n)$ having the distance $n-1$ consists of the class containing a chain and the class containing an antichain.*

P r o o f. Evidently $c(P) \geqslant 1$, $a(P) \geqslant 1$ for every non-empty poset P and therefore $d(P_1, P_2) \leqslant n - 1$ for any two posets P_1, P_2 from $\mathscr{P}_n$. According to Theorem 2, if C is a chain and A is an antichain with n vertices, then the maximum number of elements of a poset isomorphic simultaneously to a subposet of C and to a subposet of A is 1 and $d(C, A) = n - 1$. If P is a poset with n vertices being neither a chain, nor an antichain, then P contains a chain with two elements and an antichain with two elements as its subposet and hence $d(P_1, P_2) \leqslant n - 2$ whenever $\{P_1, P_2\} \neq \{C, A\}$. □

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