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A MODIFICATION OF THE MEDIAN OF A TREE

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Summary. The concept of median of a tree is modified, considering only distances from the terminal vertices instead of distances from all vertices.

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Let $T$ be a finite tree with $n$ vertices, let $V(T)$ be its vertex set. Let $d(x, y)$ denote the distance between vertices $x, y$ of $T$, i.e. the length of the (unique) path in $T$ connecting $x$ and $y$.

The median of $T$ was defined in [1]. It is a vertex $x$ at which the functional

$$a(x) = \frac{1}{n} \sum_{y \in V(T)} d(x, y)$$

attains its minimum. In [2] it was proved that in every finite tree there is either exactly one median, or exactly two medians joint by an edge. In this paper we will modify the concept of median, considering only distances from the terminal vertices instead of distances form all vertices.

Let $K(T)$ be the set of all terminal vertices of $T$, i.e. the vertices of degree 1. We may consider the functional

$$b(x) = \frac{1}{|K(T)|} \sum_{y \in K(T)} d(x, y)$$

and look for its minimum. A vertex at which this minimum is attained will be called the $K$-median of $T$.

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Obviously $|K(T)|$ is a constant and therefore the minimum of $b(x)$ is attained at the same vertices as the minimum of

$$b_0(x) = |K(T)|b(x) = \sum_{y \in K(T)} d(x, y);$$

we shall use this simpler functional.

**Lemma 1.** Let $v_0$ be a vertex of a finite tree $T$, let $v_1, v_2$ be two distinct vertices adjacent to $v_0$ in $T$. Then

$$b_0(v_0) \leq \max (b_0(v_1), b_0(v_2)),$$

where the equality may occur only if the degree of $v_0$ in $T$ is 2 and $b_0(v_0) = b_0(v_1) = b_0(v_2)$.

**Proof.** Let $B_1$ (or $B_2$) be the branch of $T$ at $v_0$ which contains $v_1$ (or $v_2$, respectively). Let $B_3$ be the union of all branches of $T$ at $v_0$ different from $B_1$ and $B_2$ (it may be empty). For $i \in \{1, 2, 3\}$ let $t_i$ be the number of terminal vertices of $T$ belonging to $B_i$. Let $x \in K(T)$. If $x$ belongs to $B_1$, then

$$d(v_1, x) = d(v_0, x) - 1;$$

in the opposite case

$$d(v_1, x) = d(v_0, x) + 1.$$

Therefore

$$b_0(v_1) = \sum_{x \in K(T)} d(v_1, x)$$

$$= \sum_{x \in K(T) \cap V(B_1)} (d(v_0, x) - 1)$$

$$+ \sum_{x \in K(T) \cap (V(B_2) \cup V(B_3))} (d(v_0, x) + 1)$$

$$= b_0(v_0) - t_1 + t_2 + t_3.$$ 

Analogously

$$b_0(v_2) = b_0(v_0) + t_1 - t_2 + t_3.$$ 

It is not possible that both the numbers $-t_1 + t_2 + t_3, t_1 - t_2 + t_3$ were negative, because then also their sum $2t_3$ would be negative, which is impossible. Therefore either $b_0(v_1) \geq b_0(v_0)$, to $b_0(v_2) \geq b_0(v_0)$ and thus

$$b_0(v_0) \leq \max (b_0(v_1), b_0(v_2)).$$
Suppose that
\[ b_0(v_0) = \max (b_0(v_1), b_0(v_2)). \]
Without loss of generality we may suppose \( b_0(v_1) \geq b_0(v_2) \) and therefore
\[ b_0(v_1) = \max (b_0(v_1), b_0(v_2)). \]
We have \( b_0(v_0) = b_0(v_1) = b_0(v_0) - t_1 + t_2 + t_3 \), which implies \( t_1 = t_2 + t_3 \). On the other hand, we have
\[ b_0(v_0) = b_0(v_1) = b_0(v_1) - t_1 + t_2 + t_3 \geq b_0(v_0) + t_1 - t_2 + t_3 = b_0(v_2), \]
which implies \( t_2 \geq t_1 \) and, together with the preceding, \( t_3 \leq 0 \). As \( t_3 \) cannot be negative, we have \( t_3 = 0 \), therefore there are no branches of \( T \) at \( v_0 \) except \( B_1 \) and \( B_2 \), and the degree of \( v_0 \) in \( T \) is 2. Further \( t_1 = t_2 \) and thus
\[ b_0(v_0) = b_0(v_1) = b_0(v_1) = b_0(v_2). \]

**Theorem 1.** Let \( T \) be a finite tree. Then \( T \) has either exactly one \( K \)-median, or all \( K \)-medians of \( T \) form a path in \( T \) whose inner vertices (if any) have degree 2 in \( T \).

**Proof.** As \( T \) is finite, the minimum of \( b_0(T) \) must be attained at least at one vertex. Now let \( v_1, v_2 \) be two non-adjacent \( K \)-medians of \( T \). Let \( P \) be the path in \( T \) connecting \( v_1 \) and \( v_2 \). Let \( v_3 \) be the inner vertex of \( P \) such that \( b_0(v_3) \) is the maximum value of \( b_0(x) \) among all vertices of \( P \). Obviously \( b_0(v_3) \geq b_0(v_1) = b_0(v_2) \). Let \( v_4, v_5 \) be the vertices of \( P \) adjacent to \( v_3 \). According to Lemma 1 we have
\[ b_0(v_3) \leq \max (b_0(v_4), b_0(v_5)). \]
However, the maximality of \( b_0(v_3) \) implies that the degree of \( v_3 \) in \( T \) is 2 and \( b_0(v_3) = b_0(v_4) = b_0(v_5) \), again according to Lemma 1. If \( v_4 = v_1 \) or \( v_4 = v_2 \), then \( b_0(v_3) = b_0(v_1) = b_0(v_2) \). If not, we proceed in a similar way with \( v_4 \) instead of \( v_3 \); after a finite number of steps we obtain the above equality. As \( b_0(v_1) \) is minimum and \( b_0(v_3) \) is maximum, all vertices \( x \) of \( P \) have the same value of \( b_0(x) \); all of them are \( K \)-medians of \( T \) and all inner vertices of \( P \) have the degree 2 in \( T \). This implies the assertion. \( \square \)
The following propositions are easy to prove.

**Proposition 1.** All vertices of a finite tree $T$ are its $K$-medians if and only if $T$ is a path.

**Proposition 2.** For each vertex $x$ of a tree $T$ with $n$ vertices the inequality $b_0(x) \geq n - 1$ holds.

**Proposition 3.** If a tree $T$ with $n$ vertices contains exactly one vertex $x$ of degree greater than 2, then this vertex $x$ is its unique $K$-median and $b_0(x) = n - 1$.

**Proposition 4.** Let a finite tree $T$ have more than one $K$-median and be different from a path, let $P$ be the path in $T$ induced by all $K$-medians of $T$. By deleting all edges and inner vertices of $P$ two trees with the same number of terminal vertices are obtained.

The proof of this proposition follows from the proof of Lemma 1, where it was shown that $t_1 = t_2$.

Now we will prove a theorem concerning $K$-medians and medians.

**Theorem 2.** For every positive integer $h$ there exists a tree $T$ which has exactly one $K$-median and exactly one median, the distance between these vertices in $T$ being $h$.

**Proof.** Let the vertex set of $T$ be $V(T) = \{v_0, v_1, \ldots, v_{2h+2}, x, y\}$ and let its edges be $v_0x$, $v_0y$ and $v_iu_i+1$ for $i = 0, \ldots, 2h + 1$. Then $T$ has the unique $K$-median $v_0$ by Proposition 3, the unique median of $T$ is $v_h$; the reader may verify it himself. Their distance is $d(v_0, v_h) = h$.

Analogously to the $K$-median, also the $K$-gravity center may be introduced. The gravity center [1] of a tree $T$ with $n$ vertices is the vertex of $T$ at which the functional

$$f(x) = \frac{1}{n} \sqrt{\sum_{y \in V(T)} (d(x,y))^2}$$

attains its minimum. Now the $K$-gravity center of $T$ is a vertex at which the functional

$$g(x) = \frac{1}{|K(T)|} \sqrt{\sum_{y \in K(T)} (d(x,y))^2}$$

attains its minimum. For the sake of simplicity instead of $g(x)$ we shall use

$$g_0(x) = \sum_{y \in K(T)} (d(x,y))^2$$

which attains its minimum at the same vertices as $g(x)$ does. \qed
Lemma 2. Let $v_0$ be a vertex of a finite tree $T$, let $v_1, v_2$ be two distinct vertices adjacent to $v_0$ in $T$. Then

$$g_0(v_0) < \max(g_0(v_1), g_0(v_2)).$$

Proof. Let the notation be the same as in the proof of Lemma 1. Then

$$g_0(v_1) = \sum_{x \in K(T) \cap V(B_1)} (d(v_0, x) - 1)^2 + \sum_{x \in K(T) \cap (V(B_2) \cup V(B_3))} (d(v_0, x) + 1)^2$$

$$= \sum_{x \in K(T)} (d(v_0, x))^2 - 2 \sum_{x \in K(T) \cap V(B_1)} d(v_0, x)$$

$$+ 2 \sum_{x \in K(T) \cap (V(B_2) \cup V(B_3))} d(v_0, x) + t_1 + t_2 + t_3$$

If $g_0(v_0) < g_0(v_1)$, then the assertion holds. Suppose that $g_0(v_0) \geq g_0(v_1)$. Then

$$-2 \sum_{x \in K(T) \cap V(B_1)} d(v_0, x) + 2 \sum_{x \in K(T) \cap (V(B_2) \cup V(B_3))} d(v_0, x) + t_1 + t_2 + t_3 \leq 0,$$

which implies

$$2 \sum_{x \in K(T) \cap (V(B_2) \cup V(B_3))} d(v_0, x) + t_1 + t_2 + t_3 \leq 2 \sum_{x \in K(T) \cap V(B_1)} d(v_0, x)$$

and then $g_0(v_0) < g_0(v_2)$, because $t_1 + t_2 + t_3 \geq 2 > 0$. □

Theorem 3. Let $T$ be a finite tree. Then $T$ has either exactly one $K$-gravity center, or exactly two $K$-gravity centers which are adjacent.

Proof. Suppose that $T$ has two non-adjacent $K$-gravity centers $v_1, v_2$. Let $P$ be the path in $T$ connecting $v_1$ and $v_2$. Let $v_3$ be the inner vertex of $P$ such that $g_0(v_3)$ is the maximum value of $g_0(x)$ among all vertices of $P$. Let $v_4, v_5$ be the vertices of $P$ adjacent to $v_3$. Then

$$g_0(v_3) < \max(g_0(v_4), g_0(v_5))$$

according to Lemma 2. Hence either $g_0(v_3) < g_0(v_4)$, or $g_0(v_3) < g_0(v_5)$. Without loss of generality let $g_0(v_3) < g_0(v_4)$. The equalities $v_4 = v_1$ or $v_4 = v_2$ contradict the minimality of $g_0(v_1) = g_0(v_2)$. The hypothesis that $v_4$ is an inner vertex of $P$ contradicts the maximality of $g_0(v_3)$ among the inner vertices of $P$. Hence two non-adjacent $K$-gravity centers cannot exist. As a tree does not contain triangles, this implies the assertion. □
References


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