Mathematica Bohemica

Jaroslav Ivančo; Bohdan Zelinka

Domination in Kneser graphs

Mathematica Bohemica, Vol. 118 (1993), No. 2, 147–152

Persistent URL: http://dml.cz/dmlcz/126050

DOMINATION IN KNESER GRAPHS

JAROSLAV IVANČO, Košice, BOHDAN ZELINKA, Liberec

(Received June 6, 1991)

Summary. The domination number and the domatic number of a certain special type of Kneser graphs are determined.

Keywords: domination number, domatic number, total domination number, total domatic number, Kneser graph

AMS classification: 05C35

Here we will determine the domination number and the domatic number of Kneser graphs $K(n, 2)$.

Let G be a finite undirected graph without loops and multiple edges. The vertex set of G is denoted by $V(G)$, its edge set by $E(G)$. Two edges are called adjacent, if they have a common end vertex.

A set $D \subseteq V(G)$ is called dominating in G, if for each vertex $x \in V(G) - D$ there exists a vertex $y \in D$ adjacent to x. The minimum number of vertices of a dominating set in G is called the domination number of G and denoted by $\delta(G)$.

A partition of $V(G)$, all of whose classes are dominating sets in G, is called a domatic partition of G. The maximum number of classes of a domatic partition of G is called the domatic number of G and denoted by $d(G)$.

The domatic number of a graph was introduced by E. J. Cockayne and S. T. Hedetniemi in [1].

Now let k, n be two integers such that $2 \leqslant k < n$. Then the Kneser graph $K(n, k)$ is defined in the following way. Let M be a set, $|M| = n$. Let $V(K(n, k))$ be the set of all subsets of M which have the cardinality k. The vertex set of $K(n, k)$ is $V(K(n, k))$ and two vertices are adjacent in $K(n, k)$ if and only if they are disjoint (as sets).

This concept was introduced by M. Kneser [4] and studied by L. Lovász [5]. A particular case when $n = 2k + 1$ was studied by H. M. Mulder [6] under the name of an odd graph.

In this paper we shall consider the particular case when $k = 2$. At proving theorems we shall use the following proposition whose proof is straightforward.

Proposition 1. *The Kneser graph $K(n,2)$ for each $n \geqslant 3$ is isomorphic to the complement of the line graph of the complete graph K_n.*

We determine the domination number of $K(n, 2)$.

Theorem 1. *The domination number of the Kneser graph $K(n,2)$ for each $n \geqslant 3$ is equal to 3. The set $D = \{u_1, u_2, u_3\}$ is dominating in $K(n,2)$ for $n \geqslant 5$ if and only if either $u_1 \cap u_2 = u_1 \cap u_3 = u_2 \cap u_3 = \emptyset$, or $|u_1 \cup u_2 \cup u_3| = 3$.*

Proof. If D fulfils the above described condition, then each set from $V(K(n,2)) - D$ is disjoint at least with one element of D, therefore D is a dominating set in $K(n,2)$. As such a set exists in each $K(n,2)$ for $n \geqslant 3$, we have $\delta(K(n,2)) \leqslant 3$. Suppose that there exists a two-element dominating set $\{v_1, v_2\}$ in $K(n,2)$. The vertices v_1, v_2 are distinct two-element subsets of M and thus the set differences $v_1 - v_2$, $v_2 - v_1$ are non-empty. If $a \in v_1 - v_2$, $b \in v_2 - v_1$, then the set $\{a,b\}$ is an element of $V(K(n,2)) - \{v_1, v_2\}$ and has non-empty intersections with both v_1, v_2. This is a contradiction with the assumption that $\{v_1, v_2\}$ is a dominating set in $K(n,2)$. Hence $\delta(K(n,2)) = 3$.

In the rest of the proof we use Proposition 1 for simplifying the considerations. We shall consider $K(n,2)$ as the complement of the line graph of K_n. Let $D = \{u_1, u_2, u_3\}$ be a dominating set in $K(n,2)$. Then u_1, u_2, u_3 are such edges of K_n that no other edge of K_n is adjacent to all of them. The condition described in the theorem means that u_1, u_2, u_3 either are pairwise non-adjacent, or form a triangle. Let us look at the other cases which can occur for three edges. If these edges form a path of length 3, then, without loss of generality, $u_1 = \{a,b\}$, $u_2 = \{b,c\}$, $u_3 = \{c,d\}$, where a, b, c, d are pairwise different elements of M. Then $\{a,c\}$ has non-empty intersections with each of u_1, u_2, u_3. If u_1, u_2, u_3 form two disjoint paths of lengths 2 and 1, then, without loss of generality, $u_1 = \{a,b\}$, $u_2 = \{b,c\}$, $u_3 = \{d,e\}$ and $\{b,d\}$ has non-empty intersections with each of u_1, u_2, u_3. If u_1, u_2, u_3 form a star, then, without loss of generality, $u_1 = \{a,b\}$, $u_2 = \{a,c\}$, $u_3 = \{a,d\}$. As we have supposed $n \geqslant 5$, there exists $e \in M - \{a,b,c,d\}$ and $\{a,e\}$ has non-empty intersections with all u_1, u_2, u_3. We have exhausted all possible cases and thus we have proved the necessity of the above mentioned condition. □

Now we shall study the domatic number. We shall start with a proposition.

Proposition 2. *The domatic number satisfies*

$$d(K(5,2)) = 2.$$

Proof. Consider the complete graph K_5. In it no three edges are pairwise non-adjacent. Therefore three edges of K_5 form a dominating set in the complement of the line graph of K_5 if and only if they form a triangle. In K_5 there exists a pair of edge-disjoint triangles, therefore $d(K(5,2)) \geqslant 2$. But if we take any two edge-disjoint triangles in K_5, then the set of all edges not belonging to them forms a quadrangle. The set of all edges of a quadrangle in K_5 is not dominating in the complement of the line graph of K_5, because an edge forming a diagonal of this quadrangle is adjacent to all these edges. As $K(5,2)$ has ten edges, any partition of its edge set into three classes of cardinalities at least 3 has two three-element classes and one four-element class. Therefore there is no domatic partition of $K(5,2)$ with three classes and $d(K(5,2)) = 2$. □

Note that $K(5,2)$ is the Petersen graph.
Now we shall prove a theorem.

Theorem 2. *Let n be an integer, $n \geqslant 3$, $n \neq 5$. Then the domatic number satisfies*

$$d(K(n,2)) = \left\lfloor \frac{1}{6} n(n-1) \right\rfloor.$$

Proof. The graph $K(3,2)$ consists of three isolated vertices and thus $d(K(3,2)) = 1$. The graph $K(4,2)$ consists of three pairwise disjoint copies of K_2 and thus $d(K(4,2)) = 2$. Now consider $n \geqslant 6$. We shall again use Proposition 1.

Let $n \equiv 0 \pmod 6$. Then we may write $n = 6p$, where p is an integer. The complete graph K_{6p} can be decomposed into $6p-1$ pairwise edge-disjoint linear factors; each of them has $3p$ edges. In each of these factors we choose a partition of its edge set into p classes, each having three elements. Each class consists of three pairwise non-adjacent edges and therefore these classes form a domatic partition of the complement of the line graph of K_n. The total number of these classes is $p(6p-1) = \frac{1}{6}n(n-1)$. As $\delta(K(n,2)) = 3$, a domatic partition cannot have more classes and $d(K(n,2)) = \frac{1}{6}n(n-1)$.

Let $n \equiv 1 \pmod 6$. Then we may write $n = 6p+1$. The graph K_{6p+1} can be decomposed into $6p+1$ maximal matchings, each having $3p$ edges. We proceed analogously as in the preceding case and obtain $p(6p+1) = \frac{1}{6}n(n-1)$ pairwise disjoint dominating sets of the complement of the line graph of K_n. Again $d(K(n,2)) = \frac{1}{6}n(n-1)$.

Let $n \equiv 2 \pmod 6$. Then we may write $n = 6p+2$. We choose two vertices u_1, u_2 of K_n and consider the complete graph K_{6p} obtained by deleting these vertices. This graph can be decomposed [3] into one linear factor and $3p-1$ Hamiltonian circuits. We choose one Hamiltonian circuit C of them and denote the vertices of K_{6p} by $v_1, \ldots, v_{6p}$ in such a way that the edges of C are $v_i v_{i+1}$ for $i = 1, \ldots, 6p$, the subscripts being taken modulo $6p$. We consider $6p$ triples of edges of the form $\{u_1 v_i, u_2 v_{i+1}, v_{i+2} v_{i+3}\}$ (as $6p > 3$, these edges are pairwise non-adjacent) for $i = 1, \ldots, 6p$, subscripts being taken modulo $6p$. These triples form a partition of the set of edges of C and all edges joining a vertex of $\{u_1, u_2\}$ with a vertex of K_{6p}. Further, for each of the Hamiltonian circuits of the decomposition which are different from C we choose a partition of its edge set into $2p$ classes, each having three pairwise non-adjacent edges (evidently such a partition exists). Finally, with the linear factor of the decomposition we proceed as in the case $n \equiv 0$; we obtain p new dominating sets. The remaining edge $u_1 u_2$ can be added to an arbitrary one of them. The total number of the dominating sets constructed is $6p + (3p-2)2p + p = p(6p+3) = \frac{1}{6}n(n-1) - \frac{1}{3} = \lfloor \frac{1}{6}n(n-1) \rfloor$.

Let $n \equiv 3 \pmod 6$. Then we may write $n = 6p+3$. The graph K_{6p+3} can be decomposed [3] into $3p+1$ pairwise edge-disjoint Hamiltonian circuits. In each of them we choose a partition of its edge set into $2p+1$ classes, each having three pairwise non-adjacent edges. These classes form a domatic partition of the complement of the line graph of K_n. The total number of these classes is $(3p+1)(2p+1) = \frac{1}{6}n(n-1)$.

Let $n \equiv 4 \pmod 6$. Then we may write $n = 6p+4$. We choose four vertices u_1, u_2, u_3, u_4 of K_n and consider the complete graph K_{6p} obtained by deleting them. This graph can be decomposed [3] into one linear factor and $3p-1$ Hamiltonian circuits. We choose two Hamiltonian circuits C_1, C_2 of them. Now we proceed analogously as in the case $n \equiv 2$, taking u_1, u_2, C_1 and u_3, u_4, C_2. We have $12p$ triples of pairwise non-adjacent edges which form a partition of the set of all edges of C_1 and C_2 and all edges joining a vertex of $\{u_1, \ldots, u_4\}$ with a vertex of K_{6p}. With all Hamiltonian circuits of the decomposition which are different from C_1 and C_2 we proceed as in the case $n \equiv 2$ with circuits; we obtain $2p(3p-3)$ triples of pairwise non-adjacent edges. Further, we take the edges of the linear factor of the decomposition and choose three of them, e_1, e_2, e_3. We consider three triples $\{e_1, u_1u_2, u_3u_4\}$, $\{e_2, u_1u_3, u_2u_4\}$, $\{e_3, u_1u_4, u_2u_3\}$ and $p-1$ triples of edges of the linear factor different from e_1, e_2, e_3. All described sets form a domatic partition of the complement of the line graph of K_n with $12p + 2p(3p-3) + 3 + p - 1 = (3p+2)(2p+1) = \frac{1}{6}n(n-1)$ classes.

Finally, let $n \equiv 5 \pmod 6$. We may write $n = (6p+3)+2$. We choose two vertices u_1, u_2 of K_n and consider the complete graph K_{6p+3} obtained by deleting these vertices. This graph can be decomposed [3] into $3p+1$ pairwise edge-disjoint Hamiltonian circuits. As $p \geqslant 1$, we may proceed analogously as in the case $n \equiv 2$ (except the linear factor). In this way we obtain $6p+3+3p(2p+1) = \lfloor \frac{1}{6}n(n-1) \rfloor$ pairwise disjoint dominating sets of the complement of the line graph of K_n. This proves the theorem. □

Now we shall add some results on the total dominating number and the total domatic number.

A subset $D \subseteq V(G)$ is called a total dominating set in G, if for each vertex $x \in V(G)$ there exists a vertex $y \in D$ adjacent to x. The minimum number of vertices of a total dominating set in G is called the total domination number of G and denoted by $\delta_t(G)$.

A partition of G, all of whose classes are total dominating sets in G, is called a total domatic partition of G. The maximum number of classes of a total domatic partition of G is called the total domatic number of G and denoted by $d_t(G)$.

The total domatic number of a graph was introduced by E. J. Cockayne, R. M. Dawes and S. T. Hedetniemi in [2]. Note that a total dominating set can exist only in a graph without isolated vertices and thus $\delta_t(G)$ and $d_t(G)$ are well-defined only for such graphs.

Theorem 3. *For the total domination number of the Kneser graph $K(n,2)$ the following holds:*

$$\delta_t(K(4,2)) = 6,$$
$$\delta_t(K(5,2)) = 4,$$
$$\delta_t(K(n,2)) = 3 \text{ for } n \geqslant 6.$$

The set $D = \{u_1, u_2, u_3\}$ is a total dominating set in $K(n,2)$ if and only if $u_1 \cap u_2 = u_1 \cap u_3 = u_2 \cap u_3 = \emptyset$.

Remark. The graph $K(3,2)$ consists of three isolated vertices and therefore $\delta_t(K(3,2))$ is not defined.

Proof. In Theorem 1 two cases were described when the set $D = \{u_1, u_2, u_3\}$ is dominating, and a set can be total dominating only if it is dominating. In the case $|u_1 \cup u_2 \cup u_3| = 3$ the set D is not total dominating, because for no element of D there exists another element of D disjoint with it. On the other hand, if $u_1 \cap u_2 = u_1 \cap u_3 = u_2 \cap u_3 = \emptyset$, this set is total dominating. In $K(4,2)$ and in $K(5,2)$ no such set exists. The graph $K(4,2)$ is regular of degree 1 and thus $\delta_t(K(4,2)) = |V(K(4,2))| = 6$. The graph $K(5,2)$ is the Petersen graph. It contains ten stars $K_{1,3}$ and the vertex set of each of them is total dominating in it; hence $\delta_t(K(5,2)) = 4$. In K_n for $n \geqslant 6$ there exist three pairwise non-adjacent edges and thus $\delta_t(K(n,2)) = 3$. □

Theorem 4. *Let n be an integer, $n \geqslant 6$. Then*

$$d_t(K(n,2)) = \left\lfloor \frac{1}{6} n(n-1) \right\rfloor .$$

Proof. The domatic partition of such a graph constructed in the proof of Theorem 2 is also a total domatic partition, which implies the assertion. □

At the end we shall express a proposition concerning the remaining cases.

Proposition 3. *The total domatic numbers satisfy*

$$d_t(K(4,2)) = 1,$$
$$d_t(K(5,2)) = 2.$$

Proof. The graph $K(4,2)$ has vertices of degree 1, therefore [2] its total domatic number is 1. The total domatic number of $K(5,2)$ cannot exceed its domatic number equal to 2. There exists a partition of $K(5,2)$ (the Petersen graph) into two classes, each of which induces a circuit of length 5. This is a total domatic partition of $K(5,2)$ and therefore $d_t(K(5,2)) = 2$. □

References

[1] *E. J. Cockayne, S. T. Hedetniemi*: Towards a theory of domination in graphs, Networks *7* (1977), 247–261.
[2] *E. J. Cockayne, R. M. Dawes, S. T. Hedetniemi*: Total domination in graphs, Networks *10* (1980), 211–219.
[3] *F. Harary*: Graph Theory, Addison-Wesley, Reading, Mass., 1969.
[4] *M. Kneser*: Aufgabe 300, Jahrber. Deutsch. Math. Verein *58* (1978), 319–324.
[5] *L. Lovász*: Kneser's conjecture, chromatic number and homotopy, J. Combin. Theory A *25* (1978), 319–324.
[6] *H. M. Mulder*: The Interval Function of a Graph, Math. Centrum Amsterdam, 1980.

Author's address: *J. Ivančo*, katedra geometrie a algebry PF UPJŠ, Jesenná 5, 041 54 Košice; *B. Zelinka*, katedra matematiky VŠST, Voroněžská 13, 461 17 Liberec.